Lipschitz spheres in the Heisenberg group

Robert Young
University of Toronto
(joint with Stefan Wenger)

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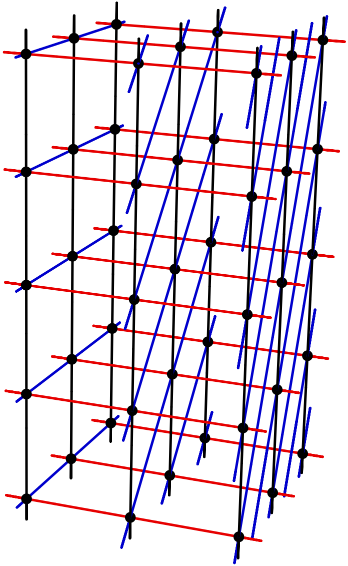
$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Cayley graph of $\mathbb{H}_{\mathbb{Z}}$ (1)



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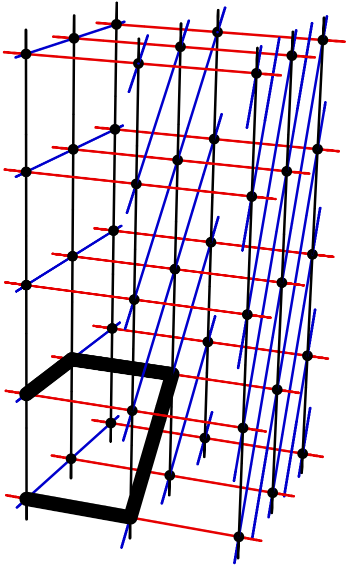
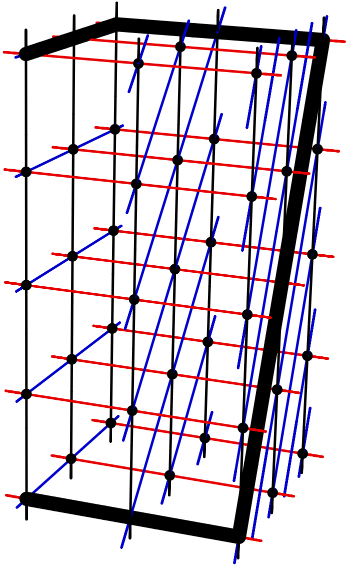
The Cayley graph of $\mathbb{H}_{\mathbb{Z}}$ (2)



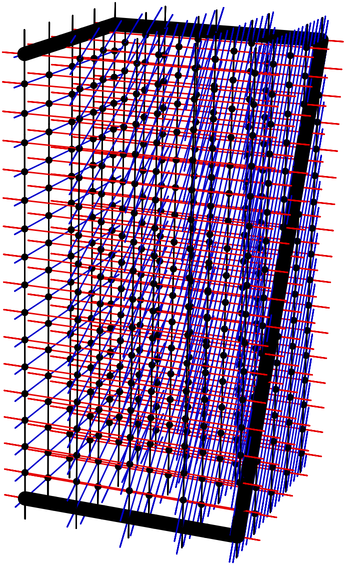
$z = xyx^{-1}y^{-1}$

The Cayley graph of $\mathbb{H}_{\mathbb{Z}}$ (3)



$$z^4 = x^2 y^2 x^{-2} y^{-2}$$

The Cayley graph of $\mathbb{H}_{\mathbb{Z}}$ (4)



$$z^{n^2} = x^n y^n x^{-n} y^{-n}$$

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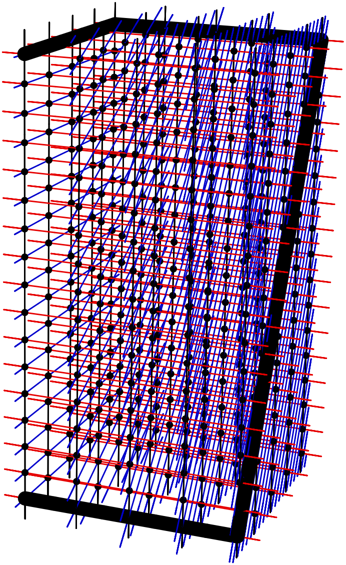
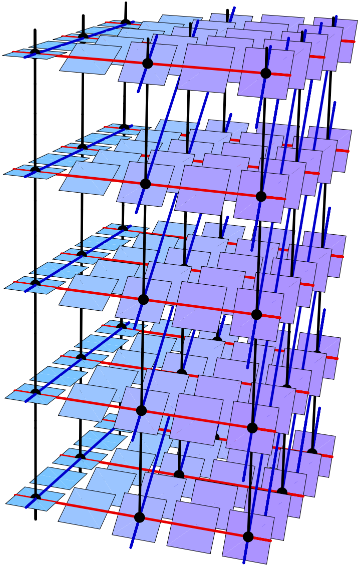


$$z^{n^2} = x^n y^n x^{-n} y^{-n}$$

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & tx & t^2 z \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix}$$

is an automorphism.

From Cayley graph to sub-riemannian metric



- Horizontal planes spanned by red and blue edges.

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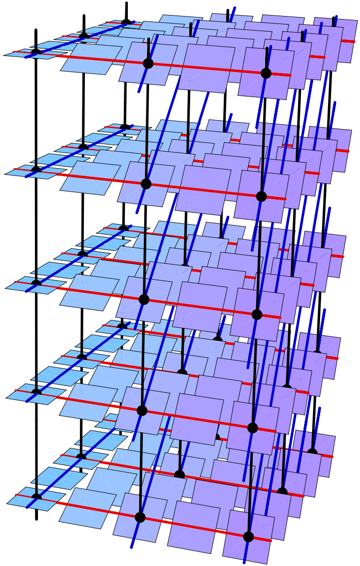

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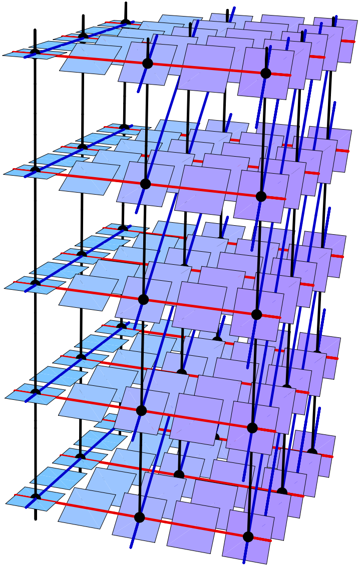

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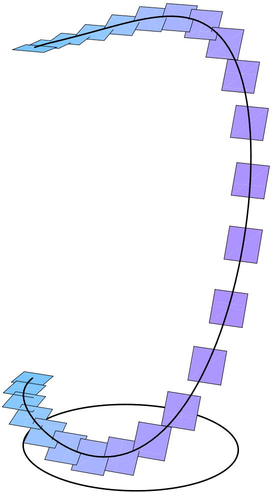
$$d(u, v) = \inf\{\ell(\gamma) \mid \gamma \text{ is a horizontal curve from } u \text{ to } v\}$$

Horizontal curves in $\mathbb{H}$



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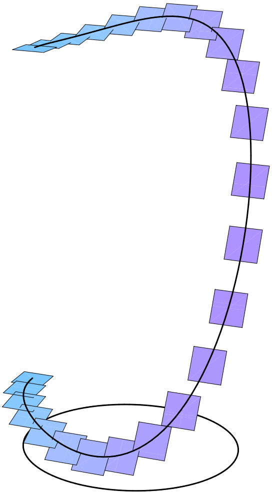

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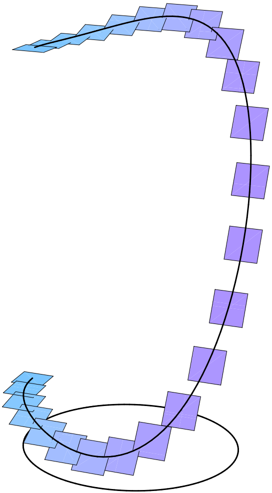

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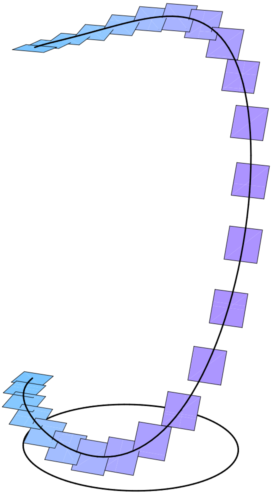

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- The length of the lift is the same as the length of the original curve.
- The change in height along the lift of a closed curve is the signed area of the curve.
- Geodesics are lifts of circles.

Horizontal surfaces in $\mathbb{H}$?

- The horizontal plane field is non-integrable, so there are no smooth horizontal surfaces in $\mathbb{H}$.

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- Even Lipschitz maps $B^n \to \mathbb{H}$ are limited:

Theorem (Pansu)

Any Lipschitz map $f : B^n \to \mathbb{H}$ from the n-ball is a.e. Pansu differentiable (differentiable, horizontal, and its differential is a homomorphism $\mathbb{R}^n \to \mathfrak{h}$ of Lie algebras.)

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- In fact,

Theorem (Wenger-Y.)

If M is a simply-connected manifold, then any Lipschitz map $f : M \to \mathbb{H}$ factors through an $\mathbb{R}$-tree.

Lipschitz homotopy groups

If X is a metric space, we define

$$\pi_n^{\mathrm{Lip}}(X) = \{\text{Lipschitz maps } S^n \to X\} / \{\text{Lipschitz homotopies } S^n \times [0,1] \to X\}$$

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If X is a riemannian manifold or simplicial complex, then $\pi_n^{\mathrm{Lip}}(X) = \pi_n(X)$.

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Question: What happens in higher-dimensional Heisenberg groups?

Higher Heisenberg groups

$$\mathbb{H}_n = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_n & z \\ 0 & 1 & \dots & 0 & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & y_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \middle| x_i, y_i, z \in \mathbb{R} \right\}$$

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- The change in height along the lift of a closed curve γ is

  $$\int_\gamma x_1\, dy_1 + \cdots + x_n\, dy_n = \int_\beta dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n,$$

  where β is a disc with $\partial\beta = \gamma$. That is, the change in height is the symplectic area of γ.

Plenty of low-dimensional horizontal surfaces

- Example: if $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, let $g : \mathbb{R}^n \to \mathbb{H}_n$

$$g(\vec{x}) = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, x_1, \dots, x_n, f(\vec{x})).$$

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If $k < n$, then any Lipschitz sphere $S^k \to \mathbb{H}_n$ can be extended to a Lipschitz map $B^{k+1} \to \mathbb{H}_n$.

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(i.e., $\pi_k^{\mathrm{Lip}}(\mathbb{H}_n) = 0$ if $k < n$.)

Not many high-dimensional horizontal surfaces

But there aren't many $(n+1)$-dimensional surfaces:

Theorem (Pansu)

Any Lipschitz map $B^k \to \mathbb{H}_n$ is differentiable and horizontal a.e., and its derivative has rank $\leq n$ a.e.

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(i.e. $\pi_n^{\mathrm{Lip}}(\mathbb{H}_n)$ is uncountably generated)

What happens in higher dimensions?

Suppose $\alpha : S^n \to \mathbb{H}_n$ is a smooth embedding and $\beta : S^k \to S^n$ is a nontrivial element of $\pi_k(S^n)$.
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Sometimes!

Main Theorem

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If $\alpha : S^n \to \mathbb{H}_n$ is a smooth embedding, $\beta : S^k \to S^n$ is an element of $\pi_k(S^n)$ which is a suspension (in particular, if $k < 2n - 1$), and $k \geq n + 2$, then $\alpha \circ \beta$ can be extended to a Lipschitz map $B^{k+1} \to \mathbb{H}_n$.

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If $\beta : S^k \to S^n$ is an element of $\pi_k(S^n)$ which is a suspension and $k \geq n + 1$, then β can be extended to a Lipschitz map $B^{k+1} \to B^{n+1}$ whose derivative has rank $\leq n$ almost everywhere.

Open questions

- What happens for other values of β?
- What are the Lipschitz homotopy groups of $\mathbb{H}_n$?