Lipschitz spheres in the Heisenberg group

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(joint with Stefan Wenger)

Apr. 2013
The Heisenberg group

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\[ x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
The Cayley graph of $\mathbb{H}_Z(1)$

\[
x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
\[
y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]
\[
z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
The Cayley graph of $\mathbb{H}_Z (2)$

\[ z = xyx^{-1}y^{-1} \]
The Cayley graph of $\mathbb{H}_\mathbb{Z} (3)$

\[ z^4 = x^2y^2x^{-2}y^{-2} \]
The Cayley graph of $\mathbb{H}_Z(4)$

$$z^{n^2} = x^n y^n x^{-n} y^{-n}$$
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$z^{n^2} = x^n y^n x^{-n} y^{-n}$

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & tx & t^2z \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix}$$

is an automorphism.
From Cayley graph to sub-riemannian metric

- Horizontal planes spanned by red and blue edges.

\[ d(u, v) = \inf \{ \ell(\gamma) | \gamma \text{ is a horizontal curve from } u \text{ to } v \} \]
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Horizontal curves in $\mathbb{H}$

- Curves in the plane lift to horizontal curves in $\mathbb{H}$.

- The length of the lift is the same as the length of the original curve.

- The change in height along the lift of a closed curve is the signed area of the curve.

- Geodesics are lifts of circles.
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Horizontal surfaces in $\mathbb{H}$?

- The horizontal plane field is non-integrable, so there are no smooth horizontal surfaces in $\mathbb{H}$. 

- Theorem (Pansu) Any Lipschitz map $f: B_n \to \mathbb{H}$ from the $n$-ball is a.e. Pansu differentiable (differentiable, horizontal, and its differential is a homomorphism $\mathbb{R}^n \to h$ of Lie algebras.)

- So any Lipschitz map $f: B_n \to \mathbb{H}$ has derivative of rank 1 a.e.

- In fact, Theorem (Wenger-Y.) If $M$ is a simply-connected manifold, then any Lipschitz map $f: M \to \mathbb{H}$ factors through an $\mathbb{R}$-tree.
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Lipschitz homotopy groups

If $X$ is a metric space, we define

$$\pi_n^{\text{Lip}}(X) = \{\text{Lipschitz maps } S^n \to X\}/$$
$$\{\text{Lipschitz homotopies } S^n \times [0, 1] \to X\}$$
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If $X$ is a riemannian manifold or simplicial complex, then $\pi_n^{\text{Lip}}(X) = \pi_n(X)$. 
Lipschitz homotopy groups of $\mathbb{H}$

$\mathbb{H} \cong \mathbb{R}^3$, so $\pi_n(\mathbb{H}) = 0$ for all $n$, but:

- $\pi_{\text{Lip}}^0(\mathbb{H}) = 0$ (if $\mathbb{H}$ is geodesic)
- $\pi_{\text{Lip}}^1(\mathbb{H})$ is uncountably generated (lots of closed curves but no surfaces)
- $\pi_{\text{Lip}}^n(\mathbb{H}) = 0$ for all $n > 1$ (higher-dimensional spheres factor through $\mathbb{R}$-trees)
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Question: What happens in higher-dimensional Heisenberg groups?
Higher Heisenberg groups

\[ H_n = \left\{ \begin{pmatrix} 1 & x_1 & \ldots & x_n & z \\ 0 & 1 & \ldots & 0 & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & y_n \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \middle| x_i, y_i, z \in \mathbb{R} \right\} \]
Plenty of horizontal curves

- Any curve $\gamma : [0, 1] \to \mathbb{R}^{2n}$ lifts to a curve in $\mathbb{H}_n$ of the same length.
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The change in height along the lift of a closed curve $\gamma$ is

$$\int_{\gamma} x_1 \, dy_1 + \cdots + x_n \, dy_n = \int_{\beta} dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n,$$

where $\beta$ is a disc with $\partial \beta = \gamma$. That is, the change in height is the symplectic area of $\gamma$. 
Plenty of low-dimensional horizontal surfaces

- Example: if $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, let $g : \mathbb{R}^n \to \mathbb{H}_n$

  $$g(\vec{x}) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, x_1, \ldots, x_n, f(\vec{x})).$$

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- Any isotropic surface \( i : \mathbb{R}^k \to \mathbb{R}^{2n} \) lifts to a horizontal surface in \( \mathbb{H}_n \).
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**Theorem (Gromov)**

*If $k < n$, then any Lipschitz sphere $S^k \to H_n$ can be extended to a Lipschitz map $B^{k+1} \to H_n$.***
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(i.e., $\pi^\text{Lip}_k(\mathbb{H}_n) = 0$ if $k < n$.)
Not many high-dimensional horizontal surfaces

But there aren’t many \((n + 1)\)-dimensional surfaces:

**Theorem (Pansu)**

*Any Lipschitz map \(B^k \to \mathbb{H}_n\) is differentiable and horizontal a.e., and its derivative has rank \(\leq n\) a.e.*
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(i.e. \(\pi_n^{\text{Lip}}(\mathbb{H}_n)\) is uncountably generated)
What happens in higher dimensions?

Suppose \( \alpha : S^n \to \mathbb{H}_n \) is a smooth embedding and \( \beta : S^k \to S^n \) is a nontrivial element of \( \pi_k(S^n) \).

Can \( \alpha \circ \beta \) be extended to a map \( B^{k+1} \to \mathbb{H}_n \)?

Theorem (DeJarnette-Hajlasz-Lukyanenko-Tyson)

There is no smooth horizontal extension.

Theorem (Wenger-Y.)

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Main Theorem

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If $\alpha : S^n \to \mathbb{H}_n$ is a smooth embedding, $\beta : S^k \to S^n$ is an element of $\pi_k(S^n)$ which is a suspension (in particular, if $k < 2n - 1$), and $k \geq n + 2$, then $\alpha \circ \beta$ can be extended to a Lipschitz map $B^{k+1} \to \mathbb{H}_n$. 
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Theorem (Wenger-Y.)

If $\beta : S^k \to S^n$ is an element of $\pi_k(S^n)$ which is a suspension and $k \geq n + 1$, then $\beta$ can be extended to a Lipschitz map $B^{k+1} \to B^{n+1}$ whose derivative has rank $\leq n$ almost everywhere.
Open questions

- What happens for other values of $\beta$?
- What are the Lipschitz homotopy groups of $\mathbb{H}_n$?