Some new results on sub-Riemannian geodesics in stratified groups

joint with E. Le Donne, G.P. Leonardi and R. Monti

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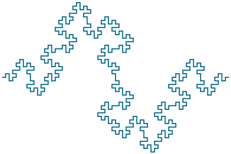

IPAM, April 30th, 2013

Part I

Sub-Riemannian geodesics

SUB-RIEMANNIAN METRIC SPACES

Let $X = (X_1, \dots, X_m)$ be smooth, linearly indep. vector fields in $\mathbb{R}^n$.

A Lipschitz curve $\gamma : [0,1] \to \mathbb{R}^n$ is *horizontal* if

$$\dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)) \quad \text{for a.e. } t \in [0,1]\,.$$

Definition (CC distance)

The *Carnot-Carathéodory distance* between $x, y \in \mathbb{R}^n$ is

$$d_c(x,y) := \inf \left\{ \ell_X(\gamma) := \int_0^1 |h(t)|\, dt : \begin{array}{l} \gamma : [0,1] \to \mathbb{R}^n \text{ horizontal} \\ \gamma(0) = x, \gamma(1) = y \end{array} \right\}.$$

In general $m < n$ ("sub"-Riemannian). If the *bracket generating condition*

$$\operatorname{rank} \mathfrak{Lie}\{X_1, \dots, X_m\}(x) = n \quad \forall\, x \in \mathbb{R}^n$$

holds, then d_c is an actual distance.

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GEODESICS

An horizontal curve $\gamma : [0,1] \to \mathbb{R}^n$ is a *geodesic* if

$$\ell_X(\gamma) = d_c(\gamma(0), \gamma(1)).$$

Geodesics exist. Are they regular?

Pontryagin maximum principle

Let $\gamma : [0,1] \to \mathbb{R}^n$ be a geodesic such that $\dot\gamma = \sum_{j=1}^m h_j X_j(\gamma)$ and $\gamma(0) = 0$. Then, there exist

$$\xi_0 \in \mathbb{R}, \qquad \xi : [0,1] \to \mathbb{R}^n \text{ Lipschitz}, \qquad (\xi_0, \xi(t)) \neq (0,0)$$

such that for any $j = 1, \dots, m$

$$\xi_0 h_j + \langle \xi, X_j(\gamma) \rangle = 0 \quad \text{a.e. on } [0,1]$$
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The function $\xi(t)$ is called *adjoint curve*; it satisfies a certain ODE.
We call *extremal* any curve satisfying the above necessary condition.

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Normal geodesics

If $\xi_0 \neq 0$ we say that γ is a *normal* geodesic and the equations

$$h_j = -\frac{1}{\xi_0}\langle \xi, X_j(\gamma)\rangle \quad \text{a.e. on } [0,1]$$

yield C^∞-smoothness. All Riemannian geodesics are normal; the same holds in *step 2* structures.

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ABNORMAL GEODESICS

If $\xi_0 = 0$, γ is an *abnormal* geodesic and satisfies

$$\langle \xi, X_j(\gamma) \rangle = 0 \quad \forall j = 1, \ldots, m.$$

Remark. A geodesic could be both normal and abnormal.

Goh condition

If γ is a strictly abnormal geodesic, then for any $i, j \in \{1, \ldots, m\}$

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REGULARITY OF GEODESICS

The regularity of sub-Riemannian geodesics (in fact, of strictly abnormal ones) is one of the main open questions: see the books by Montgomery (2002) and Agrachev-Sachkov (2004), Agrachev-Barilari-Boscain (forthcoming).

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However, strictly abnormal minimizers do exist (Montgomery 1994, Liu-Sussmann 1995, Sussmann 1996) even in stratified groups (Golé-Karidi 1995). But all these examples are smooth!

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Part II

Stratified groups

Stratified groups

Definition

A *stratified* (or *Carnot*) *group* $\mathbb{G}$ is a connected, simply connected, nilpotent Lie group whose Lie algebra admits the stratification

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

where $V_i = [V_1, V_{i-1}], i = 2, \ldots, s$ (s = "step") and $[V_1, V_s] = \{0\}$.

Concretely: $\mathbb{G} \equiv (\mathbb{R}^n, *)$. We endow $\mathbb{G}$ with the sub-Riemannian structure induced by a left-invariant, bracket-generating basis

$$X_1, \ldots, X_m$$

of the first layer V_1 ($m = \dim V_1 =$ "rank").

The induced distance d_c is left-invariant.

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Exponential coordinates of the second kind

Fix an *adapted* basis of $\mathfrak{g} \equiv T\mathbb{G}$

$$\underbrace{X_1, \ldots X_m}_{\text{basis of } V_1}, \underbrace{X_{m+1}, \ldots, X_{m_2}}_{\text{basis of } V_2}, \ldots, X_n \, .$$

We identify $\mathbb{G} \equiv (\mathbb{R}^n, *)$ through *exponential coordinates of the second kind*

$$\begin{aligned} x = (x_1, \ldots, x_n) \longleftrightarrow & \exp(x_n X_n) * \cdots * \exp(x_2 X_2) * \exp(x_1 X_1) \\ & = e^{x_1 X_1} \circ e^{x_2 X_2} \circ \cdots \circ e^{x_n X_n}(0) \, . \end{aligned}$$

In particular, $X_1 = \mathrm{e}_1$.

Grayson-Grossmann (1990) explicitly write the horizontal vector fields $X_1, \ldots, X_m$ for *free* stratified groups. Such formulae are not presently known for general stratified groups.

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Part III

Abnormal geodesics and algebraic varieties in stratified groups

DUAL CURVE IN STRATIFIED GROUPS

Let $\mathbb{G}$ be a stratified group and let θ^j be the base of 1-covectors dual to the basis $X_1, \ldots, X_n$:

$$\theta^j(X_i) = \delta_i^j .$$

Given an extremal $\gamma(t)$ with dual curve $\xi(t)$, define $\lambda(t)$ by

$$\xi_1 dx^1 + \cdots + \xi_n dx^n = \lambda_1 \theta^1(\gamma) + \cdots + \lambda_n \theta^n(\gamma) .$$

For abnormal extremals we have

$$\lambda_1 = \cdots = \lambda_m \equiv 0$$

and for strictly abnormal ones the Goh condition reads as

$$\lambda_{m+1} = \cdots = \lambda_{m_2} \equiv 0, \qquad m_2 = \dim V_1 + \dim V_2 .$$

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Main result

Set $v := \lambda(0)$.

Theorem (Le Donne-Leonardi-Monti-V.)

Let γ be an extremal (either normal or abnormal) in a stratified group $\mathbb{G} \equiv \mathbb{R}^n$ with $\gamma(0) = 0$. Then, there exist polynomials $P_1^v, \ldots, P_n^v$,

$$P_j^v(x) = \sum_{I \in \mathbb{N}^n} \sum_{\ell=1}^{n} \frac{(-1)^{|I|}}{I!} C_{j,I}^{\ell} \, v_\ell \, x^I \qquad (x^I := x_1^{I_1} x_2^{I_2} \cdots x_n^{I_n}),$$

such that $\lambda_j(t) = P_j^v(\gamma(t))$ for any $j = 1, \ldots, n$.

For $j, \ell \in \{1, \ldots, n\}$ and $I = (I_1, \ldots, I_n) \in \mathbb{N}^n$, the *generalized structure constants* $C_{j,I}^{\ell}$ are defined by

$$[\cdots[[X_j, \underbrace{X_1], \ldots, X_1]}_{I_1 \text{ times}}, \underbrace{X_2], \ldots, X_2]}_{I_2 \text{ times}}, \ldots] = \sum_{\ell=1}^{n} C_{j,I}^{\ell} X_\ell \,.$$

MAIN RESULT

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KEY PROPERTY OF THE POLYNOMIALS

Core of the proof: prove that the formulae

$$X_i P_j^v = \sum_{k=1}^{n} c_{ij}^k P_k^v$$

hold for any $i, j = 1, \ldots, n$ and $v \in \mathbb{R}^n$, where $[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$.

Due to Grayson-Grossmann result, the proof of the above formulae is much easier in free stratified groups.

Corollary

A) Let γ be an abnormal extremal in a stratified group. Then, there exist polynomials $P_1^v, \ldots, P_m^v$ such that

$$P_j^v(\gamma(t)) = 0 \quad \forall j = 1, \ldots, m$$

and at least one of them is not the zero polynomial.

B) Let γ be a strictly abnormal geodesic in a stratified group. Then, there exist polynomials $P_1^v, \ldots, P_m^v, \ldots, P_{m_2}^v$ such that

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Remark. The "converse" of A) holds as well. In other words, we provide a characterization of abnormal extremals in stratified groups.

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Comments and future directions

Our results could provide a first step towards a “dimensional reduction” argument to attack the problem of geodesics’ regularity. Moreover, they could be useful to classify the possible singularities of abnormal geodesics. One could also try to adapt the tecniques of Leonardi-Monti (2008) to exclude certain singularities for minimizers.

Applications - 1

Theorem (Tan-Yang, 2013)

Every geodesic γ in a stratified group $\mathbb{G}$ of step 3 is C^∞-smooth.

Proof. Reason by contradiction: then γ is strictly abnormal and contained in a vertical hyperplane

$$\{x \in \mathbb{R}^n : P(x) = a_1 x_1 + \cdots + a_m x_m = 0\}$$

Thus

$$\dot\gamma \in \{a_1 X_1 + \cdots + a_m X_m\}^\perp \cap V_1,$$

i.e., γ is contained in a stratified group of step 3 and rank $m-1$. Use induction on the rank. □

Remark. Tan and Yang prove that abnormal geodesics in step 3 groups are "lines" $t \mapsto \exp(tX)(P)$ for suitable $X \in V_1$, $P \in \mathbb{G}$.

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Polynomials and Tanaka prolongation

Assume one can extend $\mathfrak{g}$ to a graded Lie algebra

$$\tilde{\mathfrak{g}} = \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_s$$

in such a way that $[V_i, V_j] \subset V_{i+j}$. The biggest (possibly infinite) such extension is *Tanaka prolongation*; it is never trivial ($V_0 \neq \emptyset$).

Extend the basis $X_1, \dots, X_n$ to a graded basis $\dots, X_{-1}, X_0, X_1, \dots, X_n$ of $\tilde{\mathfrak{g}}$, then one can formally define

$$P_j^v(x) = \sum_{I \in \mathbb{N}^n} \sum_{\ell=1}^{n} \frac{(-1)^{|I|}}{I!} \widetilde{C}_{j,I}^{\ell} \, v_\ell \, x^I \qquad \forall j \leq n \, .$$

Again, $X_i P_j^v = \sum_{k=1}^n \tilde{c}_{ij}^k P_k^v$ for any $j \leq n$ and $1 \leq i \leq n$.

Corollary

If γ is an abnormal extremal, then $P_j^v(\gamma) \equiv 0$ for any $j \leq 0$ ($v := \lambda(0)$).

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Extend the basis $X_1, \ldots, X_n$ to a graded basis $\ldots, X_{-1}, X_0, X_1, \ldots, X_n$ of $\tilde{\mathfrak{g}}$, then one can formally define

$$P_j^v(x) = \sum_{I \in \mathbb{N}^n} \sum_{\ell=1}^{n} \frac{(-1)^{|I|}}{I!} \tilde{C}_{j,I}^{\ell} \, v_\ell \, x^I \qquad \forall j \leq n\,.$$

Again, $X_i P_j^v = \sum_{k=1}^n \tilde{c}_{ij}^k P_k^v$ for any $j \leq n$ and $1 \leq i \leq n$.

Corollary

If γ is an abnormal extremal, then $P_j^v(\gamma) \equiv 0$ for any $j \leq 0$ ($v := \lambda(0)$).

Applications - 2

Theorem

Let $\mathbb{G} \equiv \mathbb{R}^8$ be the free group of rank 2 and step 4. Then the set

$$\{p \in \mathbb{G} : \text{there exists an abnormal extremal from 0 to } p\}$$

is contained in an algebraic variety.

Proof. If γ is an abnormal extremal, then there exists v such that

$$P_j^v(\gamma) \equiv 0, \qquad -3 \leq j \leq 3\,.$$

Since $P_j^v(x) = \sum_{k=4}^{8} v_k Q_{jk}(x)$, one has

$$(Q_{j4}(\gamma), \ldots, Q_{j8}(\gamma)) \perp (v_4, \ldots, v_8) \quad \forall j = -3, \ldots, 3\,,$$

i.e., the 5×7 matrix $(Q_{jk}(\gamma))_{jk}$ has rank at most 4, and the 21 determinants of its 5×5 minors vanish along γ. □

This improves some results by A. Agrachev and has some applications related to the smoothness of the sub-Riemannian distance.

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An open question

Roughly speaking: rich algebraic structure $\Rightarrow$ "long" prolongation $\Rightarrow$ a lot of polynomials $\Rightarrow$ a few abnormal extremals.

Question

Assume that $\tilde{\mathfrak{g}}$ is "long enough" (e.g., if $\mathbb{G}$ is non-rigid). Is it true that the set

$$\{p \in \mathbb{G} : \text{there exists an abnormal extremal from } 0 \text{ to } p\}$$

is contained in an algebraic variety?