Isodiametric problem in Carnot groups

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Isodiametric inequality in $\mathbb{R}^n$

- Isodiametric inequality:

$$\mathcal{L}^n(A) \leq 2^{-n}\omega_n \, (\text{diam}\, A)^n$$

where $\omega_n = \mathcal{L}^n(B(0,1))$.

- Equality holds iff A is a ball (up to a null set).
- An application:

$$\mathcal{H}^n = \mathcal{S}^n = 2^n \omega_n^{-1} \mathcal{L}^n$$

where

$$\mathcal{H}^n(A) = \lim_{\delta \downarrow 0} \inf\{\sum_i (\text{diam}\, A_i)^n \; ; \; A \subset \cup_i A_i \; , \; \text{diam}\, A_i \leq \delta\}$$

$$\mathcal{S}^n(A) = \lim_{\delta \downarrow 0} \inf\{\sum_i (\text{diam}\, A_i)^n \; ; \; A \subset \cup_i A_i \; , \; A_i \text{ ball} \; , \; \text{diam}\, A_i \leq \delta\}$$

General setting

Let G be a locally compact topological group equipped with:

- dilations $\delta_\lambda : G \to G$, $\lambda > 0$, group homeomorphisms satisfying:
  - $\delta_1 = Id$
  - $\delta_{\lambda\lambda'} = \delta_\lambda \circ \delta_{\lambda'}$
- a left invariant homogeneous distance d inducing the topology of the group, i.e., satisfying:
  - $d(x.y, x.z) = d(y, z)$
  - $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$

Hausdorff measures

- Assume that for some Q

$$0 < \mathcal{H}^Q(B) < +\infty \quad \text{for any ball } B.$$

- It follows that
  - $\dim_H(G) = Q$,
  - $\mathcal{H}^Q$ and $\mathcal{S}^Q$ are Haar measures on G and hence are proportional.

- Note that

$$\mathcal{H}^Q(\delta_\lambda(A)) = \lambda^Q \mathcal{H}^Q(A) \quad \text{and} \quad \mathcal{S}^Q(\delta_\lambda(A)) = \lambda^Q \mathcal{S}^Q(A).$$

Hausdorff measures

Proposition

We have

$$\mathcal{S}^Q(B) = (\text{diam}\, B)^Q$$

for any ball B.

Proposition

We have

$$\mathcal{S}^Q = C_d\, \mathcal{H}^Q$$

where

$$C_d = \sup \left\{ \frac{\mathcal{S}^Q(A)}{(\text{diam}\, A)^Q} \ ; \ 0 < \text{diam}\, A < +\infty \right\}.$$

Isodiametric problem

One seeks for the maximal possible value of the measure of sets with a given diameter:

$$\sup\{\ \mathcal{S}^Q(A)\ ;\ \ \mathrm{diam}\, A = \lambda\ \}$$

where $\lambda > 0$ is fixed.

Questions:

- sup = max? Yes.
- which are the sets, called isodiametric, that realize the max?
- what kind of properties can be deduced from these informations?

Answers to these questions do not depend on the choice of a Haar measure. They may on the contrary depend strongly on the distance d the group G is equipped with.

Isodiametric problem

Using dilations it holds:

$$\sup\{\mathcal{S}^Q(A)\ ;\ \operatorname{diam} A = \lambda\} = \lambda^Q \sup\{\mathcal{S}^Q(A)\ ;\ \operatorname{diam} A = 1\}$$

and

$$\sup\{\mathcal{S}^Q(A);\ \operatorname{diam} A = 1\} = \sup\left\{ \frac{\mathcal{S}^Q(A)}{(\operatorname{diam} A)^Q}\ ;\ 0 < \operatorname{diam} A < +\infty \right\} = C_d.$$

Note that: $C_d \geq 1$,
$C_d = 1$ iff balls are isodiametric,
and, as a consequence, $\mathcal{S}^Q = \mathcal{H}^Q$ iff balls are isodiametric.

Heisenberg group

- The Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R}$

· group law: $[z,t] \cdot [z',t'] = [z+z', t+t'+2\,\mathrm{Im}\, z\bar{z}']$

· dilations: $\delta_\lambda([z,t]) = [\lambda z, \lambda^2 t]$

with the left invariant homogeneous distance

$$d_\infty([z,t],[z',t']) = \|[z,t]^{-1} \cdot [z',t']\|_\infty$$

where $\|[z,t]\|_\infty = \max(\|z\|, |t|^{1/2})$.

- Homogeneous dimension: $Q = 2n+2$.

Carnot groups

- A non abelian Carnot group G is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathcal{G}$ admits a stratification,

$$\mathcal{G} = \oplus_{j=1}^{k} V_j \ , \ [V_1, V_j] = V_{j+1} \ , \ V_k \neq \{0\} \ , \ V_{k+1} = \{0\} \ ,$$

  for some integer $k \geq 2$ called the step of the stratification.
- The exponential map $\exp : \mathcal{G} \to G$ is a global diffeomorphism and the group law is given by the Campbell-Hausdorff formula,

$$\exp X \cdot \exp Y = \exp H(X, Y) \ ,$$

  where $H(X, Y) = X + Y + [X, Y]/2 + \cdots$.
- Dilations on $\mathcal{G}$ are given by $\delta_\lambda(\sum_{j=1}^{k} Y_j) = \sum_{j=1}^{k} \lambda^j Y_j$, $Y_j \in V_j$, $\lambda > 0$.

A left invariant homogeneous distance d_∞

- Let $(X_1, \ldots, X_n)$ be a basis of $\mathcal{G}$ adapted to the stratification and define an Euclidean norm $\| \cdot \|$ on $\mathcal{G}$ by declaring it orthonormal.
- Choose positive coefficients c_j so that $\|H(Y,Z)\|_\infty \leq \|Y\|_\infty + \|Z\|_\infty$ where $\|Y\|_\infty = \max_j c_j \|Y_j\|^{1/j}$ whenever $Y = Y_1 + \cdots + Y_k$, $Y_j \in V_j$.
- Set $\|x\|_\infty = \| \exp^{-1} x\|_\infty$ and $d_\infty(x,y) = \|x^{-1} \cdot y\|_\infty$.

  Then d_∞ is a left invariant homogeneous distance on G.

- The homogeneous dimension is $Q = \sum_{j=1}^{k} j \dim V_j$.

Theorem [R]

Let G be a non abelian Carnot group. There exists a homogeneous distance d on G such that $C_d > 1$, i.e., for which balls are not isodiametric.

Proof.

A sufficient condition for not being an isodiametric set:

Assume that A is compact and $\operatorname{diam} A > 0$. Assume there exists $x \in \partial A$ such that $d(x, y) < \operatorname{diam} A$ for all $y \in A$. Then A is not isodiametric.

Take $d = d_\infty$ and B a ball centered at 0. Apply the lemma to $x = \exp X \in \partial B$ with $X \in V_k$.

Corollary

Let G be a non abelian Carnot group equipped with some homogeneous distance. Let $Q = \dim_{\mathrm{H}}(G)$. Then G is purely Q-unrectifiable.

Otherwise one can find a Lipschitz map $f : A \subset \mathbb{R}^Q \to (G, d_\infty)$ such that $0 < \mathcal{H}^Q(f(A)) < +\infty$. Then it holds

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^Q(f(A) \cap B(x,r))}{(2r)^Q} = 1 \quad \text{for } \mathcal{H}^Q \text{ a.e. } x \in f(A).$$

On the other hand,

$$\frac{\mathcal{H}^Q(f(A) \cap B(x,r))}{(2r)^Q} \leq \frac{\mathcal{H}^Q(B(x,r))}{(2r)^Q} = \frac{\mathcal{H}^Q(B(x,r))}{\mathcal{S}^Q(B(x,r))} = C_{d_\infty}^{-1} < 1,$$

which gives a contradiction.

Carnot groups

Theorem [R]

Let G be a non abelian Carnot group equipped with a sub-Riemannian distance d_c. Assume that there exists a length minimizing curve $\gamma : [a, b] \to G$ that stops to be minimizing after reaching $\gamma(b)$. Then balls are not isodiametric.

Example: $\mathbb{H}^n$

Open question: does there exist a Carnot group G and a homogeneous distance d on G such that $C_d = 1$?

Besicovitch 1/2-problem

- Given a metric space (M, d), let $\sigma_n(M, d)$ denote the smallest number such that every subset $A \subset M$ of finite $\mathcal{H}^n$-measure having
$$\underline{D}_n(A, x) > \sigma_n(M, d)$$
at $\mathcal{H}^n$-a.e. $x \in A$ is n-rectifiable,

  where
$$\underline{D}_n(A, x) = \liminf_{r \downarrow 0} \frac{\mathcal{H}^n(A \cap B(x, r))}{(2r)^n}.$$

- $\sigma_n(M, d) \leq 1$
- $\sigma_n(M, d) = 1$ does not give any significant information about rectifiability.

Besicovitch 1/2-problem

- $1/2 \leq \sigma_1(\mathbb{R}^2) \leq 3/4$

Conjecture: $\sigma_1(\mathbb{R}^2) = 1/2\,?$

More generally: $\sigma_n(M, d) \leq 1/2\,?$

- $\sigma_1(M, d) \leq \dfrac{2 + \sqrt{46}}{12} \sim 0.73$

- $\sup\{\sigma_k(H)\,;\ H \text{ Hilbert space}\} < 1$

Besicovitch 1/2-problem

Corollary

Let (G, d) be a non abelian Carnot group equipped with a homogeneous distance. Let $Q = \dim_{\mathrm{H}}(G)$. Then one has

$$\sigma_Q(G, d) = C_d^{-1}.$$

In particular $\sigma_Q(G, d) < 1$ iff balls are not isodiametric.

Theorem [R]

In $(\mathbb{H}^n, d_\infty)$, we have $1 < C_{d_\infty} < 2$ and hence

$$1/2 < \sigma_{2n+2}(\mathbb{H}^n, d_\infty) < 1.$$

NB: $1/2 < \sigma_{2n+2}(\mathbb{H}^n, d_c) < 1$ for $n = 1, \ldots, 8$.

Sub-Riemannian distance on $\mathbb{H}^n$

- The Heisenberg group $\mathbb{H}^n$ is a Carnot group of step 2. The stratification of the Lie algebra is given by

$$V_1 = \text{span}\,\{X_j, Y_j \;;\; j = 1, \dots, n\} \quad \text{and} \quad V_2 = \text{span}\,\{\partial_t\}$$

where $X_j = \partial_{x_j} + 2y_j\partial_t,\; Y_j = \partial_{y_j} - 2x_j\partial_t$.

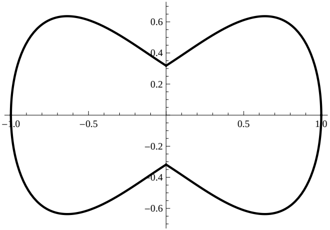
- Sub-Riemannian distance:

$$d_c(p, q) = \inf\{\mathit{length}(\gamma);\; \gamma \text{ horizontal curve joining } p \text{ to } q\}.$$

where γ is said to be horizontal if it is absolutely continuous and such that $\dot\gamma(t) \in \text{span}\{X(\gamma(t));\; X \in V_1\}$ a.e.

Isodiametric sets in $(\mathbb{H}^n, d_c)$

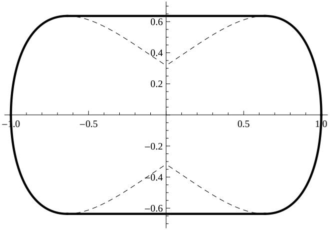
(joint work with G.P. Leonardi and D. Vittone)



Unit ball in $(\mathbb{H}^n, d_c)$

Isodiametric sets in $(\mathbb{H}^n, d_c)$

(joint work with G.P. Leonardi and D. Vittone)



The set A_2

- diam $A_2 = 2$
- $\forall\, p \in \partial A_2,\ \exists\, q \in \partial A_2$ such that $d_c(p,q) = \operatorname{diam} A_2$
- A_2 is rotationally invariant

Rotationally invariant isodiametric sets in $(\mathbb{H}^n, d_c)$

(joint work with G.P. Leonardi and D. Vittone)

- Given $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$, let $r_\theta([z,t]) = [(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n), t]$
- $\mathcal{R} = \{F \subset \mathbb{H}^n \ ; \ r_\theta(F) \subset F \text{ for all } \theta \in \mathbb{R}^n\}$
- $C_{d_c,\mathcal{R}} = \sup \left\{ \frac{\mathcal{S}^{2n+2}(F)}{(\operatorname{diam} F)^{2n+2}} \ ; \ F \in \mathcal{R}, \ 0 < \operatorname{diam} F < +\infty \right\}$
- $\mathcal{I}_\mathcal{R} = \{F \in \mathcal{R} \ ; \ F \text{ compact}, \ \operatorname{diam} F > 0, \ \mathcal{S}^{2n+2}(F) = C_{d_c,\mathcal{R}} \ (\operatorname{diam} F)^{2n+2}\}$.

Rotationally invariant isodiametric sets in $(\mathbb{H}^n, d_c)$

(joint work with G.P. Leonardi and D. Vittone)

- For $F \subset \mathbb{H}^n$, let $\mathrm{St}\, F$ denote its Steiner symetrisation w.r.t. the $\mathbb{C}^n$-plane:

$$\mathrm{St}\, F = \{[z,t] \in \mathbb{H}^n \; ; \; z \in \pi(F), \; 2|t| \leq \mathcal{L}^1(\{s \in \mathbb{R} \; ; \; [z,s] \in F\})\}$$

where $\pi([z,t]) = z$.

- Let $\sigma([z,t]) = [\bar{z}, t]$.

Lemma

Let $F \subset \mathbb{H}^n$ be compact and such that $\sigma(F) = F$. Then $\mathrm{diam}(\mathrm{St}\, F) \leq \mathrm{diam}\, F$.

Rotationally invariant isodiametric sets in $(\mathbb{H}^n, d_c)$

(joint work with G.P. Leonardi and D. Vittone)

Theorem [Leonardi-R-Vittone]

Let $E \in \mathcal{I}_{\mathcal{R}}$. Then

$$\mathrm{St}\, E \in \mathcal{I}_{\mathcal{R}} \quad \text{and} \quad \mathrm{St}\, E = A_{\operatorname{diam} E}.$$

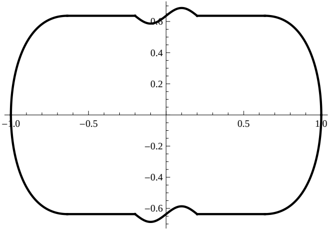
Corollary (non uniqueness in $\mathcal{I}_{\mathcal{R}}$)

There exists $E \in \mathcal{I}_{\mathcal{R}}$ such that $p \cdot E$ does not coincide with $A_{\operatorname{diam} E}$ for all $p \in \mathbb{H}^n$.

Corollary (existence of non rotationally invariant isodiametric sets)

There exists an isodiametric set E which is not rotationally invariant.

Small perturbation of A_2



A small pertubation of A_2 with same diameter and same volume.