Regularity of subRiemannian isometries

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Plan

Collaborations with Alessandro Ottazzi & Luca Capogna.

1. Isometries of Carnot groups
2. Isometries of geodesic homogeneous spaces
3. Isometries of subRiemannian manifolds

Carnot groups

A (subFinsler) **Carnot group** is a locally compact and geodesic metric space X that is

1. isometrically homogeneous: $\mathrm{Isom}(X) \curvearrowright X$ transitively,
2. self-similar: $\exists \lambda > 1$ s.t. $\lambda X = X$.

Fact: such spaces have Lie groups structures.
Aim: study the isometries of X.

An example of a Carnot group

$X = \mathbb{E}^n =$ Euclidean space

1. $f : \mathbb{E}^n \to \mathbb{E}^n$ isometry with $f(0) = 0$
   $\implies f \in O(n)$, (hence, linear)
2. $U, V \subseteq \mathbb{E}^n, f : U \to V$ isometry with $f(0) = 0$
   $\implies f$ extends to global isometry $\tilde{f} \in O(n)$.

General Carnot groups

$X = \mathbb{G} =$ Carnot group.

1. [U.Hamenstädt] $f : \mathbb{G} \to \mathbb{G}$ isometry with $f(e) = e$ $\implies f$ group isomorphism.
2. [ELD, A.Ottazzi] $U, V \subseteq \mathbb{G}$ *open*, $f : U \to V$ isometry with $f(e) = e$ $\implies f$ extends to global isometry $\tilde{f}$, that is a group isomorphism.

Rmk. U open is necessary!

Definition (Carnot group)

Let $\mathbb{G}$ be a simply connected Lie group such that

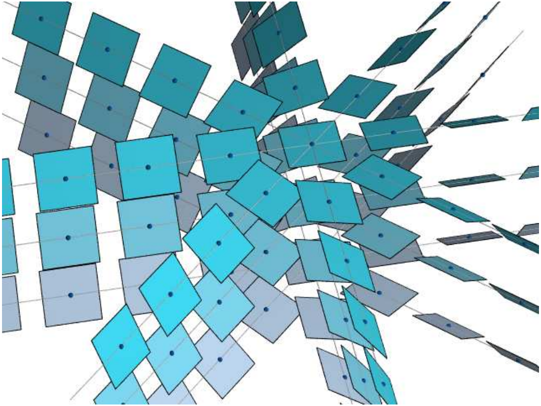
$$\mathrm{Lie}(\mathbb{G}) = V_1 \oplus \cdots \oplus V_s, \quad \text{with } [V_j, V_1] = V_{j+1}, \quad \text{for } 1 \le j \le s,$$

where $V_{s+1} = \{0\}$.
Fix $\|\cdot\|$ on V_1. Δ = left-invariant subbundle of $T\mathbb{G}$ s.t. $\Delta_e = V_1$.
The CC distance is

$$d(x,y) := \inf \left\{ \int_0^1 \|\dot\gamma(t)\| dt \, : \, \gamma \in C^\infty([0,1];\mathbb{G})_{\gamma(1)=y}^{\gamma(0)=x}, \dot\gamma \in \Delta \right\}.$$

The pair $(\mathbb{G}, d)$ is called a *subFinsler Carnot group*.

Example: $\Delta := \mathrm{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \right\} \subset T\mathbb{R}^3.$



Proof of Hamenstädt's theorem

Aim: New proof of Hamenstädt's result that can be generalized to local isometries.

Proof. $\mathcal{K}$ = Killing vector fields

$\mathcal{K}_0 := \{Z \in \mathcal{K} : Z_e = 0\}$

$\mathcal{K}_j := \{Z \in \mathcal{K}_0 : [Z, Y^R] \in \mathcal{K}_{j-1}, \forall Y \in V_1\}$

[Pansu] $\implies$ Assume $df_e|_{V_1} = \mathrm{id}_{V_1}$

Need: $f_* Y^R = Y^R$.

Lemma

1. $Z \in \mathcal{K}, Z_e \in V_1 \implies f_* Z \in Z + \mathcal{K}_0$
2. $Z \in \mathcal{K}_j \implies f_* Z \in Z + \mathcal{K}_{j+1}$

Geodesic homogeneous spaces

$X = G/H$ manifold with G Lie group, H closed subgroup.
$\Delta \subseteq T(G/H)$ G-invariant subbundle.
$\|\cdot\|$ G-invariant norm on Δ.
Assume Δ bracket generates $T(G/H)$.
Define the CC distance as

$$d_\Delta(x,y) := \inf\left\{\int_0^1 \|\dot\gamma(t)\|dt : \gamma \in C^\infty([0,1];X)_{\gamma(1)=y}^{\gamma(0)=x}, \dot\gamma \in \Delta\right\}.$$

Fact: All locally compact, geodesic, and homogeneous spaces are of this form.

Isometries of geodesic homogeneous spaces

Theorem (ELD, A.Ottazzi)

$f : (G/H, d_\Delta) \to (G/H, d_\Delta)$ isometry. Then

1. *f is analytic,*
2. *$\forall p \in G/H$, f is determined by $f(p)$ and $df_p|_{\Delta_p}$.*

Smoothness is a consequence of the fact that Hilbert 5^{th} theory applies to the transitive action $\mathrm{Isom}(X) \curvearrowright X$ of the locally compact group $\mathrm{Isom}(X)$.

SubRiemannian manifolds

X	smooth manifold.
$\Delta \subseteq TM$	subbundle.
$\|\cdot\|$	norm on Δ (coming from scalar product).

Assume Δ bracket generates TM.

Define the CC distance as

$$d_\Delta(x,y) := \inf \left\{ \int_0^1 \|\dot\gamma(t)\| dt \ : \gamma \in C^\infty([0,1];X)_{\gamma(1)=y}^{\gamma(0)=x}, \dot\gamma \in \Delta \right\}.$$

(X, d_Δ) is a **subRiemannian manifold.**

Isometries of subRiemannian manifolds

Theorem (L.Capogna, ELD)

$F : (M, d_\Delta) \to (N, d_\Delta)$ isometry.
Assume $\exists\, C^\infty$ volumes vol_M, vol_N s.t. $F_\# \mathrm{vol}_M = \mathrm{vol}_N$.
Then F is C^∞.

Rmk. There are many examples for which the Hausdorff measure is a smooth volume: groups, homogeneous spaces, several equiregular manifolds [Agrachev - Barilari - Boscain].
Problem: not always true!
Solution: On every equiregular manifolds one can always use Popp's measures!

Idea of proof.

Idea of proof

$x_1, \dots x_n$ coordinates in N.

$$\int_N \langle \nabla_{\mathrm{H}} x_i, \nabla_{\mathrm{H}} v \rangle \, \mathrm{d}\,\mathrm{vol}_N = \int_N g_i \, v \, \mathrm{d}\,\mathrm{vol}_N, \qquad \forall v \in \mathrm{Lip}_c(N).$$

$\Downarrow$ F measure-preserving isometry

$$\int_M \langle \nabla_{\mathrm{H}}(x_i \circ F), \nabla_{\mathrm{H}} \tilde{v} \rangle \, \mathrm{d}\,\mathrm{vol}_M = \int_M (g_i \circ F) \, \tilde{v} \, \mathrm{d}\,\mathrm{vol}_M, \qquad \forall \tilde{v} \in \mathrm{Lip}_c(M).$$

$$g_i \circ F \in C^{0,\alpha}_{\mathrm{H}}(M) \overset{\text{Subellipticity}}{\Longrightarrow} x_i \circ F \in C^{1,\alpha}_{\mathrm{H}}(M) \Longrightarrow$$
$$g_i \circ F \in C^{1,\alpha}_{\mathrm{H}}(M) \overset{\text{Subellipticity}}{\Longrightarrow} x_i \circ F \in C^{2,\alpha}_{\mathrm{H}}(M) \Longrightarrow \dots \Longrightarrow$$
$$x_i \circ F \in C^{\infty}_{\mathrm{H}}(M) = C^{\infty}. \qquad \square$$

Riemannian VS subRiemannian

M, N equiregular subRiemannian manifolds.

Corollary

$f : (M, d_\Delta) \to (N, d_\Delta)$ isometry. Then $\exists$ Riemannian extensions g_M and g_N s.t. $f : (M, g_M) \to (N, g_N)$ is an isometry.

Proposition

1. $\mathrm{Isom(M)}$ *is a Lie group;*
2. $\forall H < \mathrm{Isom(M)}$ *compact, $\exists$ Riemannian extension g s.t. $H < \mathrm{Isom}(M, g)$.*

THANKS