Partitions, Volume doubling property, Quasisymmetry and Heat Kernel Estimates

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Brownian motion, Laplacian, Heat equation on $\mathbb{R}^n$

$\mathcal{E}(u, v) = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx$ : Dirichlet energy

for $u, v \in H^{1,2}(\mathbb{R}^n)$: the Sobolev space.

$\mathcal{E}(u, v) = -\int_{\mathbb{R}^n} u \Delta v dx,$

where

$\Delta v = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2}$ \hspace{1cm} \text{Laplacian.}$

$\Downarrow$

Brownian motion $(\{B_t\}_{t>0}, \{P_x\}_{x \in \mathbb{R}^n})$ on $\mathbb{R}^n$ and/or

$\frac{\partial u}{\partial t} = \Delta u$ : Heat equation.
With an initial condition \( u_0 \), the solution of the heat equation is

\[
(e^{t\Delta} u_0)(x) = \int_{\mathbb{R}^n} p(t, x, y) u_0(y) dy
\]

\[
= E_x(u_0(B_t)) : \text{ Expectation among the paths starting from } x,
\]

where

\[
p(t, x, y) = \frac{c_1}{t^{n/2}} \exp \left( -c_2 \frac{d_*(x, y)^2}{t} \right),
\]

\text{(Gaussian Heat Kernel)}

where \( d_*(x, y) = |x - y| \) is the Euclidean metric.
**Time Change = Inhomogeneous media**
Let introduce $\mu$: **Density of the media**, a Radon measure on $\mathbb{R}^n$.
Assume that $\mu(A) = 0$ whenever $\text{Cap}(A) = 0$.

\[
\mu \text{ need not to be absolutely continuous to } dx
\]

Then by taking a proper modification $\mathcal{F}_\mu$ of the original domain $H^{1,2}(\mathbb{R}^n)$,

**Proposition 0.1.** $(\mathcal{E}, \mathcal{F}_\mu)$ is a local regular Dirichlet form on $L^2(\mathbb{R}^n, \mu)$

\[
\downarrow
\]

a **diffusion process** $(\{X_t\}_{t>0}, \{\tilde{P}_x\}_{x \in \mathbb{R}^n})$

the **time change** of the Brownian motion

and/or

(inhomogeneous) **Laplacian** $\Delta_\mu$: a self-adjoint operator on $L^2(\mathbb{R}^n, \mu)$

\[
\mathcal{E}(u, v) = - \int_{\mathbb{R}^n} u \Delta_\mu v d\mu
\]
The solution of
\[ \frac{\partial u}{\partial t} = \Delta_{\mu} u : \text{ Heat Equation} \]
with an initial condition \( u_0 \) is given by
\[
e^{t \Delta_{\mu}} u_0 = \int_{\mathbb{R}^n} p_{\mu}(t, x, y) u_0(y) \mu(dy)
= \mathcal{E}_x(u_0(X_t)),
\]
where \( p_{\mu}(t, x, y) \) is the heat kernel defined except on a null set.

**Special case**
\[ \mu(dx) = f(x)dx, \]
where \( f \) is non-negative measurable function, then
\[ \Delta_{\mu} u = \frac{1}{f(x)} \Delta u \]
Asymptotic behavior of $p_\mu(t, x, y)$?

- When does it behave nicely?
- How can we describe a behavior?

The original metric, the Euclidean metric, may not be the best.

**Notation**  Balls

\[ B_e(x, r) = \{ y | |x - y| < r \} : \text{Euclidean ball} \]

\[ B_d(x, r) = \{ y | d(x, y) < r \}. \]
Theorem 0.2. Assume that \( \exists c > 0, \epsilon > 0, \forall r > 0, a \in (0, 1), x \in \mathbb{R}^n, \)
\[
\mu(B_* (x, ar)) \leq ca^{n-2+\epsilon} \mu(B_* (x, r)).
\]
Then, \( \mu \) has the volume doubling property with respect to \( d_* \). \( \iff \) \( \exists \) a metric \( d \) s.t.

1. \( \mu \) is quasisymmetric to \( d_* \).
2. \( \mu \) has the volume doubling property with respect to \( d \).
3. \( p_\mu (t, x, y) \) is continuous on \((0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n\) and

\[
\exists \beta \geq 2, c_1, c_2, c_3, c_4 > 0, \forall x, y \in \mathbb{R}^n, t > 0,
\]

\[
\begin{align*}
p_\mu (t, x, y) & \leq \frac{c_1}{\mu(B_d (x, t^{1/\beta}))} \exp \left( -c_2 \left( \frac{d(x, y)^\beta}{t} \right)^{\frac{1}{1+\epsilon}} \right) \\
& \quad \text{(UHK)$_{d, \beta}$}
\end{align*}
\]

and, \( \forall t > 0 \) and \( \forall x, y \in \mathbb{R}^n \) with \( d(x, y)^\beta \leq c_4 t \),

\[
\begin{align*}
\frac{c_3}{\mu(B_d (x, t^{1/\beta}))} & \leq p_\mu (t, x, y) \\
& \quad \text{(NDLHK)$_{d, \beta}$}
\end{align*}
\]
**Volume doubling** (VD): a measure $\nu$ has the volume doubling property with respect to a metric $d$, $\iff \exists c > 0$, $\forall x \in X$ and $r > 0$,

$$\nu(B_d(x, 2r)) \leq c \nu(B_d(x, r))$$

**Quasisymmetry** (QS)

A metric $d$ is quasisymmetric to $\rho$, $d \sim_{\text{QS}} \rho$ in short.

$\iff$

Uniformly bounded distortion of Annuli

$\exists \epsilon, \delta \in (0, 1)$, $\forall x \in X$, $\forall r > 0$ and $\exists R > 0$

$$A_\rho(x, \epsilon r, r) \subseteq A_d(x, \delta R, R)$$

$$A_d(x, \epsilon r, r) \subseteq A_\rho(x, \delta R, R),$$

where $A_d(x, r_1, r_2) = B_d(x, r_2) \setminus B_d(x, r_1) = \text{Annulus}$
**Elliptic Harnack Inequality** \((EHI)_d\): \(\exists c > 0, \lambda > 1 \forall x \in \mathbb{R}^n, r > 0,\) if \(u\) is a positive harmonic function on \(B_d(x, \lambda r)\), then

\[
\exists c > 0, \lambda > 1 \forall x \in \mathbb{R}^n, r > 0, \text{ if } u \text{ is a positive harmonic function on } B_d(x, \lambda r), \text{ then } c \max_{y \in B_d(x,r)} u(y) \leq \min_{y \in B_d(x,r)} u(y)
\]

**Exit Time Estimate** \((Exit)_{d,\beta}\):

\[
\mathbb{E}_x(\tau_{B_d(x,r)}) = \int_{B_d(x,r)} g_{B_d(x,r)}(x, y) \mu(dy) \asymp r^d,
\]

where \(\tau_A = \inf\{t | X_t \notin A\}\) is the exit time from \(A\).

**Theorem 0.3** (Grigor’yan-Telcs). Let \((X, d, \mu)\) be a metric-measure space. Let \((\mathcal{E}, \mathcal{F})\) be a strong local Dirichlet form on \(L^2(X, \mu)\). Then

\[
(VD)_d + (EHI)_d + (Exit)_{d,\beta} \Rightarrow (UHK)_{d,\beta} + (NDLHK)_{d,\beta}
\]

Note that \((VD)\) and \((EHI)\) are invariant under QS, i.e., if \(\rho \sim d\), then

\[
(VD)_d \Leftrightarrow (VD)_\rho \quad \text{and} \quad (EHI)_d \Leftrightarrow (EHI)_\rho.
\]
• How to construct a metric $d$?

• How to characterize “quasisymmetry”?

• How to characterize “the volume doubling property”?

Study these questions by introducing the notions of “Partition” and “Gauge function” from a general point of view.

Please forget what you have seen for a moment!
The Cube $[0, 1]^n$ as a self-similar set

Let $S \overset{\text{def}}{=} \{1, 2, \ldots, 2^n\}$ and let $V \overset{\text{def}}{=} \{p_i | p_i \text{ is a vertex of } [0, 1]^n \text{ for } i \in S\}$.

Define $F_i : [0, 1]^n \to [0, 1]^n$ by

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i$$

Then the $n$-dim. cube $K = [0, 1]^n$ is a self-similar set associated with $\{F_i\}_{i \in S}$, i.e.

$$K = \bigcup_{i \in S} F_i(K)$$
\[ W_0 \overset{\text{def}}{=} W^0 = \{\phi\} \]
\[ W_m \overset{\text{def}}{=} W^m = \{w_1 \ldots w_m | w_i \in S\} : \text{words} \]
\[ W_s \overset{\text{def}}{=} \bigcup_{m \geq 0} W_m \]

For \( w = w_1 \ldots w_m \in W_s \), we define \(|w| = m\),
\[ F_w \overset{\text{def}}{=} F_{w_1} \circ \ldots \circ F_{w_m} \]
\[ K_w \overset{\text{def}}{=} F_w(K). \]

\[ \{K_w\}_{w \in W_s}: \text{Partition of the cube} \]
Recall that $d_s$: the Euclidean metric on $K$. Define

$$\mathcal{D}(K) = \{ d \mid d \text{ is a metric on } K, \operatorname{diam}(K, d) = 1 \}$$

$$(K, d) \text{ is homeomorphic to } (K, d_s).$$

For $d \in \mathcal{D}(K)$, define $d : W_s \to [0, 1]$ by

$$d(w) = \operatorname{diam}(K_w, d).$$

Define

$$\mathcal{M}(X) = \{ \mu \mid \mu : \text{Borel regular probability measure on } K, \forall w \in W_s, \mu(K_w) > 0, \forall \text{finite set } A, \mu(A) = 0. \}$$

For $\mu \in \mathcal{M}(K)$, define $\mu : W_s \to [0, 1]$ by

$$\mu(w) = \mu(K_w).$$

Abuse of notations!!

$d, \mu$ are gauge functions.
**Gauge functions** = sizes of $K_w$’s → “balls” $U_g(x, s)$ and “metrics” $D_g(x, y)$

**Definition** $g : W_s \rightarrow [0, 1]$ is a **gauge function**.  ⇔

1. $g(\phi) = 1, \quad g(wi) \leq g(w)$ if $w \in W_s$ and $i \in S$.
2. $\lim_{m \rightarrow \infty} \sup_{w \in W_m} g(w) = 0$.

**Example** (Euclidean gauge) $d_s(w) = 1/2^{|w|}$.

\[
\Lambda^g_s = \{w_1 \ldots w_m \in W_s | g(w_1 \ldots w_{m-1}) > s \geq g(w_1 \ldots w_m)\}
\]

= the collection of $w$’s with $g(w) \approx s$.

\[
\Lambda^g_s(x) = \{w | w \in \Lambda^g_s, \exists v \in \Lambda^g_s \text{ such that } x \in K_v \text{ and } K_w \cap K_v \neq \emptyset\}
\]
\[ U_g(x, s) \overset{\text{def}}{=} \bigcup_{w \in A_1^g(x)} K_w \]

= “ball” around \( x \) with radius \( s \)

“Good” metric = “Adapted” metric

\( d \in \mathcal{D}(K) \) is adapted to \( g \) \( \iff \exists c_1, c_2 > 0, \forall x \in K, \forall r > 0 \)

\[
U_g(x, c_1 r) \subseteq B_d(x, r) \subseteq U_g(x, c_2 r)
\]

Proposition 0.4. \( d \in \mathcal{D}(K) \): adapted to \( g \) \( \Rightarrow \) \( d \) is adapted to \( d \)

\[
d \in \mathcal{D}(K) \text{ is adapted} \overset{\text{def}}{\iff} d \text{ is adapted to } d.
\]
Definition 0.5. A gauge function $g$ is elliptic $\iff$
\[ \exists \lambda \in (0,1), \exists m, \forall w \in W_s, \forall v \in W_m, \forall i \in S, \text{if } wv, wi \in T, \text{ then} \]
\[ g(wv) \leq \lambda g(w) \leq g(wi) \]

Definition 0.6. $g, h$: gauge functions, $h \sim GE g \overset{\text{def}}{\iff}$
\[ \exists c > 0, w, v \in A_g \text{ if } K_w \cap K_v \neq \emptyset, \text{ then} \]
\[ h(w) \leq c h(v) \]

Proposition 0.7. $\sim_{GE}$ is an equivalence relation on elliptic gauge functions.

$d_s$, the Euclidean gauge function, is elliptic and adapted
Theorem 0.8. $\mu \in \mathcal{M}(K)$

\[ \mu \overset{\text{GE}}{\sim} d_* \quad \text{and} \quad \mu \text{ is elliptic.} \]

$\mu$: volume doubling with respect to $d_*$

Theorem 0.9. $d \in \mathcal{D}(K)$

\[ d \overset{\text{GE}}{\sim} d_* \quad \text{and} \quad d \text{ is elliptic and } d \text{ is adapted.} \]

Among elliptic, adapted gauge functions,

Quasisymmetry = Volume doubling = $\overset{\text{GE}}{\sim}$
Construction of a metric from a gauge function

For a gauge function \( g \), define

\[
D_g(x, y) = \inf \left\{ \sum_{i=1}^{m} g(w(i)) \left| m \geq 1, (w(1), \ldots, w(m)) \text{ a path } x \leftrightarrow y \right. \right\}
\]

\( D_g(x, y) \): symmetric, non-negative, satisfy the triangle inequality but

\( x \neq y \implies D_g(x, y) > 0 \)?

Theorem 0.10.

\[
\begin{align*}
\text{g is elliptic} \quad \text{and} \quad g \sim d_e \\
\downarrow
\quad \exists \alpha \in (0, 1], \ D_{g^\alpha}(x, y): \text{a metric on } K \text{ adapted to } g^\alpha
\end{align*}
\]
Back to the first theorem:
Let
\[ g(w) = \sqrt{2^{n-2}\mu(K_w)}. \]

A. \( \mu \) is \((VD)_d \Rightarrow g\) is elliptic and \( g \sim GE_d \).

B. Up to the similitarity, \( U_g(x,s) \) has finite types of shape.

C. From B, a scaling argument shows that
\[
\tilde{E}_x(\tau_{U_g(x,s)}) = \int_{U_g(x,s)} g_{U_g(x,s)}(x,y) \mu(dy) \asymp s^2
\]

D. \( \exists \alpha \in (0,1] \) such that \( d(x,y) = D_{g^\alpha}(x,y) \) is adapted to \( g^\alpha \), i.e.
\[
U_g(x,c_1s) \subseteq B_d(x,s^\alpha) \subseteq U_g(x,c_2s)
\]

E. C and D \( \Rightarrow \) \((\text{Exit})_{\beta,d} \) with \( \beta = 2/\alpha \geq 2 \), i.e.
\[
\tilde{E}_x(\tau_{B_d(x,r)}) \asymp r^{\beta}
\]

F. A \( \Rightarrow \mu \) is \((VD)_d \) and \( d \sim d_\mu \Rightarrow \) \((\text{EHI})_{d_\mu} \).

Finally, use Grigor’yan-Telcs.