Partitions, Volume doubling property, Quasisymmetry and Heat Kernel Estimates

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Brownian motion, Lapalcian, Heat equation on $\mathbb{R}^n$

$$\boxed{\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx} : \textbf{ Dirichlet energy}$$

for $u, v \in H^{1,2}(\mathbb{R}^n)$: the Sobolev space.

$$\mathcal{E}(u,v) = -\int_{\mathbb{R}^n} u \Delta v dx,$$

where $\boxed{\Delta v = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial {x_i}^2}}$: **Laplacian**.

$$\Downarrow$$

Brownian motion $(\{B_t\}_{t>0}, \{P_x\}_{x\in\mathbb{R}^n})$ on $\mathbb{R}^n$

and/or

$$\frac{\partial u}{\partial t} = \Delta u : \textbf{ Heat equation}.$$

With an initial condition u_0, the solution of the heat equation is

$$\begin{aligned}(e^{t\Delta}u_0)(x) &= \int_{\mathbb{R}^n} p(t,x,y)u_0(y)dy \\ &= E_x(u_0(B_t)): \text{ Expectation among the pathes starting from } x,\end{aligned}$$

where

$$\boxed{p(t,x,y) = \frac{c_1}{t^{n/2}} \exp\Big(-c_2 \frac{d_*(x,y)^2}{t}\Big)}, \qquad (\textbf{Gaussian Heat Kernel})$$

where $d_*(x,y) = |x-y|$ is **the Euclidean metric**.

Time Change = Inhomogeneous media

Let introduce μ: **Density of the media**, a Radon measure on $\mathbb{R}^n$.
Assume that $\mu(A) = 0$ whenever Cap(A) = 0.

μ need not to be absolutely continuous to dx

Then by taking a proper modification $\mathcal{F}_\mu$ of the original domain $H^{1,2}(\mathbb{R}^n)$,

Proposition 0.1. *$(\mathcal{E}, \mathcal{F}_\mu)$ is a local regular Dirichlet form on $L^2(\mathbb{R}^n, \mu)$*

$\Downarrow$

a **diffusion process** $(\{X_t\}_{t>0}, \{\widetilde{P}_x\}_{x\in\mathbb{R}^n})$
the **time change** of the Brownian motion
and/or
(inhomogeneous) **Laplacian** Δ_μ: a self-adjoint operator on $L^2(\mathbb{R}^n, \mu)$

$$\mathcal{E}(u, v) = -\int_{\mathbb{R}^n} u\Delta_\mu v d\mu$$

The solution of

$$\frac{\partial u}{\partial t} = \Delta_\mu u : \quad \textbf{Heat Equation}$$

with an initial condition u_0 is give by

$$\begin{aligned} e^{t\Delta_\mu} u_0 &= \int_{\mathbb{R}^n} p_\mu(t, x, y) u_0(y) \mu(dy) \\ &= \widetilde{E}_x(u_0(X_t)), \end{aligned}$$

where $p_\mu(t, x, y)$ is the **heat kernel** defined except on a null set.

Special case

$$\mu(dx) = f(x) dx,$$

where f is non-negative measurable function, then

$$\Delta_\mu u = \frac{1}{f(x)} \Delta u$$

Asymptotic behavior of $p_\mu(t,x,y)$?

- When does it bahave nicely?
- How can we describe a behavior?

The original metric, the Euclidean metric, may not be the best.

Notation Balls

$$B_*(x,r) \underset{\text{def}}{=} \{y \,|\, |x-y|<r\} : \text{Euclidean ball}$$

$$B_d(x,r) \underset{\text{def}}{=} \{y \,|\, d(x,y)<r\}.$$

Theorem 0.2. *Assume that* $\exists c > 0, \epsilon > 0, \forall r > 0, a \in (0,1), x \in \mathbb{R}^n$,

$$\mu(B_*(x, ar)) \leq c a^{n-2+\epsilon} \mu(B_*(x, r)).$$

Then, μ *has the volume doubling property with respect to* d_*. $\Leftrightarrow$
$\exists$*a metric* d *s.t.*
(1) μ *is quasisymmetric to* d_*.
(2) μ *has the volume doubling property with respect to* d.
(3) $p_\mu(t, x, y)$ *is continuous on* $(0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$ *and*
$\exists \beta \geq 2, c_1, c_2, c_3, c_4 > 0, \forall x, y \in \mathbb{R}^n, t > 0$,

$$\boxed{p_\mu(t, x, y) \leq \frac{c_1}{\mu(B_d(x, t^{1/\beta}))} \exp\left(- c_2 \left(\frac{d(x,y)^\beta}{t} \right)^{\frac{1}{\beta - 1}} \right)} \qquad (\mathrm{UHK})_{d,\beta}$$

and, $\forall t > 0$ *and* $\forall x, y \in \mathbb{R}^n$ *with* $d(x,y)^\beta \leq c_4 t$,

$$\boxed{\frac{c_3}{\mu(B_d(x, t^{1/\beta}))} \leq p_\mu(t, x, y)} \qquad (\mathrm{NDLHK})_{d,\beta}$$

Volume doubling $(\mathrm{VD})_d$: a measure ν has the volume doubling property with respect to a metric $d \underset{\text{def}}{\Leftrightarrow} \exists c > 0$, $\forall x \in X$ and $r > 0$,

$$\boxed{\nu(B_d(x, 2r)) \leq c\nu(B_d(x, r))}$$

Quasisymmetry(QS)

A metric d is quasisymmetric to ρ, $d \underset{\mathrm{QS}}{\sim} \rho$ in short.

$$\Updownarrow$$

Uniformly bounded distortion of **Annuli**

$\exists \epsilon, \delta \in (0, 1)$, $\forall x \in X$, $\forall r > 0$ and $\exists R > 0$

$$A_\rho(x, \epsilon r, r) \subseteq A_d(x, \delta R, R)$$
$$A_d(x, \epsilon r, r) \subseteq A_\rho(x, \delta R, R),$$

where $A_d(x, r_1, r_2) = B_d(x, r_2) \backslash B_d(x, r_1) = \text{Annulus}$

Elliptic Harnack Inequality $(\mathrm{EHI})_d$: $\exists c > 0, \lambda > 1\ \forall x \in \mathbb{R}^n, r > 0$, if u is a positive harmonic function on $B_d(x, \lambda r)$, then

$$\boxed{c \max_{y \in B_d(x,r)} u(y) \le \min_{y \in B_d(x,r)} u(y)}$$

Exit Time Estimate $(\mathrm{Exit})_{d,\beta}$:

$$\boxed{\widetilde{E}_x(\tau_{B_d(x,r)}) = \int_{B_d(x,r)} g_{B_d(x,r)}(x,y)\mu(dy) \asymp r^\beta},$$

where $\tau_A = \inf\{t | X_t \notin A\}$ is the exit time from A.

Theorem 0.3 (Grigor'yan-Telcs)**.** *Let (X, d, μ) be a metric-measure sapce. Let $(\mathcal{E}, \mathcal{F})$ be a strong local Dirichlet form on $L^2(X, \mu)$. Then*

$$\boxed{(\mathrm{VD})_d + (\mathrm{EHI})_d + (\mathrm{Exit})_{d,\beta} \Rightarrow (\mathrm{UHK})_{d,\beta} + (\mathrm{NDLHK})_{d,\beta}}$$

Note that (VD) and (EHI) are invariant under QS, i.e, if $\rho \underset{\mathrm{QS}}{\sim} d$, then $\boxed{(\mathrm{VD})_d \Leftrightarrow (\mathrm{VD})_\rho}$ and $\boxed{(\mathrm{EHI})_d \Leftrightarrow (\mathrm{EHI})_\rho}$.

- How to construct a metric d?
- How to characterize"quasisymmetry"?
- How to characterize "the volume doubling property"?

Study these questions by introducing the notions of "Partition" and "Gauge function" from a general point of view.

Please forget what you have seen for a moment!

The Cube $[0,1]^n$ as a self-similar set

Let $S \underset{\text{def}}{=} \{1, 2, \ldots, 2^n\}$ and let $V \underset{\text{def}}{=} \{p_i | p_i$ is a vertex of $[0,1]^n$ for $i \in S\}$.
Define $F_i : [0,1]^n \to [0,1]^n$ by

$$\boxed{F_i(x) \underset{\text{def}}{=} \frac{1}{2}(x - p_i) + p_i}$$

Then the n-dim. cube $K = [0,1]^n$ is a **self-similar set** associated with $\{F_i\}_{i \in S}$, i.e.

$$\boxed{K = \bigcup_{i \in S} F_i(K)}$$

$$W_0 \underset{\text{def}}{=} W^0 = \{\phi\}$$

$$W_m \underset{\text{def}}{=} W^m = \{w_1 \dots w_m | w_i \in S\} : \text{words}$$

$$W_* \underset{\text{def}}{=} \bigcup_{m \geq 0} W_m$$

For $w = w_1 \dots w_m \in W_*$, we define $|w| = m$,

$$F_w \underset{\text{def}}{=} F_{w_1} \circ \dots \circ F_{w_m}$$

$$K_w \underset{\text{def}}{=} F_w(K).$$

$\{K_w\}_{w \in W_*}$: **Partition** of the cube

Recall that d_*: the Euclidean metric on K. Define

$$\mathcal{D}(K) \underset{\text{def}}{=} \{d | d \text{ is a metric on } K, \text{ diam}(K, d) = 1$$
$$(K, d) \text{ is homeomorphic to } (K, d_*).\}$$

For $d \in \mathcal{D}(K)$, define $d : W_* \to [0, 1]$ by

$$\boxed{d(w) = \text{diam}(K_w, d)}.$$

Define

$$\mathcal{M}(X) = \{\mu | \mu\text{: Borel regualr probability measure}$$
$$\text{on } K, \forall w \in W_*, \mu(K_w) > 0, \forall \text{finite set } A, \mu(A) = 0.\}$$

For $\mu \in \mathcal{M}(K)$, define $\mu : W_* \to [0, 1]$ by

$$\boxed{\mu(w) = \mu(K_w)}.$$

Abuse of notations!!

d, μ are **gauge functions**.

Gauge functions = sizes of K_w's $\longrightarrow$

"balls" $U_g(x,s)$ and "metrics" $D_g(x,y)$

Definition $g : W_* \longrightarrow [0,1]$ is a **gauge function**. $\underset{\text{def}}{\Leftrightarrow}$

(1) $g(\phi) = 1$, $\boxed{g(wi) \leq g(w)}$ if $w \in W_*$ and $i \in S$.

(2) $\boxed{\lim_{m\to\infty} \sup_{w \in W_m} g(w) = 0}$.

Example(Euclidean gauge) $d_*(w) = 1/2^{|w|}$.

$$\Lambda_s^g \underset{\text{def}}{=} \{w_1 \dots w_m \in W_* | \; g(w_1 \dots w_{m-1}) > s \geq g(w_1 \dots w_m)\}$$

= the collection of w's with $g(w) \approx s$,

$$\Lambda_s^g(x) \underset{\text{def}}{=} \{w | w \in \Lambda_s^g(x), \exists v \in \Lambda_s^g \text{ such that } x \in K_v \text{ and } K_w \cap K_v \neq \emptyset\}$$

$$U_g(x,s) \underset{\text{def}}{=} \bigcup_{w \in \Lambda_s^g(x)} K_w$$

$$= \text{"ball" around } x \text{ with radius } s$$

"Good" metric = "Adapted" metric

$d \in \mathcal{D}(K)$ is adapted to g $\underset{\text{def}}{\Leftrightarrow}$ $\exists c_1, c_2 > 0, \forall x \in K, \forall r > 0$

$$\boxed{U_g(x, c_1 r) \subseteq B_d(x,r) \subseteq U_g(x, c_2 r)}$$

Proposition 0.4. *$d \in \mathcal{D}(K)$: adapted to g $\Rightarrow$ d is adapted to d*

$d \in \mathcal{D}(K)$ is adapted $\underset{\text{def}}{\Leftrightarrow}$ d is adapted to d.

Definition 0.5. A gauge function g is **elliptic** $\underset{\text{def}}{\Leftrightarrow}$ $\exists\lambda \in (0,1), \exists m, \forall w \in W_*, \forall v \in W_m, \forall i \in S$, if $wv, wi \in T$, then

$$g(wv) \le \lambda g(w) \le g(wi)$$

Definition 0.6. g, h: gauge functions, $\boxed{h \underset{\text{GE}}{\sim} g}$ $\underset{\text{def}}{\Leftrightarrow}$ $\exists c > 0$ $w, v \in \Lambda_s^g$ if $K_w \cap K_v \neq \emptyset$, then

$$h(w) \le c h(v)$$

Proposition 0.7. $\underset{\text{GE}}{\sim}$ *is an equivalence relation on* **elliptic** *gauge functions.*

d_*, the Euclidean gauge function, is elliptic and adapted

Theorem 0.8. $\mu \in \mathcal{M}(K)$

$$\boxed{\mu \underset{\text{GE}}{\sim} d_*} \text{ and } \boxed{\mu \text{ is elliptic.}}$$

$$\Updownarrow$$

$$\boxed{\mu\text{: volume doubling with respect to } d_*}$$

Theorem 0.9. $d \in \mathcal{D}(K)$

$$\boxed{d \underset{\text{GE}}{\sim} d_*}, \boxed{d \text{ is elliptic}} \text{ and } \boxed{d \text{ is apapted.}}$$

$$\Updownarrow$$

$$\boxed{d \underset{\text{QS}}{\sim} d_*}$$

Among elliptic, adapted gauge functions,

$$\boxed{\textbf{Quasisymmetry} = \textbf{Volume doubling} = \underset{\text{GE}}{\sim}}$$

Construction of a metric from a gauge function

For a gauge function g, define

$$D_g(x,y) = \inf\left\{ \sum_{i=1}^{m} g(w(i)) \,\middle|\, m \geq 1, (w(1), \ldots, w(m)) \text{: a path } x \leftrightarrow y \right\}$$

$D_g(x,y)$: symmetric, non-negative, satisfy the triangle inequality but

$$\boxed{x \neq y \Rightarrow D_g(x,y) > 0} \text{??}$$

Theorem 0.10.

$$\boxed{g \text{ is elliptic}} \text{ and } \boxed{g \underset{\mathrm{GE}}{\sim} d_*}$$

$$\Downarrow$$

$\exists \alpha \in (0,1]$, $D_{g^\alpha}(x,y)$: a metric on K adapted to g^α

Back to the first theorem:
Let

$$\boxed{g(w) = \sqrt{2^{n-2}\mu(K_w)}}.$$

A. μ is $(\mathrm{VD})_{d_*} \Rightarrow g$ is elliptic and $g \underset{\mathrm{GE}}{\sim} d_*$.

B. Up to the simililarity, $U_g(x,s)$ has finite types of shape.

C. From B, a scaling argumemt shows that

$$\widetilde{E}_x(\tau_{U_g(x,s)}) = \int_{U_g(x,s)} g_{U_g(x,s)}(x,y)\mu(dy) \asymp s^2$$

D. $\exists \alpha \in (0,1]$ such that $d(x,y) = D_{g^\alpha}(x,y)$ is adapted to g^α, i.e.

$$U_g(x, c_1 s) \subseteq B_d(x, s^\alpha) \subseteq U_g(x, c_2 s)$$

E. C and D $\Rightarrow (\mathrm{Exit})_{\beta,d}$ with $\beta = 2/\alpha \geq 2$, i.e.

$$\widetilde{E}_x(\tau_{B_d(x,r)}) \asymp r^\beta$$

F. A $\Rightarrow \mu$ is $(\mathrm{VD})_d$ and $d \underset{\mathrm{QS}}{\sim} d_* \Rightarrow (\mathrm{EHI})_{d_*}$.

Finally, use Grigor'yan-Telcs.