Hausdorff dimensin distortion by Sobolev maps in foliated spaces

ZB, R. Monti, J. T. Tyson, K. Wildrick

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Overview

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- *Euclidean results:* -, R. Monti, J. T. Tyson, *Frequency of Sobolev and quasiconformal dimension distortion*, J. Math. Pure. Appl. 2013

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- *Metric spaces:* -, J. T. Tyson, K. Wildrick, *Dimension distortion by Sobolev mappings in foliated metric spaces*, Preprint, 2013

- *Heisenberg groups:* -, J. T. Tyson, K. Wildrick, *Frequency of Sobolev dimension distortion of horizontal subgroups of Heisenberg groups*, Preprint 2013

Euclidean results

Morrey-Sobolev estimate

Proposition 0.1. *Let $f \in W^{1,p}(\Omega, Y)$, $p > n$, and g_f denote the minimal upper gradient for f. Then for all cubes Q compactly contained in Ω, we have*

$$\operatorname{diam} f(Q) \leq C(n,p)(\operatorname{diam} Q)^{1-n/p} \left(\int_Q g_f^p \right)^{1/p}. \tag{0.1}$$

Morrey-Sobolev estimate

Proposition 0.2. *Let $f \in W^{1,p}(\Omega, Y)$, $p > n$, and g_f denote the minimal upper gradient for f. Then for all cubes Q compactly contained in Ω, we have*

$$\operatorname{diam} f(Q) \le C(n,p)(\operatorname{diam} Q)^{1-n/p} \left(\int_Q g_f^p \right)^{1/p}. \qquad (0.1)$$

By the Morrey–Sobolev embedding theorem, each supercritical mapping $f \in W^{1,p}(\Omega, Y)$, $p > n$, has a representative which is locally $(1 - n/p)$-Hölder continuous.
In particular, if $E \subset \Omega$, $\dim E = t$ then

$$\dim f(E) \le \frac{tp}{p-n}.$$

Kaufman's theorem

Theorem 0.3 (R. Kaufman 2000). $E \subset \Omega$; $\mathcal{H}^t(E) < \infty$, $0 < t < n$. *$f \in W^{1,p}(\Omega, Y)$ for some $p > n$. Then $f(E)$ has zero $\mathcal{H}^{pt/(p-n+t)}$ measure. This statement is sharp.*

Kaufman's theorem

Theorem 0.4 (R. Kaufman 2000). $E \subset \Omega$; $\mathcal{H}^t(E) < \infty$, $0 < t < n$. *$f \in W^{1,p}(\Omega, Y)$ for some $p > n$. Then $f(E)$ has zero $\mathcal{H}^{pt/(p-n+t)}$ measure. This statement is sharp.*

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Kaufman's theorem

Theorem 0.5 (R. Kaufman 2000). $E \subset \Omega$; $\mathcal{H}^t(E) < \infty$, $0 < t < n$. *$f \in W^{1,p}(\Omega, Y)$ for some $p > n$. Then $f(E)$ has zero $\mathcal{H}^{pt/(p-n+t)}$ measure. This statement is sharp.*

In particular if $\dim E = t$ then

$$\dim f(E) \leq \frac{tp}{p-n+t}.$$

If $V \in G(n,m)$. Then

$$\dim f(V_a \cap \Omega) \leq \frac{pm}{p-n+m}. \tag{0.2}$$

Main Result

Let $\mathbb{R}^n = V^\perp \bigoplus V$ and $V_a = a + V$. Given a Sobolev map $f : \mathbb{R}^n \to Y$, how frequently can the intermediate values $m \leq \alpha \leq \frac{pm}{p-n+m}$ be exceeded by $\dim f(V_a)$?

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Theorem 0.7 (-.R. Monti, J. Tyson)**.**

$$\textit{Let} \quad \beta = \beta(p, \alpha) := (n - m) - \left(1 - \frac{m}{\alpha}\right) p.$$

$f(V_a \cap \Omega)$ *has zero* $\mathcal{H}^\alpha$ *measure for* $\mathcal{H}^\beta$*-almost every* $a \in V^\perp$.

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Theorem 0.8 (-.R. Monti, J. Tyson)**.**

$$Let\ \ \beta = \beta(p,\alpha) := (n-m) - \left(1 - \frac{m}{\alpha}\right) p.$$

$f(V_a \cap \Omega)$ has zero $\mathcal{H}^\alpha$ measure for $\mathcal{H}^\beta$-almost every $a \in V^\perp$.

In particular:

$$\dim\{a \in V^\perp : \dim f(V_a) \geq \alpha\} \leq \beta$$

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Theorem 0.9 (-.R. Monti, J. Tyson)**.**

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In particular:

$$\dim\{a \in V^\perp : \dim f(V_a) \ge \alpha\} \le \beta$$

Note: $\alpha \to m \ \Rightarrow\ \beta \to n-m$ and $\alpha \to \frac{pm}{p-n+m} \ \Rightarrow\ \beta \to 0$

Sharpness

Theorem 0.10 (-.R. Monti, J. Tyson). *Let α satisfy $m < \alpha < \frac{pm}{p-n+m}$ and $\beta = \beta(p, \alpha)$ as above. There exists a $E \subset \mathbb{R}^{n-m}$ s.th. $\mathcal{H}^{\beta}(E) > 0$ and for any integer $N > \alpha$, there exists a map $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ with the property that $\dim f(\{a\} \times \mathbb{R}^m) \geq \alpha$, for $\mathcal{H}^{\beta}$-almost every $a \in E$.*

Sharpness

Theorem 0.12 (-.R. Monti, J. Tyson). *Let* α *satisfy* $m < \alpha < \frac{pm}{p-n+m}$ *and* $\beta = \beta(p, \alpha)$ *as above. There exists a* $E \subset \mathbb{R}^{n-m}$ *s.th.* $\mathcal{H}^{\beta}(E) > 0$ *and for any integer* $N > \alpha$*, there exists a map* $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ *with the property that* $\dim f(\{a\} \times \mathbb{R}^m) \geq \alpha$*, for* $\mathcal{H}^{\beta}$*-almost every* $a \in E$*.*

Theorem 0.13 (S. Hencl, P. Honzik 2013). *Let* $m < \alpha < p \leq n$ *and* $\beta(p, \alpha) := (n-m) - \left(1 - \frac{m}{\alpha}\right) p$. *and* $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ *be* $p-$*quasicontinuous. Then* $\dim\{a \in V^{\perp} : \dim f(V_a) \geq \alpha\} \leq \beta$*.*

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Theorem 0.14 (-.R. Monti, J. Tyson). *Let α satisfy $m < \alpha < \frac{pm}{p-n+m}$ and $\beta = \beta(p, \alpha)$ as above. There exists a $E \subset \mathbb{R}^{n-m}$ s.th. $\mathcal{H}^\beta(E) > 0$ and for any integer $N > \alpha$, there exists a map $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ with the property that $\dim f(\{a\} \times \mathbb{R}^m) \geq \alpha$, for $\mathcal{H}^\beta$-almost every $a \in E$.*

Theorem 0.15 (S. Hencl, P. Honzik 2013). *Let $m < \alpha < p \leq n$ and $\beta(p, \alpha) := (n-m) - \left(1 - \frac{m}{\alpha}\right) p$. and $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ be $p-$ quasicontinuous. Then $\dim\{a \in V^\perp : \dim f(V_a) \geq \alpha\} \leq \beta$.*

Note: $\alpha \to p \;\Rightarrow\; \beta \to n-p$ The above theorem is also sharp.

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Theorem 0.16 (-.R. Monti, J. Tyson). *Let* α *satisfy* $m < \alpha < \frac{pm}{p-n+m}$ *and* $\beta = \beta(p,\alpha)$ *as above. There exists a* $E \subset \mathbb{R}^{n-m}$ *s.th.* $\mathcal{H}^\beta(E) > 0$ *and for any integer* $N > \alpha$, *there exists a map* $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ *with the property that* $\dim f(\{a\} \times \mathbb{R}^m) \geq \alpha$, *for* $\mathcal{H}^\beta$*-almost every* $a \in E$.

Theorem 0.17 (S. Hencl, P. Honzik 2013). *Let* $m < \alpha < p \leq n$ *and* $\beta(p,\alpha) := (n-m) - \left(1 - \frac{m}{\alpha}\right)p$. *and* $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ *be* $p-$*quasicontinuous. Then* $\dim\{a \in V^\perp : \dim f(V_a) \geq \alpha\} \leq \beta$.

Note: $\alpha \to p \ \Rightarrow \ \beta \to n-p$ The above theorem is also sharp.

Example: [P. Hajlasz, J. Tyson] Then there exists a continuous map $f \in W^{1,m}(\mathbb{R}^n, \ell^2)$ s.th. $\dim f(\{a\} \times [0,1]^m) = \infty$ for all $a \in [0,1]^{n-m}$.

Metric space results

Metric case

Theorem 0.18 (-J. T., K. W.). *X: proper, loc. Q-homogeneous, local Q-Poincaré inequality. If $f : X \to Y$ is a continuous mapping that has an upper gradient in $L^p_{loc}(X)$, $p > Q$, then*

$$\dim_Y f(E) \leq \frac{p \dim_X E}{p - Q + \dim_X E} \tag{0.3}$$

for any subset $E \subseteq X$.

Metric case

Theorem 0.20 (-J. T., K. W.). *X: proper, loc. Q-homogeneous, local Q-Poincaré inequality. If $f : X \to Y$ is a continuous mapping that has an upper gradient in $L^p_{loc}(X)$, $p > Q$, then*

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for any subset $E \subseteq X$.

Theorem 0.21 (-J. T., K. W.). *Let (X, d, μ), Ahlfors Q-regular. For $0 \leq s \leq Q$ and $p > Q$, set $\alpha = \frac{ps}{p-Q+s}$. Let E be a compact subset of X such that $\mathcal{H}^s(E) > 0$. Then for all $N > \alpha$, there exists a continuous $f : X \to \mathbb{R}^N$ such that f has an upper gradient in $\mathrm{L}^p(X)$ and $\dim f(E) \geq \alpha$.*

Prevalence

Theorem 0.22 (-J. T., K. W.). *The set of functions*

$$S_\alpha := \{f \in \mathrm{N}^{1,p}(X;\mathbb{R}^N) : \dim f(E) \geq \alpha\}$$

is a prevalent set in the Newtonian–Sobolev space $\mathrm{N}^{1,p}(X;\mathbb{R}^N)$.

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Prevalence is an important notion in dynamics. Was introduced and developed by J. Yorke, V. Kaloshin, B. Hunt, W. Ott, T. Sauer

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Theorem 0.24 (-J. T., K. W.). *The set of functions*

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There exists a compactly supported Borel probability measure λ on $\mathrm{N}^{1,p}(X;\mathbb{R}^N)$ such that

$$\lambda(\mathrm{N}^{1,p}(X;\mathbb{R}^N) \setminus (S_\alpha + g)) = 0, \ \forall g \in \mathrm{N}^{1,p}(X;\mathbb{R}^N).$$

Idea of Proof

Idea: let μ be a Frostman measure on E and find first $h \in N^{1,p}(X;\mathbb{R}^N)$ with $I_\alpha(h_\sharp \mu) < \infty$ i.e.

$$\int_E \int_E \frac{1}{|h(x) - h(y)|^\alpha} d\mu(x) d\mu(y) < \infty.$$

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Let $\mathcal{M} = \{M \in M(N \times N, \mathbb{R}), ||M|| \leq 1\}$ and ν to be the normalized N^2-dimensional Lebesgue measure on $\mathcal{M}$.

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Let $\mathcal{M} = \{M \in M(N \times N, \mathbb{R}), ||M|| \leq 1\}$ and ν to be the normalized N^2-dimensional Lebesgue measure on $\mathcal{M}$. Consider

$$\Phi : \mathcal{M} \to N^{1,p}(X;\mathbb{R}^N), \Phi(M) := M \circ h,$$

and define: $\lambda := \Phi_\sharp \nu$. Let $g \in N^{1,p}(X;\mathbb{R}^N)$ and $f_M = M \circ h + g$.

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and define: $\lambda := \Phi_\sharp \nu$. Let $g \in N^{1,p}(X;\mathbb{R}^N)$ and $f_M = M \circ h + g$. Have to prove:

$$\int_{\mathcal{M}} \int_E \int_E \frac{1}{|f_M(x) - f_M(y)|^\alpha} d\mu(x) d\mu(y) d\nu(M) < \infty.$$

Regular foliations

Definition[G. David, S. Semmes] A surjection $\pi\colon X \to W$ is called s-regular if for any compact $K \subseteq X$

- $\pi|_K$ is Lipschitz
- $\pi^{-1}(B) \cap K$ can be covered by at most Cr^{-s} balls in X of radius Cr.

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- $\pi|_K$ is Lipschitz
- $\pi^{-1}(B) \cap K$ can be covered by at most Cr^{-s} balls in X of radius Cr.

Note that $\mathcal{H}^s_X(\pi^{-1}(a) \cap K) \leq C$, in particular, for the leaves $\pi^{-1}(a)$ it follows

$$\dim \pi^{-1}(a) \leq s,$$

and this inequality can be sometimes strict. The triple (X, W, π) will be called an s-*foliation* of X.

Dimension distortion for metric foliations

Theorem 0.25. *Let (X, d_X, μ) be a proper, loc. Q-homogeneous, with local Q-Poincaré inequality, and is equipped with an s-foliation (X, W, π). If $f : X \to Y$ is a continuous mapping with upper gradient in $L^p_{loc}(X)$, $p > Q$, then*

$$\dim\{a \in W : \dim(f(\pi^{-1}(a))) \geq \alpha\} \leq (Q - s) - (1 - \frac{s}{\alpha})$$

for $s < \alpha \leq \frac{ps}{p-Q+s}$.

Dimension distortion for metric foliations

Theorem 0.26. *Let (X, d_X, μ) be a proper, loc. Q-homogeneous, with local Q-Poincaré inequality, and is equipped with an s-foliation (X, W, π). If $f : X \to Y$ is a continuous mapping with upper gradient in $L^p_{loc}(X)$, $p > Q$, then*

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for $s < \alpha \leq \frac{ps}{p-Q+s}$.

If $s = \dim \pi^{-1}(a)$ then this is a good theorem to apply. Examples:

Dimension distortion for metric foliations

Theorem 0.27. *Let (X, d_X, μ) be a proper, loc. Q-homogeneous, with local Q-Poincaré inequality, and is equipped with an s-foliation (X, W, π). If $f : X \to Y$ is a continuous mapping with upper gradient in $L^p_{loc}(X)$, $p > Q$, then*

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If $s = \dim \pi^{-1}(a)$ then this is a good theorem to apply. Examples:

1. $X = \mathbb{R}^n, W = \mathbb{R}^{m-n}$ with the orthogonal projection $\pi : \mathbb{R}^n \to W$ defines a regular $s = m$-foliation.

Dimension distortion for metric foliations

Theorem 0.28. *Let (X, d_X, μ) be a proper, loc. Q-homogeneous, with local Q-Poincaré inequality, and is equipped with an s-foliation (X, W, π). If $f : X \to Y$ is a continuous mapping with upper gradient in $L^p_{loc}(X)$, $p > Q$, then*

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1. $X = \mathbb{R}^n, W = \mathbb{R}^{m-n}$ with the orthogonal projection $\pi : \mathbb{R}^n \to W$ defines a regular $s = m$-foliation.

2. $X = \mathbb{H}^n = \mathbb{V}^\perp \ltimes \mathbb{V}$, $\mathbb{V}$-horizontal subgroup of $\mathbb{H}^n$.

Heisenberg group results

Heisenberg foliations

$\mathbb{H}^n \simeq \mathbb{R}^{2n+1} \ni p = (x_1, y_1, \ldots, x_n, y_n, t) = (z, t),$

$$p * p' = (z + z', t + t' + 2\omega(z, z'))$$

$\omega(z, z') = \sum_{i=1}^n (x_i y_i' - x_i' y_i),$

$$d_{\mathbb{H}^n}(p, p') = ||p^{-1} * p'||_{\mathbb{H}^n}, \; ||p||_{\mathbb{H}^n} = (||z||^4 + |t|^2)^{1/4}.$$

$\delta_r p = (rz, r^2 t), \; Q = 2n + 2$. Ahlfors, Poincare OK

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Isotropic subspace of $\mathbb{V} \leq \mathbb{R}^{2n}$: $\omega(u, v) = 0 \; \forall \; u, v \in \mathbb{V} \Rightarrow$ horizontal homogenous subgroups of $\dim \mathbb{V} = m \leq n$.
Vertical complementary subspace: $\mathbb{V}^\perp = V^\perp \times \mathbb{R}$ is a normal subgroup generating the semidirect product.

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Vertical complementary subspace: $\mathbb{V}^\perp = V^\perp \times \mathbb{R}$ is a normal subgroup generating the semidirect product.

$\mathbb{H}^n = \mathbb{V}^\perp \ltimes \mathbb{V}$, $p = p_{\mathbb{V}^\perp} * p_{\mathbb{V}}$ defines two types of folitations:

$\pi_{\mathbb{V}} \colon \mathbb{H}^n \to \mathbb{V}\ \ , \pi_{\mathbb{V}}^{-1}(a) = \mathbb{V}^\perp * a$ and $\pi_{\mathbb{V}^\perp} \colon \mathbb{H}^n \to \mathbb{V}^\perp,\ \pi_{\mathbb{V}^\perp}^{-1}(a) = a * \mathbb{V}$

Two types of foliations

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Good news:

$$(\mathbb{H}^n, \mathbb{V}, \pi_{\mathbb{V}})$$

is a regular $(Q - m)$- foliation. Moreover:

$$\dim(\pi_{\mathbb{V}}^{-1}(a)) = \dim(\mathbb{V}^{\perp} * a) = Q - m$$

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Improvement: $(\mathbb{H}^n, (\mathbb{V}^{\perp}, d_E), \pi_{\mathbb{V}^{\perp}})$ is a regular $m+1$- foliation, but $s = m+1 > m = \dim(\pi_{\mathbb{V}^{\perp}}^{-1}(a))$, which for

$$m \leq \alpha \leq \frac{Qm}{p-Q+m} \text{ and } \beta = (Q-s) - p(1 - \frac{s}{\alpha})$$

does not imply sharp results at endpoints.

Heisenberg Foliation Distortion

Heisenberg Foliation Distortion

Redefine β as:

$$\beta(p,m,\alpha) = \begin{cases} (Q-1-m) - \frac{p}{2}\left(1-\frac{m}{\alpha}\right) & \alpha \in \left[m, \frac{pm}{p-2}\right], \\ (Q-m) - p\left(1-\frac{m}{\alpha}\right) & \alpha \in \left[\frac{pm}{p-2}, \frac{pm}{p-(Q-m)}\right]. \end{cases}$$

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Theorem 0.31. *Let $f \in W^{1,p}_{loc}(\mathbb{H}^n; Y)$, $p > Q$. Given a horizontal subgroup $\mathbb{V}$ of $\mathbb{H}^n$ of dimension $1 \leq m \leq n$, and*

$$m \leq \alpha \leq \frac{pm}{p-(Q-m)},$$

it holds that

$$\dim_{\mathbb{R}}\{a \in \mathbb{V}^{\perp} : \dim f(a * \mathbb{V}) \geq \alpha\} \leq \beta(p,m,\alpha).$$

Sharpness in $\mathbb{H}$

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Theorem 0.33. *Let $\mathbb{V}_x$ denote the horizontal subgroup defined by the x-axis in $\mathbb{H}$, and let $p > 4$. For each*

$$1 < \alpha < \frac{p}{p-2} < \frac{p}{p-3}$$

there is a compact set $E_\alpha \subset \mathbb{V}_x^\perp$ and a continuous mapping $f \in W^{1,p}(\mathbb{H}; \mathbb{R}^2)$ such that

$$0 < \mathcal{H}_{\mathbb{R}^3}^{2-p\left(1-\frac{1}{\alpha}\right)}(E_\alpha) < \infty$$

and $\dim f(a * \mathbb{V}) \geq \alpha$ for every $a \in E_\alpha$.

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and $\dim f(a * \mathbb{V}) \geq \alpha$ *for every* $a \in E_\alpha$*.*

Note: $2 - p(1 - \frac{1}{\alpha}) < \beta = 2 - \frac{p}{2}(1 - p(1 - \frac{1}{\alpha})$, however the example is asymptotically sharp

$$2 - p(1 - \frac{1}{\alpha}) \to 2, \text{ if } \alpha \to 1.$$