Hausdorff dimension distortion by Sobolev maps in foliated spaces

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Overview
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- **Metric spaces:** J. T. Tyson, K. Wildrick, *Dimension distortion by Sobolev mappings in foliated metric spaces*, Preprint, 2013


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- **Metric spaces:** -, J. T. Tyson, K. Wildrick, *Dimension distortion by Sobolev mappings in foliated metric spaces*, Preprint, 2013

- **Heisenberg groups:** -, J. T. Tyson, K. Wildrick, *Frequency of Sobolev dimension distortion of horizontal subgroups of Heisenberg groups*, Preprint 2013
Euclidean results
Proposition 0.1. Let \( f \in W^{1,p}(\Omega, Y) \), \( p > n \), and \( g_f \) denote the minimal upper gradient for \( f \). Then for all cubes \( Q \) compactly contained in \( \Omega \), we have

\[
\text{diam } f(Q) \leq C(n, p)(\text{diam } Q)^{1-n/p} \left( \int_Q g_f^p \right)^{1/p}.
\]
**Proposition 0.2.** Let \( f \in W^{1,p}(\Omega, Y) \), \( p > n \), and \( g_f \) denote the minimal upper gradient for \( f \). Then for all cubes \( Q \) compactly contained in \( \Omega \), we have

\[
\text{diam } f(Q) \leq C(n, p)(\text{diam } Q)^{1-n/p} \left( \int_Q g_f^p \right)^{1/p}.
\]

(0.1)

By the Morrey–Sobolev embedding theorem, each supercritical mapping \( f \in W^{1,p}(\Omega, Y) \), \( p > n \), has a representative which is locally \((1 - n/p)\)-Hölder continuous.

In particular, if \( E \subset \Omega \), \( \dim E = t \) then

\[
\dim f(E) \leq \frac{tp}{p - n}.
\]
Kaufman’s theorem

Theorem 0.3 (R. Kaufman 2000). \( E \subset \Omega; \mathcal{H}^t(E) < \infty, 0 < t < n. \) \( f \in W^{1,p}(\Omega, Y) \) for some \( p > n. \) Then \( f(E) \) has zero \( \mathcal{H}^{pt/(p-n+t)} \) measure. This statement is sharp.
Theorem 0.4 (R. Kaufman 2000). \( E \subset \Omega; \mathcal{H}^t(E) < \infty, 0 < t < n. \)
\( f \in W^{1,p}(\Omega, Y) \) for some \( p > n. \) Then \( f(E) \) has zero \( \mathcal{H}^{pt/(p-n+t)} \) measure. This statement is sharp.

In particular if \( \dim E = t \) then

\[
\dim f(E) \leq \frac{tp}{p - n + t}.
\]
**Kaufman’s theorem**

**Theorem 0.5** (R. Kaufman 2000). \( E \subset \Omega; \mathcal{H}^{t}(E) < \infty, 0 < t < n. \) \( f \in W^{1,p}(\Omega, Y) \) for some \( p > n. \) Then \( f(E) \) has zero \( \mathcal{H}^{pt/(p-n+t)} \) measure. This statement is sharp.

In particular if \( \dim E = t \) then

\[
\dim f(E) \leq \frac{tp}{p - n + t}.
\]

If \( V \in G(n, m). \) Then

\[
\dim f(V_a \cap \Omega) \leq \frac{pm}{p - n + m}. \tag{0.2}
\]
Let $\mathbb{R}^n = V^\perp \bigoplus V$ and $V_a = a + V$. Given a Sobolev map $f : \mathbb{R}^n \to Y$, how frequently can the intermediate values $m \leq \alpha \leq \frac{pm}{p-n+m}$ be exceeded by $\dim f(V_a)$?
Main Result

Let $\mathbb{R}^n = V^\perp \oplus V$ and $V_a = a + V$. Given a Sobolev map $f : \mathbb{R}^n \to Y$, how frequently can the intermediate values $m \leq \alpha \leq \frac{pm}{p-n+m}$ be exceeded by $\dim f(V_a)$?

**Theorem 0.7** (-.R. Monti, J. Tyson).

Let $\beta = \beta(p, \alpha) := (n - m) - \left(1 - \frac{m}{\alpha}\right)p$.

$f(V_a \cap \Omega)$ has zero $\mathcal{H}^\alpha$ measure for $\mathcal{H}^\beta$-almost every $a \in V^\perp$. 
Main Result

Let $\mathbb{R}^n = V^\perp \oplus V$ and $V_a = a + V$. Given a Sobolev map $f : \mathbb{R}^n \to Y$, how frequently can the intermediate values $m \leq \alpha \leq \frac{pm}{p-n+m}$ be exceeded by $\dim f(V_a)$?

**Theorem 0.8** (-.R. Monti, J. Tyson).

Let $\beta = \beta(p, \alpha) := (n - m) - \left(1 - \frac{m}{\alpha}\right)p$.

$f(V_a \cap \Omega)$ has zero $\mathcal{H}^\alpha$ measure for $\mathcal{H}^\beta$-almost every $a \in V^\perp$.

In particular:

$$\dim \{a \in V^\perp : \dim f(V_a) \geq \alpha\} \leq \beta$$
Let $\mathbb{R}^n = V^\perp \bigoplus V$ and $V_a = a + V$. Given a Sobolev map $f : \mathbb{R}^n \to Y$, how frequently can the intermediate values $m \leq \alpha \leq \frac{pm}{p-n+m}$ be exceeded by $\dim f(V_a)$?

**Theorem 0.9** (-R. Monti, J. Tyson).

Let $\beta = \beta(p, \alpha) := (n - m) - \left(1 - \frac{m}{\alpha}\right)p$.

$f(V_a \cap \Omega)$ has zero $\mathcal{H}^\alpha$ measure for $\mathcal{H}^\beta$-almost every $a \in V^\perp$.

In particular:

$$\dim\{a \in V^\perp : \dim f(V_a) \geq \alpha\} \leq \beta$$

**Note:** $\alpha \to m \Rightarrow \beta \to n - m$ and $\alpha \to \frac{pm}{p-n+m} \Rightarrow \beta \to 0$
**Theorem 0.10** (-.R. Monti, J. Tyson). Let $\alpha$ satisfy $m < \alpha < \frac{pm}{p-n+m}$ and $\beta = \beta(p, \alpha)$ as above. There exists a $E \subset \mathbb{R}^{n-m}$ s.th. $\mathcal{H}^\beta(E) > 0$ and for any integer $N > \alpha$, there exists a map $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ with the property that $\dim f(\{a\} \times \mathbb{R}^m) \geq \alpha$, for $\mathcal{H}^\beta$-almost every $a \in E$. 
Sharpness

**Theorem 0.12** (R. Monti, J. Tyson). Let $\alpha$ satisfy $m < \alpha < \frac{pm}{p-n+m}$ and $\beta = \beta(p, \alpha)$ as above. There exists a $E \subset \mathbb{R}^{n-m}$ s.th. $\mathcal{H}^\beta(E) > 0$ and for any integer $N > \alpha$, there exists a map $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ with the property that $\dim f(\{a\} \times \mathbb{R}^m) \geq \alpha$, for $\mathcal{H}^\beta$-almost every $a \in E$.

**Theorem 0.13** (S. Hencl, P. Honzik 2013). Let $m < \alpha < p \leq n$ and $\beta(p, \alpha) := (n - m) - (1 - \frac{m}{\alpha}) p$. and $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ be $p$-quasicontinuous. Then $\dim \{a \in V^\perp : \dim f(V_a) \geq \alpha\} \leq \beta$. 
**Sharpness**

**Theorem 0.14** (R. Monti, J. Tyson). Let $\alpha$ satisfy $m < \alpha < \frac{pm}{p-n+m}$ and $\beta = \beta(p, \alpha)$ as above. There exists a $E \subset \mathbb{R}^{n-m}$ s.th. $\mathcal{H}^\beta(E) > 0$ and for any integer $N > \alpha$, there exists a map $f \in W^{1,p}(\mathbb{R}^{n}, \mathbb{R}^{N})$ with the property that $\dim f(\{a\} \times \mathbb{R}^{m}) \geq \alpha$, for $\mathcal{H}^\beta$-almost every $a \in E$.

**Theorem 0.15** (S. Hencl, P. Honzik 2013). Let $m < \alpha < p \leq n$ and $\beta(p, \alpha) := (n-m) - \left(1 - \frac{m}{\alpha}\right)p$. and $f \in W^{1,p}(\mathbb{R}^{n}, \mathbb{R}^{k})$ be $p$-quasicontinuous. Then $\dim\{a \in V^\perp : \dim f(V_a) \geq \alpha\} \leq \beta$.

Note: $\alpha \to p \Rightarrow \beta \to n - p$ The above theorem is also sharp.
Sharpness

**Theorem 0.16** (-.R. Monti, J. Tyson). Let $\alpha$ satisfy $m < \alpha < \frac{pm}{p-n+m}$ and $\beta = \beta(p, \alpha)$ as above. There exists a $E \subset \mathbb{R}^{n-m}$ s.th. $H^\beta(E) > 0$ and for any integer $N > \alpha$, there exists a map $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$ with the property that $\dim f(\{a\} \times \mathbb{R}^m) \geq \alpha$, for $H^\beta$-almost every $a \in E$.

**Theorem 0.17** (S. Hencl, P. Honzik 2013). Let $m < \alpha < p \leq n$ and $\beta(p, \alpha) := (n-m) - (1 - \frac{m}{\alpha}) p$. \textit{and} $f \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^k)$ be $p$-quasicontinuous. Then $\dim \{a \in V^\perp : \dim f(V_a) \geq \alpha\} \leq \beta$.

Note: $\alpha \to p \Rightarrow \beta \to n - p$ The above theorem is also sharp.

**Example:** [P. Hajlasz, J. Tyson] Then there exists a continuous map $f \in W^{1,m}(\mathbb{R}^n, \ell^2)$ s.th. $\dim f(\{a\} \times [0, 1]^m) = \infty$ for all $a \in [0, 1]^{n-m}$. 
Metric space results
**Theorem 0.18** (-J. T., K. W.). \( X: \) proper, loc. \( Q \)-homogeneous, local \( Q \)-Poincaré inequality. If \( f: X \to Y \) is a continuous mapping that has an upper gradient in \( L^p_{\text{loc}}(X), p > Q \), then

\[
\dim_Y f(E) \leq \frac{p \dim_X E}{p - Q + \dim_X E}
\]  

(0.3)

for any subset \( E \subseteq X \).
**Theorem 0.20** (-J. T., K. W.). \( X: \) proper, loc. \( Q \)-homogeneous, local \( Q \)-Poincaré inequality. If \( f: X \to Y \) is a continuous mapping that has an upper gradient in \( L^p_{loc}(X), \ p > Q \), then

\[
\dim_Y f(E) \leq \frac{p \dim_X E}{p - Q + \dim_X E}
\]

(0.3)

for any subset \( E \subseteq X \).

**Theorem 0.21** (-J. T., K. W.). Let \((X, d, \mu), \) Ahlfors \( Q \)-regular. For \( 0 \leq s \leq Q \) and \( p > Q \), set \( \alpha = \frac{ps}{p - Q + s} \). Let \( E \) be a compact subset of \( X \) such that \( H^s(E) > 0 \). Then for all \( N > \alpha \), there exists a continuous \( f: X \to \mathbb{R}^N \) such that \( f \) has an upper gradient in \( L^p(X) \) and \( \dim f(E) \geq \alpha \).
Prevalence

Theorem 0.22 (-J. T., K. W.). The set of functions

\[ S_\alpha := \{ f \in N^{1,p}(X; \mathbb{R}^N) : \dim f(E) \geq \alpha \} \]

is a prevalent set in the Newtonian–Sobolev space \( N^{1,p}(X; \mathbb{R}^N) \).
Prevalence

**Theorem 0.23** (-J. T., K. W.). *The set of functions*

\[ S_\alpha := \{ f \in N^{1,p}(X; \mathbb{R}^N) : \text{dim } f(E) \geq \alpha \} \]

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Prevalence is an important notion in dynamics. Was introduced and developed by J. Yorke, V. Kaloshin, B. Hunt, W. Ott, T. Sauer.
Theorem 0.24 (-J. T., K. W.). The set of functions

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Prevalence is an important notion in dynamics. Was introduced and developed by J. Yorke, V. Kaloshin, B. Hunt, W. Ott, T. Sauer.

There exists a compactly supported Borel probability measure \( \lambda \) on \( N^{1,p}(X; \mathbb{R}^N) \) such that

\[ \lambda(N^{1,p}(X; \mathbb{R}^N) \setminus (S_\alpha + g)) = 0, \quad \forall g \in N^{1,p}(X; \mathbb{R}^N). \]
Idea of Proof

Idea: let $\mu$ be a Frostman measure on $E$ and find first $h \in N^{1,p}(X; \mathbb{R}^N)$ with $I_\alpha(h_\#\mu) < \infty$ i.e.

$$\int_E \int_E \frac{1}{|h(x) - h(y)|^\alpha} d\mu(x) d\mu(y) < \infty.$$
Idea of Proof

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Let $\mathcal{M} = \{M \in M(N \times N, \mathbb{R}), ||M|| \leq 1\}$ and $\nu$ to be the normalized $N^2$-dimensional Lebesgue measure on $\mathcal{M}$. 
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Let $\mathcal{M} = \{M \in M(N \times N, \mathbb{R}), ||M|| \leq 1\}$ and $\nu$ to be the normalized $N^2$-dimensional Lebesgue measure on $\mathcal{M}$. Consider

$$\Phi : \mathcal{M} \to N^{1,p}(X; \mathbb{R}^N), \Phi(M) := M \circ h,$$

and define: $\lambda := \Phi_\# \nu$. Let $g \in N^{1,p}(X; \mathbb{R}^N)$ and $f_M = M \circ h + g.$
Idea of Proof

Idea: let $\mu$ be a Frostman measure on $E$ and find first $h \in N^{1,p}(X; \mathbb{R}^N)$ with $I_\alpha(h_\# \mu) < \infty$ i.e.

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$$\Phi : \mathcal{M} \to N^{1,p}(X; \mathbb{R}^N), \Phi(M) := M \circ h,$$

and define: $\lambda := \Phi_\# \nu$. Let $g \in N^{1,p}(X; \mathbb{R}^N)$ and $f_M = M \circ h + g$. Have to prove:

$$\int_{\mathcal{M}} \int_E \int_E \frac{1}{|f_M(x) - f_M(y)|^\alpha} d\mu(x) d\mu(y) d\nu(M) < \infty.$$
Regular foliations

Definition [G. David, S. Semmes] A surjection \( \pi : X \rightarrow W \) is called \( s \)-regular if for any compact \( K \subseteq X \):

- \( \pi \mid_K \) is Lipschitz
- \( \pi^{-1}(B) \cap K \) can be covered by at most \( Cr^{-s} \) balls in \( X \) of radius \( Cr \).
Regular foliations

**Definition** [G. David, S. Semmes] A surjection $\pi : X \to W$ is called $s$-regular if for any compact $K \subseteq X$

- $\pi|_K$ is Lipschitz
- $\pi^{-1}(B) \cap K$ can be covered by at most $Cr^{-s}$ balls in $X$ of radius $Cr$.

Note that $\mathcal{H}^s_X(\pi^{-1}(a) \cap K) \leq C$, in particular, for the leaves $\pi^{-1}(a)$ it follows

$$\dim \pi^{-1}(a) \leq s,$$

and this inequality can be sometimes strict. The triple $(X, W, \pi)$ will be called an $s$-foliation of $X$. 
**Theorem 0.25.** Let \((X, d_X, \mu)\) be a proper, loc. \(Q\)-homogeneous, with local \(Q\)-Poincaré inequality, and is equipped with an \(s\)-foliation \((X, W, \pi)\). If \(f : X \to Y\) is a continuous mapping with upper gradient in \(L^p_{loc}(X)\), \(p > Q\), then

\[
\dim \{ a \in W : \dim (f(\pi^{-1}(a))) \geq \alpha \} \leq (Q - s) - \left(1 - \frac{s}{\alpha}\right)
\]

for \(s < \alpha \leq \frac{ps}{p-Q+s}\).
Theorem 0.26. Let \((X, d_X, \mu)\) be a proper, loc. \(Q\)-homogeneous, with local \(Q\)-Poincaré inequality, and is equipped with an \(s\)-foliation \((X, W, \pi)\). If \(f : X \to Y\) is a continuous mapping with upper gradient in \(L^{p}_{loc}(X)\), \(p > Q\), then

\[
\dim\{a \in W : \dim(f(\pi^{-1}(a))) \geq \alpha\} \leq (Q - s) - \left(1 - \frac{s}{\alpha}\right)
\]

for \(s < \alpha \leq \frac{ps}{p-Q+s}\).

If \(s = \dim \pi^{-1}(a)\) then this is a good theorem to apply. Examples:
**Theorem 0.27.** Let $(X, d_X, \mu)$ be a proper, loc. $Q$-homogeneous, with local $Q$-Poincaré inequality, and is equipped with an $s$-foliation $(X, W, \pi)$. If $f : X \to Y$ is a continuous mapping with upper gradient in $L^p_{loc}(X)$, $p > Q$, then

$$\dim\{a \in W : \dim(f(\pi^{-1}(a))) \geq \alpha\} \leq (Q - s) - (1 - \frac{s}{\alpha})$$

for $s < \alpha \leq \frac{ps}{p - Q + s}$.

If $s = \dim \pi^{-1}(a)$ then this is a good theorem to apply. Examples:

1. $X = \mathbb{R}^n$, $W = \mathbb{R}^{m-n}$ with the orthogonal projection $\pi : \mathbb{R}^n \to W$ defines a regular $s = m$-foliation.
Theorem 0.28. Let \((X, d_X, \mu)\) be a proper, loc. \(Q\)-homogeneous, with local \(Q\)-Poincaré inequality, and is equipped with an \(s\)-foliation \((X, W, \pi)\). If \(f: X \to Y\) is a continuous mapping with upper gradient in \(L^p_{loc}(X), p > Q\), then

\[
\dim\{a \in W : \dim(f(\pi^{-1}(a))) \geq \alpha\} \leq (Q - s) - (1 - \frac{s}{\alpha})
\]

for \(s < \alpha \leq \frac{ps}{pQ+s}\).

If \(s = \dim \pi^{-1}(a)\) then this is a good theorem to apply. Examples:

1. \(X = \mathbb{R}^n, W = \mathbb{R}^{m-n}\) with the orthogonal projection \(\pi: \mathbb{R}^n \to W\) defines a regular \(s = m\)-foliation.

2. \(X = \mathbb{H}^n = V^\perp \ltimes V, V\)-horizontal subgroup of \(\mathbb{H}^n\).
Heisenberg group results
Heisenberg foliations

\[ \mathbb{H}^n \cong \mathbb{R}^{2n+1} \ni p = (x_1, y_1, \ldots, x_n, y_n, t) = (z, t), \]

\[ p \ast p' = (z + z', t + t' + 2\omega(z, z')) \]

\[ \omega(z, z') = \sum_{i=1}^{n} (x_i y'_i - x'_i y_i), \]

\[ d_{\mathbb{H}^n}(p, p') = \|p^{-1} \ast p'\|_{\mathbb{H}^n}, \|p\|_{\mathbb{H}^n} = (\|z\|^4 + |t|^2)^{1/4}. \]

\[ \delta_r p = (rz, r^2t), Q = 2n + 2. \text{ Ahlfors, Poincare OK} \]
Heisenberg foliations

\[ \mathbb{H}^n \cong \mathbb{R}^{2n+1} \ni p = (x_1, y_1, \ldots, x_n, y_n, t) = (z, t), \]

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\[ \omega(z, z') = \sum_{i=1}^{n} (x_i y'_i - x'_i y_i), \]

\[ d_{\mathbb{H}^n}(p, p') = ||p^{-1} \ast p'||_{\mathbb{H}^n}, \quad ||p||_{\mathbb{H}^n} = (||z||^4 + |t|^2)^{1/4}. \]

\[ \delta_{r}p = (rz, r^2t), \quad Q = 2n + 2. \text{ Ahlfors, Poincare OK} \]

Isotropic subspace of \( \mathbb{V} \leq \mathbb{R}^{2n} \): \( \omega(u, v) = 0 \ \forall \ u, v \in \mathbb{V} \Rightarrow \) horizontal homogenous subgroups of \( \dim \mathbb{V} = m \leq n. \)

Vertical complementary subspace: \( \mathbb{V}^\perp = \mathbb{V}^\perp \times \mathbb{R} \) is a normal subgroup generating the semidirect product.
Heisenberg foliations

\[ \mathbb{H}^n \simeq \mathbb{R}^{2n+1} \ni p = (x_1, y_1, \ldots, x_n, y_n, t) = (z, t), \]

\[ p \ast p' = (z + z', t + t' + 2\omega(z, z')) \]

\[ \omega(z, z') = \sum_{i=1}^{n} (x_iy_i' - x_i'y_i), \]

\[ d_{\mathbb{H}^n}(p, p') = ||p^{-1} \ast p'||_{\mathbb{H}^n}, ||p||_{\mathbb{H}^n} = (||z||^4 + |t|^2)^{1/4}. \]

\[ \delta_r p = (rz, r^2t), \quad Q = 2n + 2. \quad \text{Ahlfors, Poincare OK} \]

Isotropic subspace of \( \mathbb{V} \leq \mathbb{R}^{2n} \): \( \omega(u, v) = 0 \ \forall \ u, v \in \mathbb{V} \Rightarrow \) horizontal homogenous subgroups of \( \dim \mathbb{V} = m \leq n. \)

Vertical complementary subspace: \( \mathbb{V}^\perp = \mathbb{V}^\perp \times \mathbb{R} \) is a normal subgroup generating the semidirect product.

\[ \mathbb{H}^n = \mathbb{V}^\perp \ltimes \mathbb{V}, p = p_{\mathbb{V}^\perp} \ast p_{\mathbb{V}} \text{ defines two types of foliations:} \]

\[ \pi_{\mathbb{V}} : \mathbb{H}^n \to \mathbb{V}, \pi_{\mathbb{V}}^{-1}(a) = \mathbb{V}^\perp \ast a \text{ and } \pi_{\mathbb{V}^\perp} : \mathbb{H}^n \to \mathbb{V}^\perp, \pi_{\mathbb{V}^\perp}^{-1}(a) = a \ast \mathbb{V} \]
Two types of foliations
Good news:

\((\mathbb{H}^n, \mathbb{V}, \pi_\mathbb{V})\)

is a regular \((Q - m)\)-foliation. Moreover:

\[
\dim(\pi_\mathbb{V}^{-1}(a)) = \dim(\mathbb{V}^\perp \ast a) = Q - m
\]
Good news:

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is a regular \((Q - m)\)-foliation. Moreover:

\[ \dim(\pi_{\mathbb{V}}^{-1}(a)) = \dim(\mathbb{V}^\perp \ast a) = Q - m \]

Bad news:

\((\mathbb{H}^n, \mathbb{V}^\perp, \pi_{\mathbb{V}^\perp})\)

is NOT a regular \(m\)-foliation.
Two types of foliations

Good news: \((\mathbb{H}^n, \mathcal{V}, \pi_{\mathcal{V}})\)

is a regular \((Q - m)\)-foliation. Moreover:

\[
\dim(\pi_{\mathcal{V}}^{-1}(a)) = \dim(\mathcal{V}^\perp \ast a) = Q - m
\]

Bad news: \((\mathbb{H}^n, \mathcal{V}^\perp, \pi_{\mathcal{V}^\perp})\)

is NOT a regular \(m\)-foliation.

Improvement: \((\mathbb{H}^n, (\mathcal{V}^\perp, d_E), \pi_{\mathcal{V}^\perp})\) is a regular \(m + 1\)-foliation, but

\[
s = m + 1 > m = \dim(\pi_{\mathcal{V}^\perp}^{-1}(a)),
\]

which for

\[
m \leq \alpha \leq \frac{Qm}{p - Q + m} \text{ and } \beta = (Q - s) - p(1 - \frac{s}{\alpha})
\]

does not imply sharp results at endpoints.
Heisenberg Foliation Distortion

Overview
Introduction and Notations
Redefine $\beta$ as:

$$\beta(p, m, \alpha) = \begin{cases} 
(Q - 1 - m) - \frac{p}{2} \left(1 - \frac{m}{\alpha}\right) & \alpha \in \left[m, \frac{pm}{p-2}\right], \\
(Q - m) - p \left(1 - \frac{m}{\alpha}\right) & \alpha \in \left[\frac{pm}{p-2}, \frac{pm}{p-(Q-m)}\right].
\end{cases}$$
Redefine $\beta$ as:

$$
\beta(p, m, \alpha) = \begin{cases} 
(Q - 1 - m) - \frac{p}{2} \left(1 - \frac{m}{\alpha}\right) & \alpha \in \left[m, \frac{pm}{p-2}\right], \\
(Q - m) - p \left(1 - \frac{m}{\alpha}\right) & \alpha \in \left[\frac{pm}{p-2}, \frac{pm}{p - (Q-m)}\right].
\end{cases}
$$

**Theorem 0.31.** Let $f \in W_{loc}^{1,p}(\mathbb{H}^n; Y)$, $p > Q$. Given a horizontal subgroup $\mathbb{V}$ of $\mathbb{H}^n$ of dimension $1 \leq m \leq n$, and

$$
m \leq \alpha \leq \frac{pm}{p - (Q - m)},
$$

it holds that

$$
\dim_{\mathbb{R}} \{a \in \mathbb{V}^\perp : \dim f(a \ast \mathbb{V}) \geq \alpha\} \leq \beta(p, m, \alpha).
$$
Overview

Introduction and Notations
Theorem 0.33. Let $\mathbb{V}_x$ denote the horizontal subgroup defined by the $x$-axis in $\mathbb{H}$, and let $p > 4$. For each
\[
1 < \alpha < \frac{p}{p-2} < \frac{p}{p-3}
\]
there is a compact set $E_\alpha \subset \mathbb{V}_x^\perp$ and a continuous mapping $f \in W^{1,p}(\mathbb{H}; \mathbb{R}^2)$ such that
\[
0 < \mathcal{H}_{\mathbb{R}^3}^{2-p(1-\frac{1}{\alpha})}(E_\alpha) < \infty
\]
and $\dim f(a \ast \mathbb{V}) \geq \alpha$ for every $a \in E_\alpha$. 

Sharpness in $\mathbb{H}$
Theorem 0.34. Let $V_x$ denote the horizontal subgroup defined by the $x$-axis in $\mathbb{H}$, and let $p > 4$. For each

$$1 < \alpha < \frac{p}{p-2} < \frac{p}{p-3}$$

there is a compact set $E_\alpha \subset V_x^\perp$ and a continuous mapping $f \in W^{1,p}(\mathbb{H}; \mathbb{R}^2)$ such that

$$0 < H^{2-p\left(1-\frac{1}{\alpha}\right)}_{\mathbb{R}^3}(E_\alpha) < \infty$$

and $\dim f(a \ast V) \geq \alpha$ for every $a \in E_\alpha$.

Note: $2 - p(1 - \frac{1}{\alpha}) < \beta = 2 - \frac{p}{2}(1 - p(1 - \frac{1}{\alpha}))$, however the example is asymptotically sharp

$$2 - p(1 - \frac{1}{\alpha}) \to 2, \text{ if } \alpha \to 1.$$