Right-angled billiards, pillowcase covers, and volumes of the moduli spaces

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(joint work with Jayadev Athreya, Alex Eskin
with an important contribution by Jon Chaika)

Dynamics of Groups and Rational Maps
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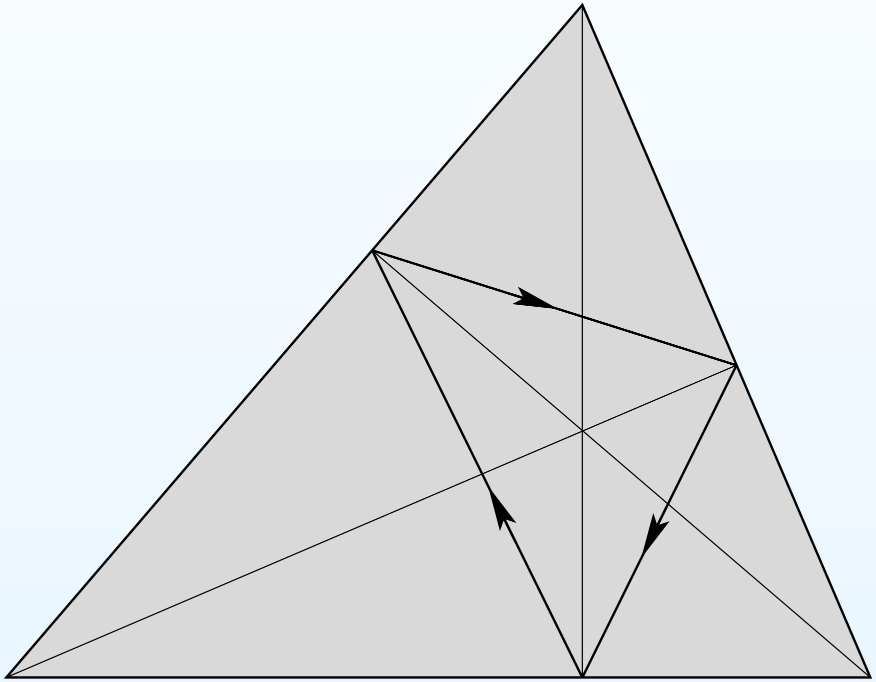
I. Billiards in right-angled polygons

Closed billiard trajectories

It is easy to find a periodic trajectory in an acute triangle:



Exercise. Show that the broken line joining the base points of the heights in an acute triangle is a closed billiard trajectory (called *Fagnano trajectory*). Show that it is an inscribed triangle of the minimal possible perimeter.

Challenge

It is difficult to believe, but for an obtuse triangle the problem is open:

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The answer might be affirmative (for triangles with obtuse angle at most 100^o R. Schwartz has verified it by a rigorous heavily computer-assisted proof). But even if it is affirmative, the natural question "And how many?.." is completely and desperately open already for acute triangles.

Open Problem. *Estimate the number $N(\Pi, L)$ of periodic trajectories of length at most L in a polygon Π as $L \to +\infty$.*

Billiards in rational polygons.

Life is better for *rational* polygons with all angles rational multiples of π.

Theorem (H. Masur). *For any rational polygon Π there exist constants c, C such that for L large enough the number $N(\Pi, L)$ of closed trajectories satisfies*

$$c \cdot L^2 \leq N(\Pi, L) \leq C \cdot L^2 \,.$$

For several exceptional rational polygons (namely, for regular polygons; for certain very special triangles; for squares with a vertical barrier; for L-shaped polygons (possibly with a barrier) with ratios of the horizontal and vertical sides in the same quadratic field; and for the finite covers of the above ones) an exact quadratic asymptotics is proved:

$$N(\Pi, L) \sim \mathrm{const} \cdot L^2 \quad \text{as} \quad L \to \infty \,.$$

These polygons correspond to *Teichmüller curves* or to *Teichmüller surfaces*. The proofs of exact asymptotics and the computation of the values of the constants requires a heavy machinery performed in the papers of Veech, Eskin-Markloff-Morris, Eskin-Masur-Schmoll, Bouw-Möller, Hooper, Bainbridge.

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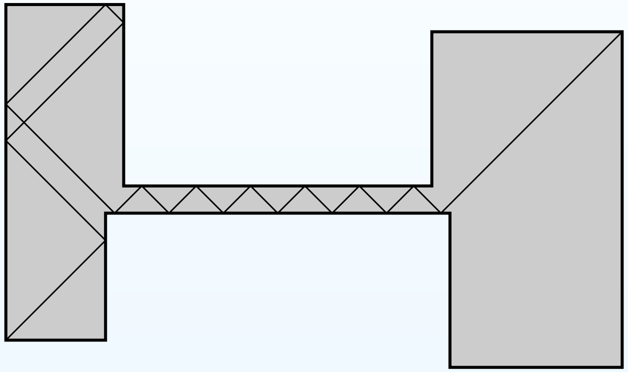
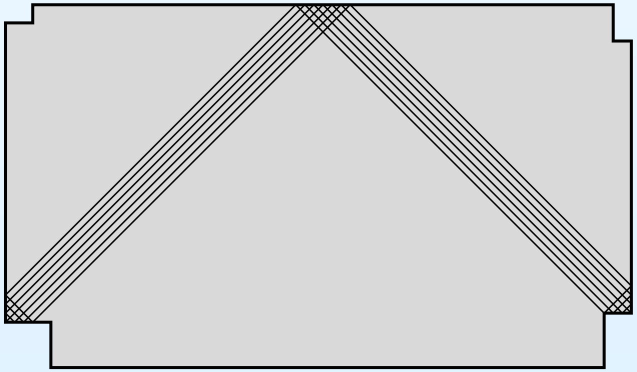
Following Moon Duchin
we shall play on
right-angled billiard tables.

Closed trajectories and generalized diagonals

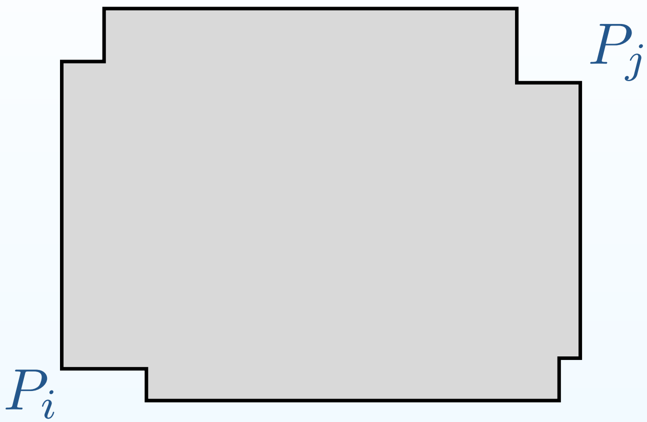
We count the asymptotic number of trajectories of bounded length joining a given pair of corners ("*generalized diagonals*") as the bound L tends to infinity.

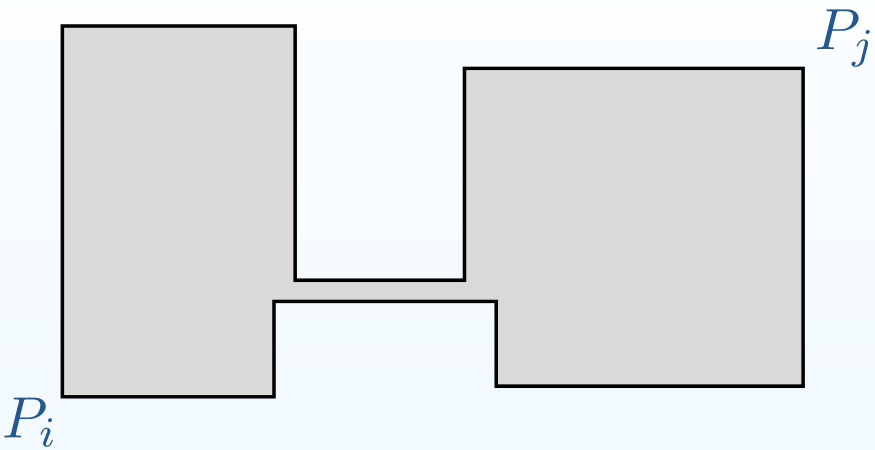


We also want to count the number of periodic trajectories of length at most L, or rather the number of *bands* of periodic trajectories. We might also count the bands with the weight representing the "thickness" of the band.



Number of generalized diagonals



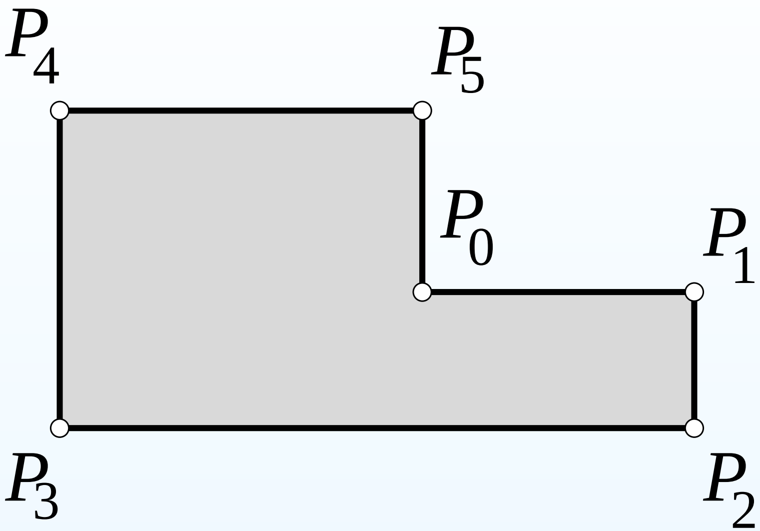


Example of a Theorem. *For almost any right-angled polygon Π in any family $\mathcal{B}(k_1, \dots, k_n)$ of right-angled polygons with angles $k_1 \frac{\pi}{2}, \dots k_n \frac{\pi}{2}$, the number $N_{i,j}(\Pi, L)$ of trajectories of length bounded by L joining any two fixed corners with true right angles $\frac{\pi}{2}$ is asymptotically the same as for a rectangle:*

$$N_{i,j}(\Pi, L) \sim \frac{1}{2\pi} \cdot \frac{(\text{bound } L \text{ for the length})^2}{\text{area of the table}} \quad \text{as} \quad L \to \infty$$

and does not depend on the shape of the polygon Π.

Naive intuition does not help...



However, say, for almost any L-shaped polygon Π the number $N_{0,j}(\Pi, L)$ of trajectories joining the corner P_0 with the angle $3\frac{\pi}{2}$ to some other corner P_j has asymptotics

$$N_{0,j}(\Pi, L) \sim \frac{2}{\pi} \cdot \frac{(\text{bound } L \text{ for the length})^2}{\text{area of the table}} \quad \text{as} \quad L \to \infty\,,$$

which is 4 times (and not 3) times bigger than the number of trajectories joining a fixed pair of right corners...

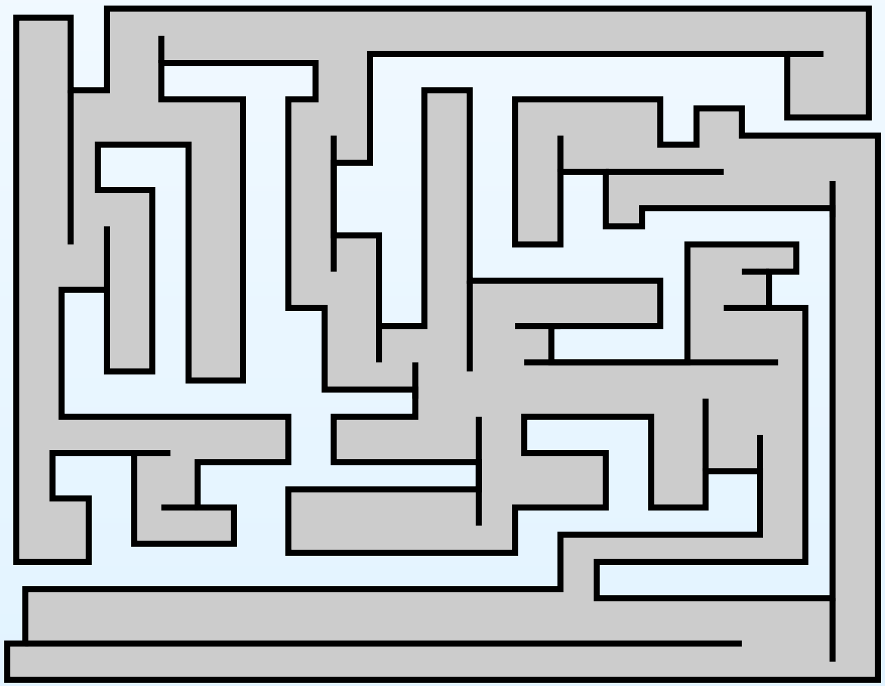
Billiard in a right-angled polygon: general answer

For each family $\mathcal{B}(k_1, \ldots, k_n)$ of right-angled polygons we find all topological types of "admissible" generalized diagonals (closed trajectories). We show that a billiard table Π outside of a zero measure set in $\mathcal{B}(k_1, \ldots, k_n)$ does not contain a single "non-admissible" generalized diagonal (closed trajectory).

Billiard in a right-angled polygon: general answer

For each topological type we explicitly compute the coefficient in the exact quadratic asymptotics for the corresponding number of generalized diagonals (number of closed trajectories) of bounded length L, which is the same for almost all Π in the billiard family. Say, the coefficients in the exact quadratic asymptotics for the number of generalized diagonals joining a pair of distinct fixed vertices of one of the angles $\frac{\pi}{2}, 3\frac{\pi}{2}, 4\frac{\pi}{2}$ is described by the following table:

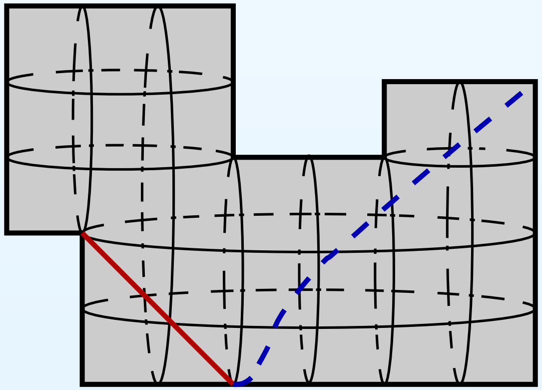
$angle$	$\frac{4\pi}{2}$	$\frac{3\pi}{2}$	$\frac{\pi}{2}$
$\frac{4\pi}{2}$	$\frac{9}{10}$	$\frac{45}{64}$	$\frac{9}{32}$
$\frac{3\pi}{2}$	$\frac{45}{64}$	$\frac{16}{3\pi^2}$	$\frac{2}{\pi^2}$
$\frac{\pi}{2}$	$\frac{9}{32}$	$\frac{2}{\pi^2}$	$\frac{1}{2\pi^2}$

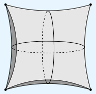


II. Pillowcase covers and volumes of the moduli spaces

Billiards in right-angled polygons versus quadratic differentials on $\mathbb{CP}^1$

The topological sphere obtained by gluing two copies of the billiard table by the boundary is naturally endowed with a flat metric. This metric has conical singularities at the points coming from vertices of the polygon, otherwise it is nonsingular. In the case of a “rectangular polygon” the flat metric has holonomy in $\mathbb{Z}/(2\mathbb{Z})$. Hence it corresponds to a meromorphic quadratic differential with at most simple poles on $\mathbb{CP}^1$. Moreover, geodesics on this flat sphere project to billiard trajectories! Thus, to count billiard trajectories we may count geodesics on flat spheres!





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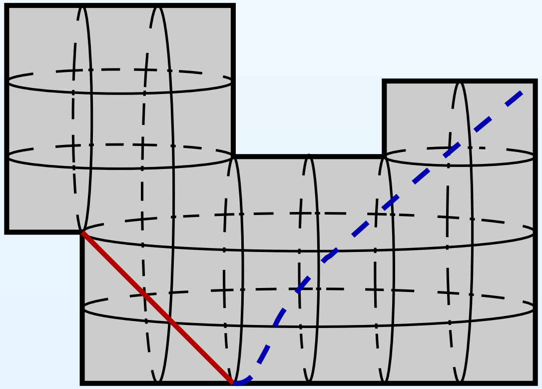

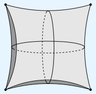

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Integer points in the moduli space of Abelian differentials

When a flat metric has on a surface S trivial holonomy, it defines a quadratic differential which is a global square of a holomorphic 1-form. The moduli space $\mathcal{H}(m_1,\dots,m_n)$ of holomorphic 1-forms with zeroes of multiplicities $m_1,\dots,m_n$, where $\sum m_i = 2g-2$, is modelled on the vector space $H^1(S,\{P_1,\dots,P_n\};\mathbb{C})$. The latter vector space contains a natural lattice $H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$. The points of this lattice are represented by *square-tiled surfaces*.

Indeed, if a flat surface S is defined by an holomorphic 1-form ω such that $[\omega]\in H^1(S,\{P_1,\dots,P_n\};\mathbb{Z}\oplus i\mathbb{Z})$, it has a canonical structure of a ramified cover over the torus $\mathbb{T}=\mathbb{R}^2/(\mathbb{Z}\oplus i\mathbb{Z})$ defined by the map

$$P\mapsto \int_{P_1}^{P}\omega \mod \mathbb{Z}\oplus i\mathbb{Z}\,.$$

Integer points in the strata $\mathcal{Q}(d_1,\dots,d_n)$ of quadratic differentials are represented by “pillowcase covers” over $\mathbb{C}\mathrm{P}^1$ branched at four points.

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Calculation the volume of a "sphere" through counting integer points inside a "ball" of large radius.

The volume of the "unit sphere" $\mathcal{H}_1(m_1,\dots,m_n)$ of flat surfaces of area 1, is a multiple of the volume of the "unit ball" $\mathcal{H}_{\le 1}(m_1,\dots,m_n)$ of flat surfaces of area at most 1 by a dimensional factor:

$$\mathrm{Vol}(\mathcal{H}_1(m_1,\dots,m_n)) = \dim_{\mathbb{R}} \mathcal{H}(m_1,\dots,m_n)\cdot \mu((\mathcal{H}_{\le 1}(m_1,\dots,m_n))\,.$$

The volume of the "unit ball" is equal to the coefficient in the asymptotics of the number of lattice points captured inside the unit ball for the lattice with a grid $1/N$ when $N\to\infty$. The latter number is the same as the number of integer points inside a "ball of radius N".

Thus, to compute the volume of a stratum of flat surface, it is sufficient to find the asymptotics for the number $\mathrm{Sq}_N(m_1,\dots,m_n)$ of square-tiled surfaces tiled with at most N squares:

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Historical remarks.

For Abelian differentials this counting problem was successfully solved by A. Eskin and A. Okounkov in 2001. Their formula implies that the volume $\operatorname{Vol}\mathcal{H}_1(m_1,\dots,m_n)$ of every connected component of every stratum of Abelian differentials is equal to $r\cdot\pi^{2g}$, where r is a rational number. It also provides an efficient algorithm allowing to compute r for all strata up to genus 10 and for some strata (like the principal one) up to genus 200.

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A. Eskin and A. Okounkov also obtained a formula for the volumes $\operatorname{Vol}\mathcal{Q}_1(d_1,\dots,d_k)$ of the moduli spaces of quadratic differentials. However, this time the resulting expressions in terms of characters of the symmetric group are more complicated, and the formula is not translated into a computer code yet. This is why we know the numerical values of the volumes only for several low-dimensional strata, where they are computed by hands.

Kontsevich conjecture

Let $$v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \geq -1 \text{ is odd} \\ 2 & \text{when } n \geq 0 \text{ is even} \end{cases}$$

By convention we set $(-1)!! := 0!! := 1$, so $v(-1) = 1$ and $v(0) = 2$.

Theorem. *The volume of any stratum $\mathcal{Q}_1(d_1, \ldots, d_k)$ of meromorphic quadratic differentials with at most simple poles on $\mathbb{C}\mathrm{P}^1$ (i.e. when $d_i \in \{-1\,;\,0\} \cup \mathbb{N}$ for $i = 1, \ldots, k$, and $\sum_{i=1}^k d_i = -4$) is equal to*

$$\operatorname{Vol} \mathcal{Q}_1(d_1, \ldots, d_k) = 2\pi \cdot \prod_{i=1}^{k} v(d_i)\,.$$

Corollary. *The number of pillowcase covers of degree at most N with ramification pattern corresponding to $\mathcal{Q}(d_1, \ldots, d_k)$ has the following leading term in the asymptotics as $N \to \infty$*

$$\text{Number of pillowcase covers } \sim \frac{\pi}{k-2} \prod_{i=1}^{k} v(d_i) \cdot N^{k-2}\,.$$

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Various approaches to a proof.

M. Kontsevich conjectured this formula about ten years ago. Using Lyapunov exponents of the Teichmüller geodesic flow, he predicted volumes of the special strata $\mathcal{Q}(d, -1^{d+4})$ and then made an ambitious guess for general case.

In ten years we made numerous attempts to prove the Conjecture, for example, following an approach based on Kontsevich' solution to the Witten conjecture. We also tried to developing ideas of Eskin–Okounkov. And not only. Each time we ran into very interesting combinatorics, but we were never able to get any simple expressions. Finally we had to come back to the approach based on the formula for the Lyapunov exponents of the Teichmüller geodesic flow. The latter formula is derived using serious analytic, geometrical, and dynamical inputs.

Challenge. *Find an alternative proof of the formula for the volumes.*

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III. Siegel–Veech constants and Lyapunov exponents

Siegel—Veech constant

Closed regular geodesics on flat surfaces appear in families of parallel closed geodesics sharing the same length. Every such family fills a *maximal cylinder* having conical points on each of the boundary components.

Denote by $N_{area}(S,L)$ the sum of areas of all cylinders spanned by geodesics of length at most L.

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where constant $c_{area}(d_1,\dots,d_k)$ is called the Siegel–Veech constant.

Analogous formulae are valid for other counting functions considered above, say for $N(S,L)$ or for $N_{\mathcal{C}}(S,L)$, where $\mathcal{C}$ is any admissible *configuration* of closed geodesics or of saddle connections.

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Theorem (A. Eskin, H. Masur.) *For almost any flat surface S in any stratum $\mathcal{Q}(d_1, \dots, d_k)$ one has*

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$$= (\text{explicit combinatorial factor}) \cdot \frac{\prod \text{Vol}(\text{adjacent simpler strata})}{\text{Vol}\, \mathcal{Q}_1(d_1, \dots, d_k)}\,.$$

(The proof develops ideas from joint works of Eskin and Zorich with H. Masur.)

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Lyapunov exponents and alternative expression for the Siegel–Veech constant

Theorem (A. Eskin, M. Kontsevich, A. Z.) *Let $d\nu_1$ be a $\mathrm{PSL}(2,\mathbb{R})$ ergodic probability measure supported on a ~~regular~~ [1] suborbifold $\mathcal{M}_1$ in some stratum $\mathcal{Q}_1(d_1,\dots,d_n)$. The Lyapunov exponents of the Hodge bundle H^1_- along the Teichmüller flow restricted to $\mathcal{M}_1$ satisfy the following relation:*

$$\lambda_1+\lambda_2+\dots+\lambda_g \;=\; \frac{1}{24}\cdot\sum_{i=1}^{n}\frac{d_i(d_i+4)}{d_i+2}+\frac{\pi^2}{3}\cdot c_{area}(d\nu_1)\,,$$

where $c_{\text{area}}(d\nu_1)$ is the Siegel–Veech constant.

Corollary. *For any $\mathrm{PSL}(2,\mathbb{R})$-invariant manifold in any stratum of quadratic differentials in genus zero one has*

$$c_{area}(d_1,\dots,d_n)=-\frac{1}{8\pi^2}\sum_{j=1}^{n}\frac{d_j(d_j+4)}{d_j+2}\,.$$

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Combinatorial identities

Combining two expressions for $c_{area}(d_1,\dots,d_n)$ we get series of combinatorial identities recursively defining volumes of all strata:

$$\big(\text{explicit combinatorial factor}\big)\cdot\frac{\prod \operatorname{Vol}(\text{adjacent simpler strata})}{\operatorname{Vol}\mathcal{Q}_1(d_1,\dots,d_k)}=$$
$$=-\frac{1}{8\pi^2}\sum_{j=1}^{n}\frac{d_j(d_j+4)}{d_j+2}.$$

It remains to verify that the guessed answer satisfy these identities. The verification is reduced to verifying some cute combinatorial identities for multinomial coefficients; it is based on manipulations with appropriate generating functions.

Having proved the formula for the volumes of strata in genus 0, we can plug the values into the formulae for the Siegel–Veech constants, and obtain their numerical values.

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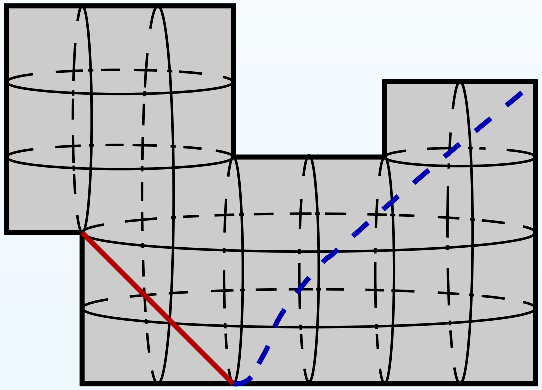
IV. Back to billiards in right-angled polygons

Transversality

We have solved the counting problem for almost all flat spheres in any family $\mathcal{Q}_1(d_1, \ldots, d_k)$ in genus zero. The trouble is that the subspace of those flat spheres which correspond to right-angled billiards has large codimension: it is a bit larger than half dimension of the in the ambient family.



Proposition. *Consider the canonical local embedding*

$$\mathcal{B}(k_1, \ldots, k_n) \subset \mathcal{Q}(k_1 - 2, \ldots, k_n - 2).$$

For almost all directional billiards in $\mathcal{B}(k_1, \ldots, k_n)$ the projection of the tangent space $T_*\mathcal{B}(k_1, \ldots, k_n)$ to the unstable subspace of the Teichmüller geodesic flow is a surjective map.

Ergodic Theorem by J. Chaika

J. Chaika used a variation of an argument of Margulis to prove equidistribution in the ambient stratum $\mathcal{Q}_1$ of large circles centered at almost all points of the billiard submanifold $\mathcal{B}_1$. This approach is similar in spirit to the approach of Eskin–Margulis–Mozes.

Theorem (J. Chaika). *Let f be any bounded 1-Lipschitz function with a zero mean on a stratum $\mathcal{Q}_1(k_1-2,\dots,k_n-2)$ of quadratic differentials in genus zero. Then for $\mu_{\mathcal{B}}$-almost every right angled billiard Π in $\mathcal{B}_1(k_1, dots, k_n)$ one has:*

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f(g_T r_\theta q_\Pi)\,d\theta = \frac{1}{\mu_1(\mathcal{Q}_1)}\int_{\mathcal{Q}_1} f\,d\mu_1\,.$$

The proof uses, in particular, the result of Athreya on quantitative recurrence of the Teichmüller geodesic flow, and the result of Avila–Resende on exponential mixing of the Teichmüller geodesic flow on $\mathcal{Q}_1$.

Applying these results, one proves the exact quadratic asymptotics for $\mu_{\mathcal{B}}$-almost all quadratic differentials $q(\Pi)$ in the billiard submanifold.

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Georges Braque, Le Billard (1944). Centre Pompidou, Paris