Parametrizing hyperbolic structures using shears

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$\mathbf{H}$ =the hyperbolic plane

Definition: The universal Teichmüller space
$T(\mathbf{H}) = \{f : \mathbf{H} \to \mathbf{H} | \ f \ \textit{is quasiconformal}\}$ /homotopy rel.
$\partial\mathbf{H} = S^1$ and post-composition by $PSL_2(\mathbb{R})$, or equivalently

$$T(\mathbf{H}) = \{h : S^1 \to S^1 | \ h \ \textit{is quasisymmetric}\}/PSL_2(\mathbf{R})$$

Facts about $T(\mathbf{H})$:

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Facts about $T(\mathbf{H})$:

- complete, complex, infinite dimensional, non-separable Banach manifold
- $T(\mathbf{H})$ contains (multiple copies of) Teichmüller spaces of all hyperbolic Riemann surface
- geodesically complete for the Teichmüller metric but no uniqueness for the geodesics between two points (extremal vs. uniquely extremal qc maps starts with the work Reich-Strebel, Earle, Kruskhal, culminates Bozin-Lakic-Markovic-Mateljevic);

More recently:

- Markovic proved that if $T_{qc}(X_0)$ and $T_{qc}(X_1)$ are isometric for the Teichmüller metric for two geometrically infinite hyperbolic surfaces X_0 and X_1 (without assumption that X_0 and X_1 are homeomorphic) then there exists a qc map $f : X_0 \to X_1$

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- Fletcher further proved that any $T_{qc}(X_0)$ of an infinite hyperbolic surface X_0 is locally biLipschitz to l^∞

We consider a question of a parametrization of $T(\mathbf{H})$.

For closed and finite area hyperbolic surfaces a parametrization is given by the **Fenchel-Nielsen coordinates** on a fixed pants decomposition of the base surface. Namely, two real numbers are assigned to each cuff of the pants decomposition: the hyperbolic lengths of the cuff and the twist.

Given a **maximal geodesic lamination** on a **closed** (or finite) hyperbolic surface, the hyperbolic metric induces a **finitely additive** (real valued) transverse measure to the lamination (Thurston,Bonahon). The construction of the finitely additive transverse measure is subtle and depends on the structure of geodesic laminations on closed hyperbolic surfaces. Moreover, the hyperbolic metric can be recovered from the transverse measure. A parametrization of the Teichmüller space of a closed (or finite) Riemann surface is given in terms of finitely additive transverse measures to a fixed maximal geodesic lamination on the base surface (Thurston,Bonahon).

Given an arbitrary maximal geodesic lamination λ on $\mathbf{H}$, it is not known (at least to me) what kind of transverse structure to λ is induced by the hyperbolic metric.

If a maximal geodesic lamination on a finite (punctured) hyperbolic surface contains finitely many leaves which all end at punctures, the transverse measure is an assignment of real numbers to the geodesics of the lamination. Penner parametrized Teichmüller spaces of finite punctured hyperbolic surfaces using the transverse measure to the finite lamination whose all geodesics end at the punctures.

The assignment of real numbers to each geodesic of the finite lamination λ on a punctured surface is obtained by lifting λ to the universal covering $\mathbf{H}$. The complement of the lift $\tilde{\lambda}$ consists of ideal hyperbolic triangles. Each geodesic $\tilde{g}$ of $\tilde{\lambda}$ is on the boundary of two complementary triangles and the assignment of the transverse measure (called the **shear**) to the geodesic is (essentially) the logarithm of the cross-ratio of the four endpoints of the two complementary triangles adjacent to the geodesic.

Alternatively, the shear on $\tilde{g}$ is the signed hyperbolic distance between the foots on $\tilde{g}$ of the perpendiculars to $\tilde{g}$ from the third vertices of the two adjacent triangles to $\tilde{g}$.

In other words, the shear measures how far are the two adjacent triangles to $\tilde{g}$ from being reflections of each other in the common side $\tilde{g}$.

More precisely, the shear is the translation length of the hyperbolic translation with axis $\tilde{g}$ that moves the reflection (across $\tilde{g}$) of one triangle onto the other triangle.

This definition of the shear extends to the locally finite maximal geodesic laminations in the hyperbolic plane $\mathbf{H}$.

A homeomorphism f of $\partial\mathbf{H} = S^1$ induces a map between geodesics of $\mathbf{H}$ by identifying the geodesics with the pairs of its endpoints. Given a maximal locally finite geodesic lamination λ of $\mathbf{H}$, the image $f(\lambda)$ is also a maximal locally finite geodesic lamination of $\mathbf{H}$. Such laminations are necessary ideal triangulations of $\mathbf{H}$.

Given a maximal locally finite geodesic lamination λ and the shears on $f(\lambda)$, the homeomorphism f can be recovered (up to post-composition by elements $PSL_2(\mathbf{R})$). This is in analogy to recovering the hyperbolic metric on closed surfaces from the transverse measures.

Let $\mathcal{F}$ be the Farey tesselation. Recall that $\mathcal{F}$ is obtained by reflecting a single ideal hyperbolic triangle along its sides. Thus the shears of $\mathcal{F}$ are all equal to 0.

Penner posed a problem of characterizing which shears on $\mathcal{F}$ give rise to homeomorphisms, quasisymmetric maps, symmetric maps, smooth maps, ...

Penner and Sullivan gave a sufficient coefficient on the shears on $\mathcal{F}$ to give quasisymmetric maps.

We give necessary and sufficient conditions on shears to obtain homeomorphism, quasisymmetric and symmetric maps.

Definition

A **chain of geodesics** in the Farey tesselation $\mathcal{F}$ is a sequence of consecutive edges without repetitions.

A chain of geodesics accumulates to a single point on S^1 and each point on S^1 is an accumulation point of a chain of geodesics.

Assume two edges E and E_1 of $\mathcal{F}$ have a common endpoint $p \in S^1$. Let C be a horocircle with center at p oriented such that the endpoints of the corresponding hornbill stay on the left when we traverse C in the given orientation. Then $E < E_1$ if $E \cap C$ comes before $E_1 \cap C$ for the orientation of C; otherwise $E_1 < E$.

Definition

A **fan of geodesics** of $\mathcal{F}$ is a two-sided sequence E_n which shares a common endpoint p. The common endpoint is called the *tip* of the fan.

Theorem (Š.)

Let $s : \mathcal{F} \to \mathbf{R}$ be a function. Then s induces a homeomorphism of S^1 if and only if for each chain of geodesics $\{E_n\}_{n\in\mathbf{N}}$ in $\mathcal{F}$ we have

$$\sum_{n=1}^{\infty} e^{s_{(n,1)}+s_{(n,2)}+\cdots+s_{(n,n)}} = \infty,$$

where $s_{(n,i)} = \pm s(E_i)$ and the sign is chosen according to the combinatorics of the finite chain $E_1, \ldots, E_n$.
Namely, if $E_n < E_{n+1}$ then $s_{(n,n)} = s(E_n)$; otherwise $s_{(n,n)} = -s(E_n)$. For $i < n$, we set $s_{(n,i)} = s(E_i)$ if $E_i < E_{i+1}$ and the number of times we switch fans from E_i to E_{n+1} is even, or if $E_{i+1} < E_i$ and the number of times we switch fans from E_i to E_{n+1} is odd. Otherwise we set $s_{(n,i)} = -s(E_i)$.

Corollary (Š.)

If $s : \mathcal{F} \to \mathbf{R}$ induces a homeomorphism of S^1, then $t \times s : \mathcal{F} \to \mathbf{R}$ defined by $(t \times s)(E) := t \times (s(E))$ for all $0 \leq t \leq 1$ also induces homeomorphism of S^1. When $t = 0$ the induced homeomorphism is the identity. Thus we obtain a path of homeomorphism which connect any given homeomorphism(given by the shear function s) to the identity via a path of homeomorphism given by $t \mapsto t \times s$.

Recall that quasisymmetric maps of S^1 are orientation preserving homeomorphisms which move symmetric triples of points on S^1 (at every scale) into triples which are not too asymmetric. A symmetric map of S^1 is an orientation preserving homeomorphism which move symmetric triples at small scales into almost symmetric triples.

The Farey tesselation $\mathcal{F}$ is obtained by reflections of the initial ideal hyperbolic triangle in its sides. The edges of the initial triangle are said to be of **generation** 0. The edges of $\mathcal{F}$ obtained by one reflection of the initial triangle are said to be of generation 1. The edges of $\mathcal{F}$ obtained by reflecting the edges of n-th generation are said to be $(n+1)$-st generation. Thus there are 3 edges of generation 0, $2\times 3=6$ edges of generation 1, and in general $2^n\times 3$ edges of generation n in the Farey tesselation.

Let $\{E_n\}_{n\in\mathbf{Z}}$ be a fan of geodesics in $\mathbf{F}$ with the tip p. Given $k\in\mathbf{Z}$ and $m\in\mathbf{N}$ we define

$$s(p;k,m)=e^{s(E_k)}\times$$
$$\times\frac{1+e^{s(E_{k+1})}+e^{s(E_{k+1})+s(E_{k+2})}+\cdots+e^{s(E_{k+1})+\cdots+s(E_{k+m})}}{1+e^{-s(E_{k-1})}+e^{-s(E_{k-1})-s(E_{k-2})}+\cdots+e^{-s(E_{k+1})-\cdots-s(E_{k-m})}}$$

The quantity $s(p;k,m)$ measures the distortion of the endpoints of three geodesics in the fan with tip p which intersect any horocircle based at p in a symmetric fashion. Recall that the hyperbolic metric on horocircles identifies them with the real line and that the limit of horocircles (when their sizes go to ∞) is S^1. The symmetries of $\mathcal{F}$ allowed us to conclude that this necessary condition for quasisymmetry is also sufficient.

Theorem (Š.)

A function $s:\mathcal{F}\to\mathbf{R}$ induces a quasisymmetric map of S^1 if and only if there exists a constant $M>1$ such that for each tip p and for every $k\in\mathbf{Z}$ and every $m\in\mathbf{N}$ we have

$$\frac{1}{M}\leq s(p;k,m)\leq M.$$

The above theorem gives the parametrization of the universal Teichmüller space $T(\mathbf{H}) := QS(S^1)/PSL_2(\mathbf{R})$ in terms of functions $s : \mathcal{F} \to \mathbf{R}$ which satisfy the boundedness condition on the fans. We are also able to parametrize the space $T_0(\mathbf{H}) := Sym(S^1)/PSL_2(\mathbf{R})$ as follows

Theorem (Š.)

A function $s : \mathcal{F} \to \mathbf{R}$ induces a symmetric map of S^1 if and only if

$$s(p; k, m) \rightrightarrows 1$$

as the generations of E_{k+m} and E_{k-m} go to infinity independently of the fan.

We also find the topology for this parametrization to be a homeomorphism. Given $h_1, h_2 \in QS(S^1)$, let $s_1, s_2 : \mathcal{F} \to \mathbf{R}$ be their corresponding shears. Define

$$M(s_1,s_2) = \sup_p \sup_{k,m} \max\{\frac{s_1(p;k,m)}{s_2(p;k,m)}, \frac{s_2(p;k,m)}{s_1(p;k,m)}\}$$

Theorem (Š.)

A sequence $h_n \to h$ in the Teichmüller metric on $T(\mathbf{H})$ if and only if $M_{s_n,s} \to 1$ as $n \to \infty$, where s_n, s are shears of h_n, h.

Let $t \mapsto h_t$ for $t \in (-\epsilon, \epsilon)$ be a differentiable path in $T(\mathbf{H})$ with $h_0 = id$. Then

$$\frac{d}{dt} h_t|_{t=0} = V : S^1 \to \mathcal{T} S^1$$

is a Zygmund vector field. Let s_t be the path of shears and define

$$\dot{s}(E) = \frac{d}{dt} s_t(E)|_{t=0}$$

the shear for V, where $E \in \mathcal{F}$. Given a shear function $\dot{s} : \mathcal{F} \to \mathbf{R}$ for V, the vector field V can be recovered (up to adding a quadratic polynomial). Not all shear function correspond to (Zygmund or continuous) vector fields. We parametrize Zygmund vector fields

Theorem (Š.)

A function $\dot{s}:\mathcal{F}\to\mathbf{R}$ induces a Zygmund vector field V on S^1 if and only if there exists a constant $C>0$ such that for all fans $\{E_n\}_{n\in\mathbf{Z}}$ and for all $m,k\in\mathbf{Z}$, $k\geq 0$, we have

$$\Big|\dot{s}(E_m)+\frac{k}{k+1}[\dot{s}(E_{m+1})+\dot{s}(E_{m-1})]+\frac{k-1}{k+1}[\dot{s}(E_{m+2})+ \\ +\dot{s}(E_{m-2})]+\cdots+\frac{1}{k+1}[\dot{s}(E_{m+k})+\dot{s}(E_{m-k})]\Big|\leq C.$$

The (almost) complex structure on the universal Teichmüller space at the tangent space at the identity is given by the (properly normalized) Hilbert transform on Zygmund maps (Kerckhoff, Nag-Verjovsky). We use the above parametrization of the (normalized) Zygmund maps in terms of shears to obtain the almost complex structure in terms of the hyperbolic geometry.

Let $s : \mathcal{F} \to \mathbf{R}$ be the shear function of a Zygmund vector field V on S^1. Given a fan $\mathcal{F}_p = \{E_n\}$ of geodesics with tip p we denote by V_p the vector field on S^1 obtained from the shear function $s_p : \mathcal{F} \to \mathbf{R}$ defined by $s_p(E) := \frac{1}{2}s(E)$ if $E \in \mathcal{F}_p$, and $s_p(E) = 0$ otherwise. Then V_p is a Zygmund vector field.

Theorem (Š.)

The Hilbert transform of V is given by the series

$$HV(x) = \sum_p HV_p(x)$$

where HV_p is the Hilbert transform of V_p and V_p is defined as above using the shears in the fan $\mathcal{F}_p$. The series converges uniformly on S^1.

Moreover, the Hilbert transform $H(V_p)$ of the vector field V_p corresponding to the fan $\mathcal{F}_p = \{E_n^p\}_n$ is given by the formula

$$HV_p(z) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \dot{s}(E_n^p)\frac{(z-a_n^p)(z-b_n^p)}{a_n^p - b_n^p}\log\left|\frac{z-b_n^p}{z-a_n^p}\right|$$

where a_n^p is the initial point and b_n^p is the terminal point of E_n^p.

When we restrict to a finite punctured surface, the Hilbert transform gives the almost complex structure. The Weil-Petersson metric is obtained by taking the Weil-Petersson symplectic two form of the tangent vector and its image under the almost complex structure operator. Since the symplectic two form in shear coordinates is simply the Thurston symplectic form, the main difficulty in computing is to find the almost complex structure. Our formula gives a way to compute the WP metric. Ongoing work with Todd Gaugler an undergraduate at Queens College to have a computer implementation for once punctured torus.

Motivation: one of the main ingredients in the proof of surface subgroup conjecture (Kahn-Markovic) (Every closed hyperbolic 3-manifold has an immersed closed surface of some (possibly large) genus.)
From the ergodicity of the geodesic flow, it follows that there exist a large family of immersed pairs of pants in the 3-manifold which have large cuffs of approximately equal and large length, and that they glued to each other such that the complex Fenchel-Nielsen twist-bends have small imaginary parts. The pairs of pants can be glued together to form an abstract closed surface. To show that there is no additional gluings (which means that the abstract surface is immersed) it remained to show that the natural developing map from the Fuchsian representation to the Kleinian representation with the given complex Fenchel-Nielsen coordinates is injective (and thus the representation is quasiFuchsian).

Thus it is an important problem to give sufficient conditions on complex Fenchel-Nielsen coordinates to guarantee that the representation is injective. We consider a more general situation when the deformation of the Fuchsian structure is along a geodesic lamination. Thurston and Bonahon defined the transverse cocycle to the bending locus of a pleated surface as a finitely additive $\mathbf{C}/2\pi i\mathbf{Z}$-valued measure, and they proved that the pleated surface is completely determined by the cocycle. Bonahon proved that the image of the quasiFuchsian pleated surfaces is an open set in the space of all $\mathbf{C}/2\pi i\mathbf{Z}$-valued cocycles that contains real cocycles determining Fuchsian pleated surfaces as a subset. To find a necessary and sufficient condition on the transverse cocycle in order to guarantee that the representations are quasiFuchsian is hopeless.

We find a sufficient condition on $\mathbf{C}/2\pi i\mathbf{Z}$-valued transverse cocycle such that the pleated surface is quasiFuchsian and that there exists an open neighborhood of all Fuchsian representations that satisfy our sufficient condition. This is generalization of the sufficient condition on complex Fenchel-Nielsen coordinates given by Kahn and Markovic. Note that their (Fuchsian) surfaces are special because they have pants decomposition that have large and nearly equal cuffs, while our condition works starting from any (Fuchsian) surface.

Let λ be the maximal geodesic lamination on the hyperbolic surface S(with Fuchsian holonomy) along which the pleating is to be performed. We record the geometric information about S and λ as follows.

Let $\{k_1,\dots,k_n\}$ be finite closed hyperbolic arcs transversely intersecting λ (with endpoints in $S-\lambda$) at angles between $\pi/4$ and $3\pi/4$, and that their lengths are at most $1/20$. Assume that collapsing each arc to a point and each homotopy class of arcs in $\lambda-\cup k_i$ to a single arc we obtained a trivalent train track on S that carries λ. Such collection of arcs $\{k_1,\dots,k_n\}$ is said to be geometric.

Let m_0 and m_* be the maximal and the minimal distance between two arcs in the geometric set $\{k_1,\dots,k_n\}$. Let k^* and k_* be the maximal and the minimal lengths of the arcs.

For a transverse cocycle β, define $\|\beta\|_{max}=\max_{i=1,\dots,n}\{|\beta(k_i)|\}$. Orient each k_i arbitrary and for each component d of $k_i-\lambda$ denote by k_d the subarc of k_i with the initial point the initial point of k_i and the endpoint an arbitrary point in d.
Let $\delta>0$ and define $\|\beta\|_{var_{\delta},k_i}=\max_{1\leq l\leq l_i}|\beta(k_{d_l})|$, where d_l's are component of $k_i-\lambda$ such that the length of $k_i-\cup_l k_{d_l}$ is less than $\delta|k_i|$. Define $\|\beta\|_{var_{\delta}}=\max_i\|\beta\|_{var_{\delta},k_i}$.

Theorem (Š.)

There exist $\epsilon > 0$ and $\delta > 0$ such that for any closed hyperbolic surface S and a maximal geodesic lamination λ on S the following holds. Let $\{k_1, \dots, k_n\}$ be a geometric set of arcs for λ such that

$$k^* < \frac{e^{-2m_0} \tanh \frac{m_*}{2}}{8\pi}. \quad (1)$$

If an $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle β to λ satisfies

$$\|\beta\|_{max} < \epsilon k_* \quad (2)$$

and

$$\|\beta\|_{var_\delta} < \epsilon \quad (3)$$

then the developing map $\tilde{f}_\beta : \mathbb{H}^2 \to \mathbb{H}^3$ continuously extends to an injective map $\tilde{f}_\beta : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$.

Let α be an $\mathbb{R}$-valued transverse cocycle to λ which is induced by the hyperbolic metric on S. For $w \in \mathbb{C}$, define the transverse cocycle α_w by $\alpha_w = (1+w)\alpha \mod (2\pi i\mathbb{Z})$. Let $\tilde{f}_w$ be the corresponding developing map.

Corollary (Š.)

Let α be an $\mathbb{R}$-valued transverse cocycle to a geodesic lamination λ corresponding to a hyperbolic metric on a closed surface S and let $\tilde{f}_w$ be the shear-bend map for α_w. Then there exists $\epsilon > 0$ such that the shear-bend map

$$\tilde{f}_w : \mathbb{H}^2 \to \mathbb{H}^3$$

extends by continuity to a holomorphic motion of $\partial_\infty \mathbb{H}^2$ in $\partial_\infty \mathbb{H}^3$ for the parameter $\{w \in \mathbb{C} : |w| < \epsilon\}$.

Thank you!

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