Linear and Isometry Cocycles over Hyperbolic Systems

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Linear cocycles over hyperbolic systems

$f : \mathcal{M} \to \mathcal{M}$ an Anosov diffeomorphism or a hyperbolic system:

locally maximal hyperbolic set, subshift of finite type, Anosov flow. Also, some accessible partially hyperbolic diffeomorphisms.

$P : \mathcal{E} \to \mathcal{M}$ is a Hölder continuous finite dim'l vector bundle;

$F : \mathcal{E} \to \mathcal{E}$ is a Hölder continuous linear cocycle over f, i.e.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E} \\ P\downarrow & & P\downarrow \\ \mathcal{M} & \xrightarrow{f} & \mathcal{M} \end{array}$$

and $F_x : \mathcal{E}_x \to \mathcal{E}_{fx}$ is a linear isomorphism which depends Hölder continuously on x.

Derivative cocycle: $F = Df$ is a cocycle on $\mathcal{E} = T\mathcal{M}$, more generally: $Df|_{\mathcal{E}'}$, where $\mathcal{E}'$ is a Df-invariant sub-bundle.

Trivial bundle: $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$, $\mathcal{E}_x = \mathbb{R}^d$, $F : \mathcal{M} \to GL(d, \mathbb{R})$.

Cohomology and classification of cocycles

Let $F, \tilde{F} : \mathcal{E} \to \mathcal{E}$ be linear cocycles over f.

Definition

F and $\tilde{F}$ are *cohomologous* if there exists $C_x \in GL(E_x)$ such that

$$\tilde{F}_x = C_{fx} \, F_x \, C_x^{-1}$$

Problems: Classification. Reduction to a smaller group.

Oseledets Multiplicative Ergodic Theorem: There exist numbers $\lambda_1 < \cdots < \lambda_l$, an f-invariant set X with $\mu(X) = 1$, and an F-invariant decomposition of $\mathcal{E}$ for $x \in X$

$$\mathcal{E}_x = E_{\lambda_1}(x) \oplus \cdots \oplus E_{\lambda_l}(x)$$

such that for any nonzero $v \in E_{\lambda_i}(x)$, $\lim_{n \to \pm\infty} n^{-1} \log \| F_x^n v \| = \lambda_i$.

Zimmer's Amenable Reduction

Theorem

f – an ergodic transformation of a measure space (X, μ),
$F : X \to GL(d, \mathbb{R})$ – a measurable cocycle.

Then F is **measurably** *cohomologous to a cocycle with values in an amenable subgroup of $GL(d, \mathbb{R})$.*

There are 2^{d-1} "standard" amenable subgroups of $GL(d, \mathbb{R})$:

$$\begin{bmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_k \end{bmatrix}$$

$A_i = c_i \cdot (d_i \times d_i$ orthogonal matrix$)$
and $d_1 + \dots + d_k = d$.

Periodic data

For a periodic point $p = f^n p$ in $\mathcal{M}$, consider the return map

$$F_p^n = F_{f^{n-1}p} \circ \cdots \circ F_{fp} \circ F_p : \quad \mathcal{E}_p \to \mathcal{E}_p$$

Question: What can be said about F based on its **periodic data** $\{F_p^n\}$?

Question: What assumptions on the periodic data ensure that is F conformal (isometric)?

In particular, suppose that whenever $f^n p = p$,

- F_p^n is an **isometry** with respect to an inner product on $\mathcal{E}_p$;
- F_p^n is **conformal** with respect to an inner product on $\mathcal{E}_p$;
- The eigenvalues of F_p^n are of the same modulus, equivalently, F_p^n has only one Lyapunov exponent.

Conformal and isometric cocycles

Let F be a Hölder continuous linear cocycle over Anosov f.

Theorem (B.K., V. Sadovskaya.)

Let $d = 2$. If whenever $p = f^n p$, F_p^n is conformal (isometric) with respect to an inner product on $\mathcal{E}_p$,

then F is conformal (isometric) with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.

If $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$, F is Hölder cohomologous to a cocycle with values in the conformal (orthogonal) subgroup of $GL(d, \mathbb{R})$.

For $d \geq 3$ the Theorem holds if the conformal structures on $\mathcal{E}_p$ are uniformly bounded.

Conformal structure on $\mathbb{R}^d$ is a class of proportional inner products.
Conformal structures $\approx$ real symmetric positive definite $d \times d$ matrices with determinant 1 $\approx$ $SL(d, \mathbb{R})/SO(d, \mathbb{R})$.

Isometric at periodic points, but not conformal, in dim 3

Example (dim $\mathcal{E}_x \geq 3$)

There exists $F : \mathcal{E} \to \mathcal{E}$ such that whenever $f^n p = p$
F_p^n is **isometric** with respect to an inner product on $\mathcal{E}_p$, but
F is **not conformal** with respect to any continuous metric on $\mathcal{E}$.

Let $\mathcal{E} = \mathcal{M} \times \mathbb{R}^3$, $\qquad F_x = \begin{bmatrix} \cos\alpha(x) & -\sin\alpha(x) & \epsilon \\ \sin\alpha(x) & \cos\alpha(x) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Let S be a closed f-invariant set in $\mathcal{M}$ without periodic points;

$\alpha : \mathcal{M} \to \mathbb{R}$, $\quad \alpha(x) = 0$ for $x \in S$ and $0 < \alpha(x) \leq \epsilon$ for $x \notin S$

For $x \in S$, $\quad F_x^n = \begin{bmatrix} 1 & 0 & \mathbf{n}\epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

F is not conformal w.r. to any continuous Riemannian metric.

At $p = f^n p$, $\quad F_p^n$ is diagonalizable with eigenvalues of modulus 1.

Outline of the proof - general case

Quasiconformal distortion

$$K_F(x,n) = \|F_x^n\| \cdot \|(F_x^n)^{-1}\| = \frac{\max\{\, \| F_x^n(v) \| : v \in \mathcal{E}_x,\ \|v\|=1 \,\}}{\min\{\, \| F_x^n(v) \| : v \in \mathcal{E}_x,\ \|v\|=1 \,\}}$$

Theorem

If there is C_{per} such that $(*)$ $K_F(p,n) \le C_{per}$ whenever $f^n p = p$, then F is conformal with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.

$(*) \implies F$ has only one Lyapunov exponent at each $p = f^n p$
$\implies (**)$ $K_F(x,n) \le C_\epsilon e^{\epsilon |n|}$ for all $x \in \mathcal{M}$ and $n \in \mathbb{Z}$

$(*), (**) \implies K_F(x,n)$ is uniformly bounded for all x and n,
i.e. F is uniformly quasiconformal

$\implies F$ preserves a measurable bounded conformal structure on $\mathcal{E}$

$\implies F$ preserves a Hölder continuous conformal structure on $\mathcal{E}$

Lyapunov exponents

The largest and smallest Lyapunov exponents of F w.r.t. μ:

$$\lambda_+(F,\mu) = \lim_{n\to\infty} \frac{1}{n} \log \|F_x^n\|, \qquad \text{for } \mu \text{ a.e. } x \in \mathcal{M},$$

$$\lambda_-(F,\mu) = \lim_{n\to\infty} \frac{1}{n} \log \|(F_x^n)^{-1}\|^{-1} \qquad \text{for } \mu \text{ a.e. } x \in \mathcal{M}.$$

Theorem

Suppose that there exists $\gamma \geq 0$ such that $\lambda_+(F,p) - \lambda_-(F,p) \leq \gamma$ for every f-periodic point p. Then $\lambda_+(F,\mu) - \lambda_-(F,\mu) \leq \gamma$ for any ergodic f-invariant measure μ.

Moreover, for any $\epsilon > 0$ there exists C_ϵ such that

$$K_F(x,n) \leq C_\epsilon e^{(\gamma+\epsilon)|n|} \quad \textit{for all } x \in \mathcal{M} \textit{ and } n \in \mathbb{Z}.$$

Fiber bunched cocycles

$f : \mathcal{M} \to \mathcal{M}$ transitive Anosov: $\quad T\mathcal{M} = E^s \oplus E^u$, where

$$\|Df(\mathbf{v}^s)\| < \nu(x) < 1 < \gamma(x) < \|Df(\mathbf{v}^u)\|$$

Definition

A β-Hölder cocycle F over f is **fiber bunched** if
$\|F_x\| \cdot \|F_x^{-1}\| \cdot \nu(x)^\beta < 1$ and $\|F_x\| \cdot \|F_x^{-1}\| \cdot (1/\gamma(x))^\beta < 1$.

Fiber-bunching ensures convergence of $(F_y^n)^{-1} F_x^n$ for $y \in W^s(x)$, which gives linear maps $H_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$.

Theorem (B.K., V. Sadovskaya)

If F be a fiber bunched and μ is an ergodic measure with full support and local product structure then any μ-measurable F-invariant conformal structure on $\mathcal{E}$ is Hölder continuous.

If $\lambda_+(F,\mu) = \lambda_-(F,\mu)$, then any μ-measurable F-invariant sub-bundle of $\mathcal{E}$ is Hölder continuous.

Continuous Amenable Reduction

Theorem (B.K., V. Sadovskaya)

Suppose that F is fiber bunched and $\lambda_+(F,\mu) = \lambda_-(F,\mu)$.
Then there exists a finite cover $\tilde{F} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ of F, $N \in \mathbb{N}$, and a flag of continuous $\tilde{F}^N$-invariant sub-bundles

$$\{0\} = \tilde{\mathcal{E}}^0 \subset \tilde{\mathcal{E}}^1 \subset \ldots \subset \tilde{\mathcal{E}}^{k-1} \subset \tilde{\mathcal{E}}^k = \tilde{\mathcal{E}}$$

such that each factor-cocycle induced by $\tilde{F}^N$ on $\tilde{\mathcal{E}}^i/\tilde{\mathcal{E}}^{i-1}$, preserves a continuous conformal structure, i.e. is conformal w.r.t. a continuous Riemannian metric on $\tilde{\mathcal{E}}^i/\tilde{\mathcal{E}}^{i-1}$.

For any finite cover $p : \tilde{\mathcal{M}} \to \mathcal{M}$, the pullback of $\mathcal{E}$ is a vector bundle $\tilde{\mathcal{E}}$ over $\tilde{\mathcal{M}}$. If $\tilde{f} : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ is a cover of f, then F lifts uniquely to a linear cocycle $\tilde{F} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ over $\tilde{f}$.

Amenable subgroups of $GL(d, \mathbb{R})$ [Moore, 1979]

There are 2^{d-1} "standard " amenable subgroups of $GL(d, \mathbb{R})$ corresponding to distinct compositions of d, $d = d_1 + \cdots + d_k$. Each subgroup consists of all matrices

$$\begin{bmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_k \end{bmatrix}$$

where each A_i is a scalar multiple of a $d_i \times d_i$ orthogonal matrix and $d_1 + \cdots + d_k = d$.

Any amenable subgroup of $GL(d, \mathbb{R})$ has a finite index subgroup contained in a conjugate of "standard" amenable subgroup.

Am amenable group which does not lie in any such conjugate:

The finite extension of the diagonal subgroup that contains all permutations of the coordinate axes.

Continuous Amenable Reduction – special case

Let $f : \mathcal{M} \to \mathcal{M}$ be a transitive Anosov diffeomorphism.

$F : \mathcal{E} \to \mathcal{E}$ is a Hölder linear cocycle over f.

Theorem (B.K., V. Sadovskaya)

Suppose that for every f-periodic point the invariant measure μ_p on its orbit satisfies $\lambda_+(F, \mu_p) = \lambda_-(F, \mu_p)$, and hence $\lambda_+(F, \nu) = \lambda_-(F, \nu)$ for every ergodic f-invariant measure ν.

Then there exist a flag of Hölder F-invariant sub-bundles

$$\{0\} = \mathcal{E}^0 \subset \mathcal{E}^1 \subset ... \subset \mathcal{E}^{j-1} \subset \mathcal{E}^j = \mathcal{E}$$

and Hölder Riemannian metrics on the factor bundles $\mathcal{E}^i / \mathcal{E}^{i-1}$, so that for some positive Hölder function $\phi : \mathcal{M} \to \mathbb{R}$ the factor-cocycles induced by ϕF on $\mathcal{E}^i / \mathcal{E}^{i-1}$ are isometries.

If the flag is trivial, F is conformal on $\mathcal{E}$.

Corollary: Polynomial growth

Corollary

Let f be a transitive Anosov diffeomorphism. Suppose that for every f-periodic point the invariant measure μ_p on its orbit satisfies $\lambda_+(F,\mu_p) = \lambda_-(F,\mu_p)$.

Then the quasiconformal distortion of F satisfies

$$\mathbf{K_F(x,n)} \overset{\mathbf{def}}{=} \|\mathbf{F_x^n}\| \cdot \|(\mathbf{F_x^n})^{-1}\| \leq \mathbf{Cn^{2m}} \quad \textit{for all } x \in \mathcal{M} \textit{ and } n \in \mathbb{N}.$$

Moreover, if $\lambda_+(F,\mu_p) = \lambda_-(F,\mu_p) = 0$ for every μ_p,

then $\|\mathbf{F_x^n}\| \leq \mathbf{Cn^m}$ for all $x \in \mathcal{M}$ and $n \in \mathbb{N}$.

$\mathbf{m} = j - 1$ is the number of non-trivial sub-bundles in the flag $\leq d - 1$.

Example – no invariant sub-bundles and conf. structures

There exists an analytic cocycle F on $\mathcal{E} = \mathbb{T}^2 \times \mathbb{R}^2$ over an Anosov automorphism f of $\mathbb{T}^2$ so that F is fiber bunched and has only one Lyapunov exponent with respect to the Haar measure μ, but has no invariant μ-measurable sub-bundles and conformal structures.

$\bar{\mathbb{T}}^2 = \mathbb{R}^2/(4\mathbb{Z} \times \mathbb{Z})$ is a 4-cover of $\mathbb{T}^2$,

$$\bar{A}(x) = \begin{bmatrix} a(x) & 0 \\ 0 & b(x) \end{bmatrix}, \qquad \text{where} \quad \begin{aligned} a(x) &= 1 + \epsilon \cos(\pi x_1), \\ b(x) &= 1 - \epsilon \cos(\pi x_1) \end{aligned}$$

$\bar{C}(x) = R\left(\frac{\pi}{2} x_1\right)$, the rotation by $\frac{\pi}{2} x_1$.

A hyperbolic matrix in $SL(2, \mathbb{Z})$ $\equiv \mathrm{Id} \bmod 4$, e.g. $\begin{bmatrix} 41 & 32 \\ 32 & 25 \end{bmatrix}$,
f and $\bar{f}$ – the induced automorphisms of $\mathbb{T}^2$ and $\bar{\mathbb{T}}^2$.

$$\bar{F}(x) = \bar{C}(\bar{f} x)\, \bar{A}(x)\, \bar{C}(x)^{-1}$$

$\bar{F}$ is 1-periodic in x_1 and x_2, hence projects to an analytic F on $\mathbb{T}^2$.