Weyl's Laplacian eigenvalue asymptotics for the measurable Riemannian structure on the Sierpiński gasket

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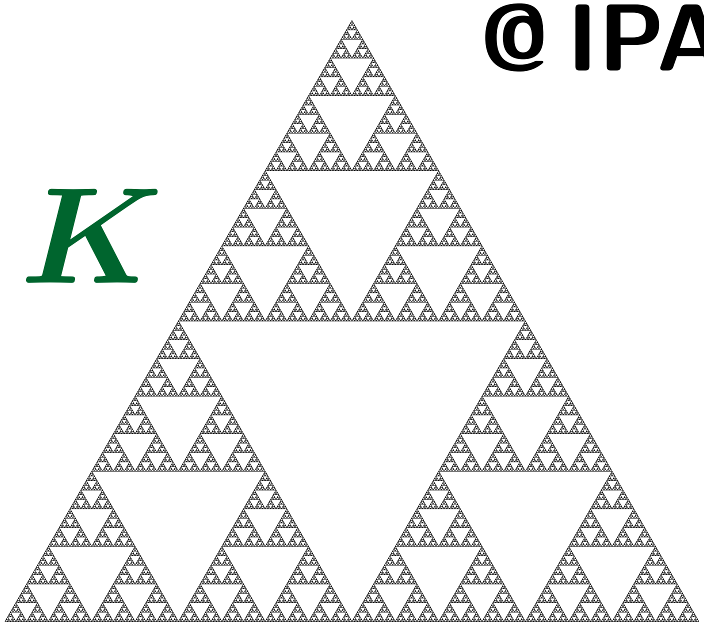

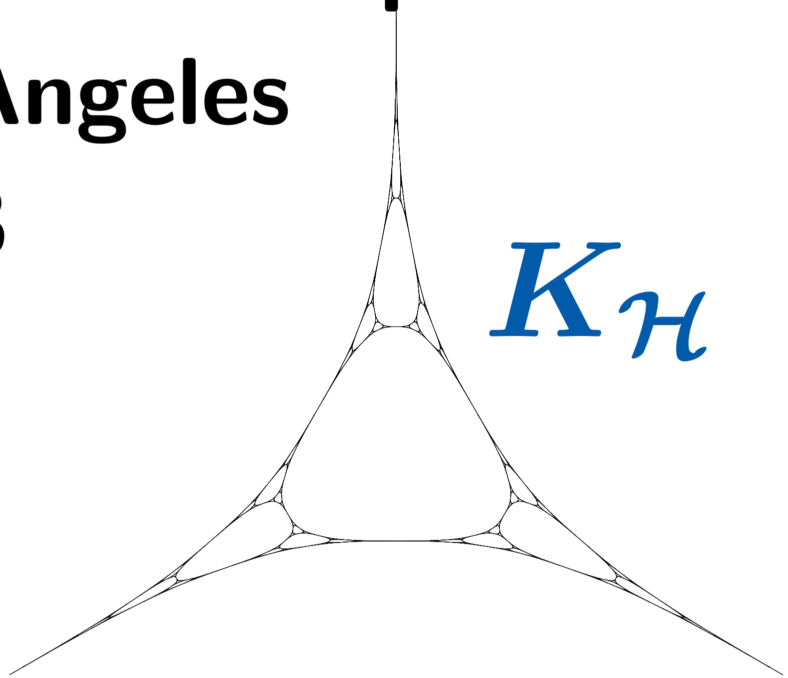

$K_{\mathcal{H}}$

0 Introduction

Harmonic Sierpiński gasket $K_{\mathcal{H}}$

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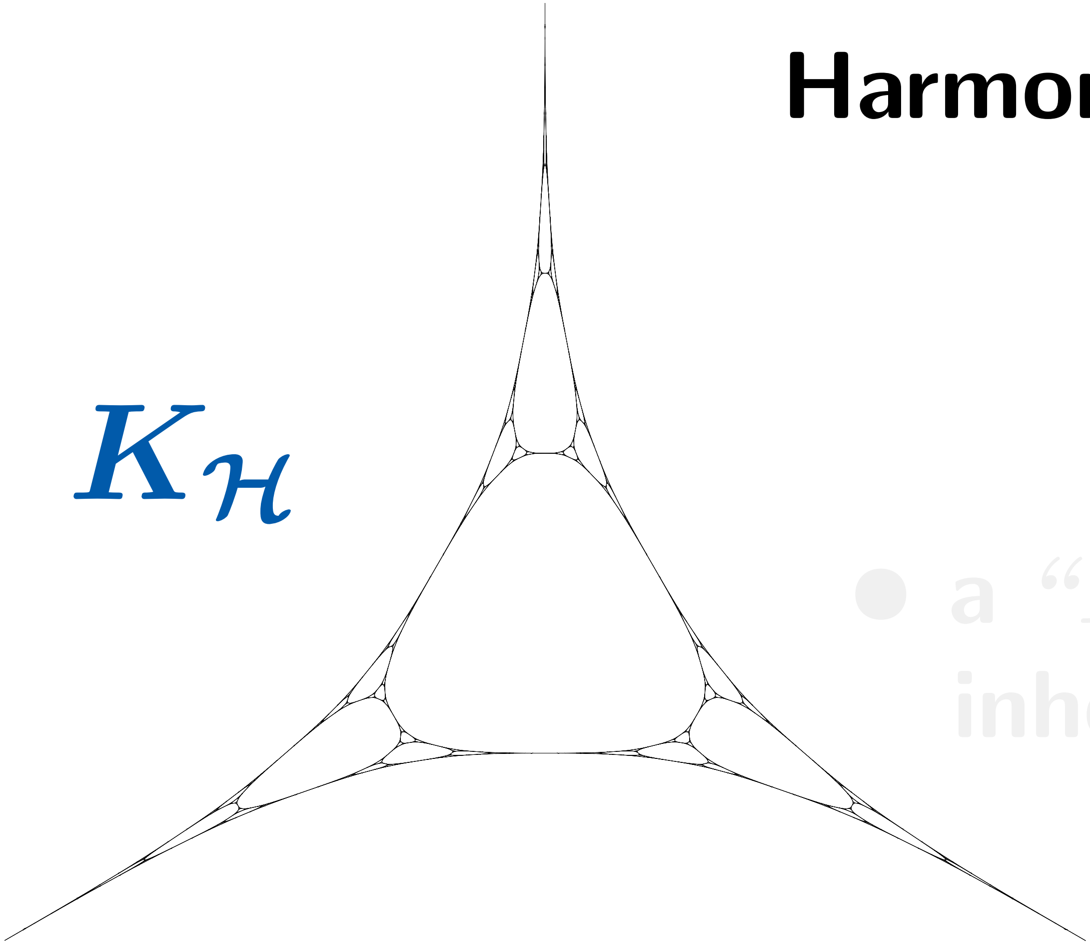

- a *"Riemannian structure"* inherited from $\mathbb{R}^2$

- (Kigami '08) GAUSSIAN bound for $p_t^{\mathcal{H}}(x,y)$
- (K.) Varadhan's asymp.: $4t \log p_t^{\mathcal{H}}(x,y) \xrightarrow{t\downarrow 0} -\rho_{\mathcal{H}}(x,y)^2$

Q. Asymp. of Laplacian eigenvalues $\{\lambda_n^{\mathcal{H}}\}_{n\in\mathbb{N}}$?

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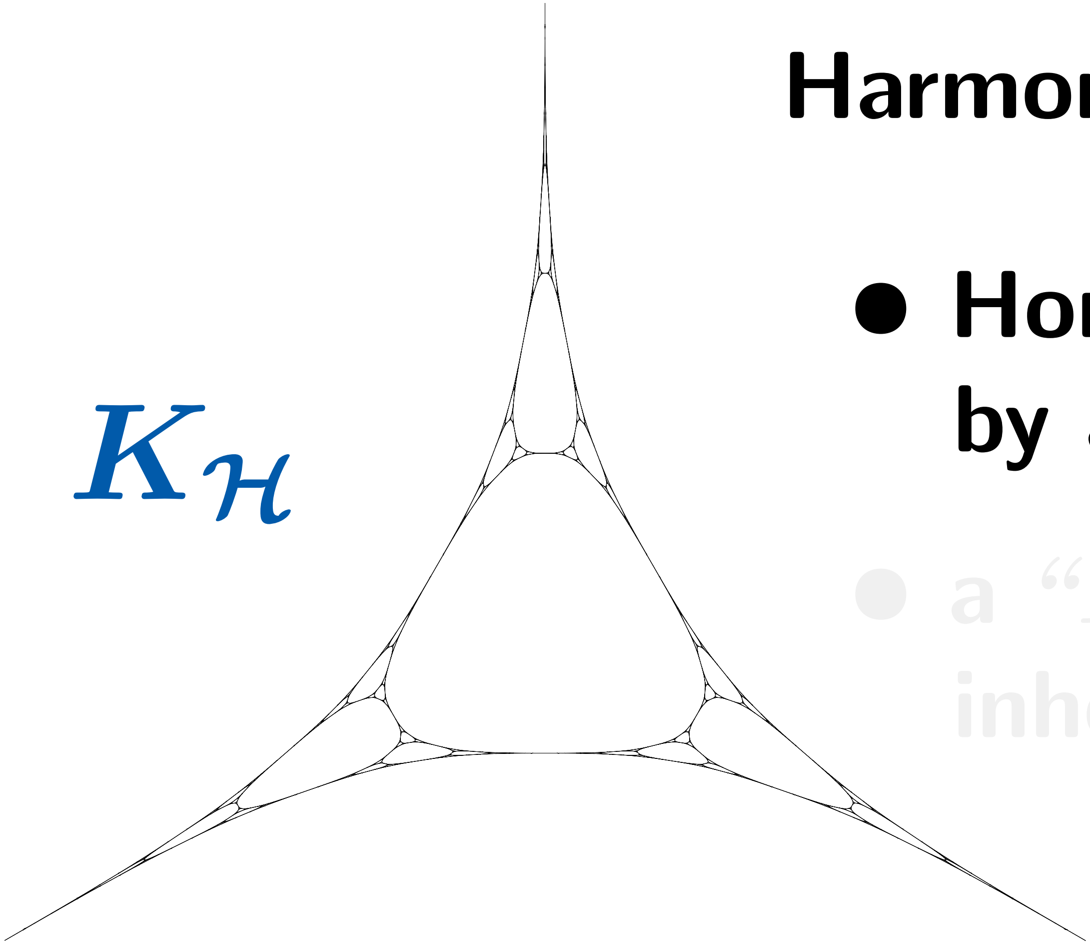

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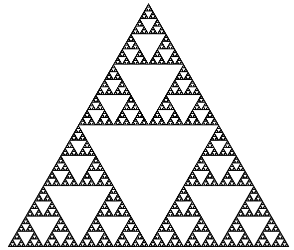
- Homeomor. to $K :=$  by a harmonic map
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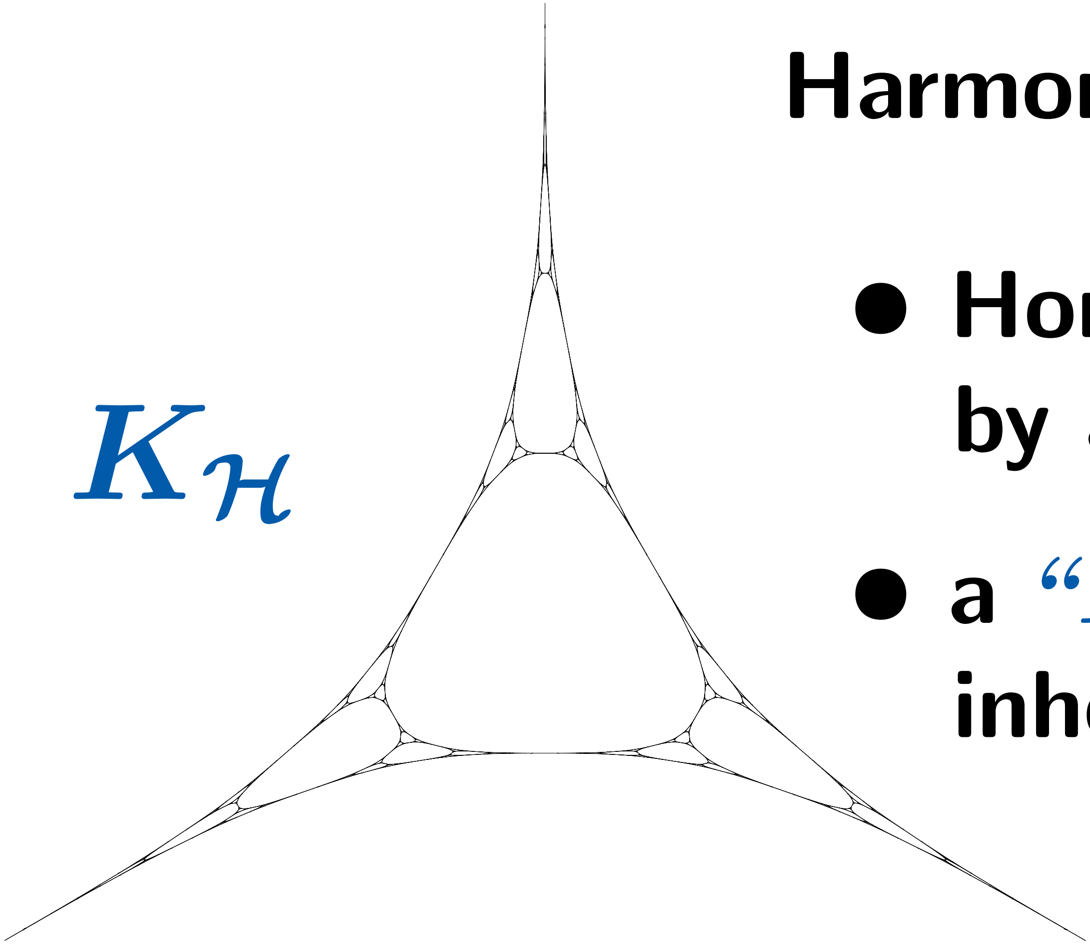

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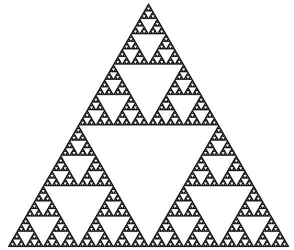

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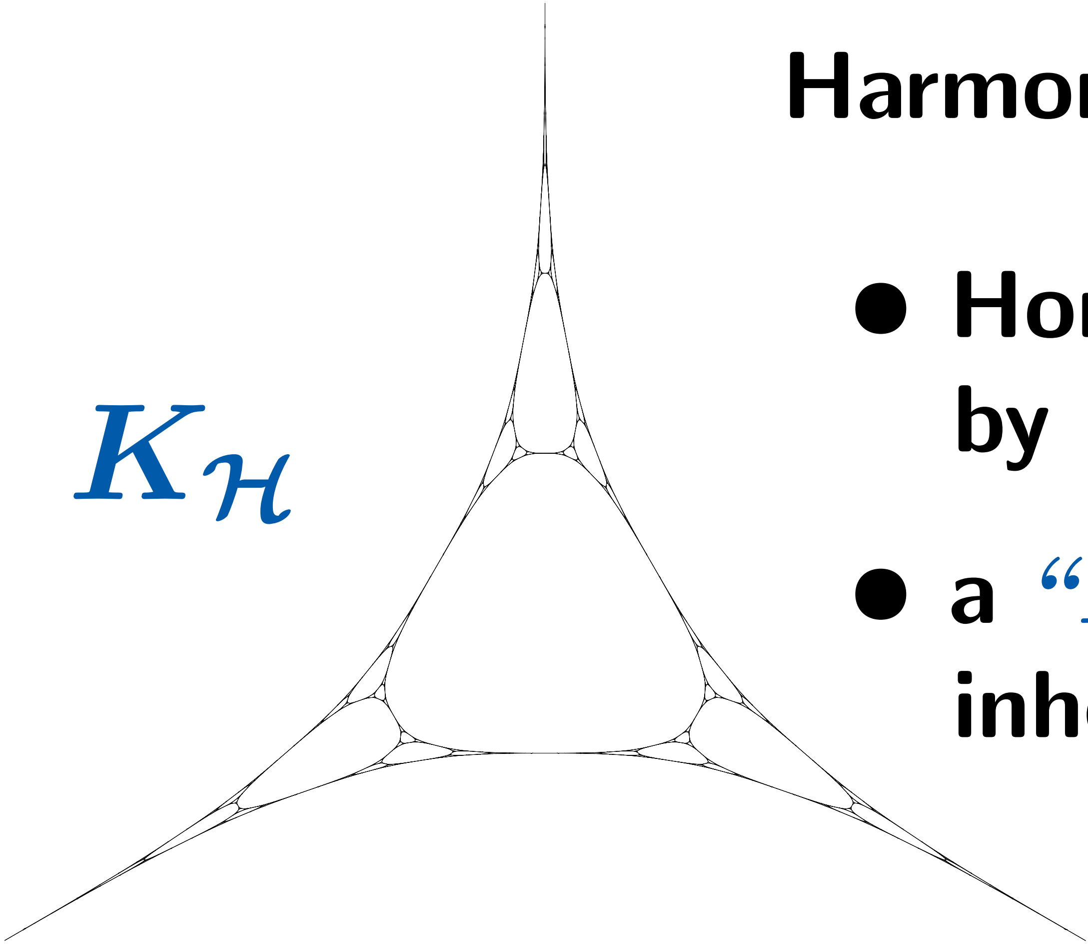

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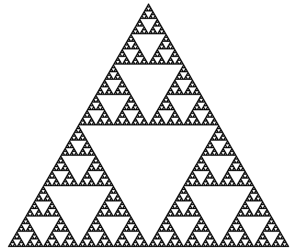

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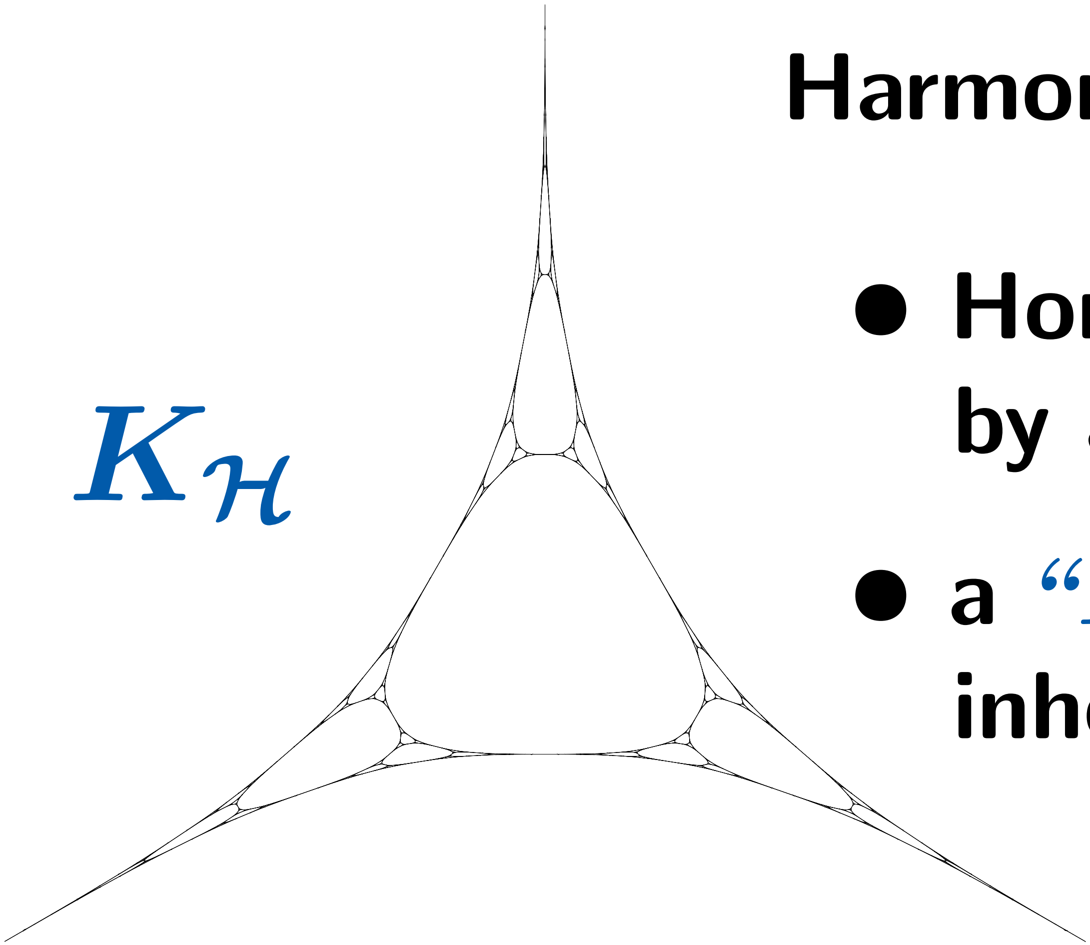

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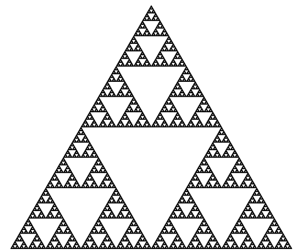

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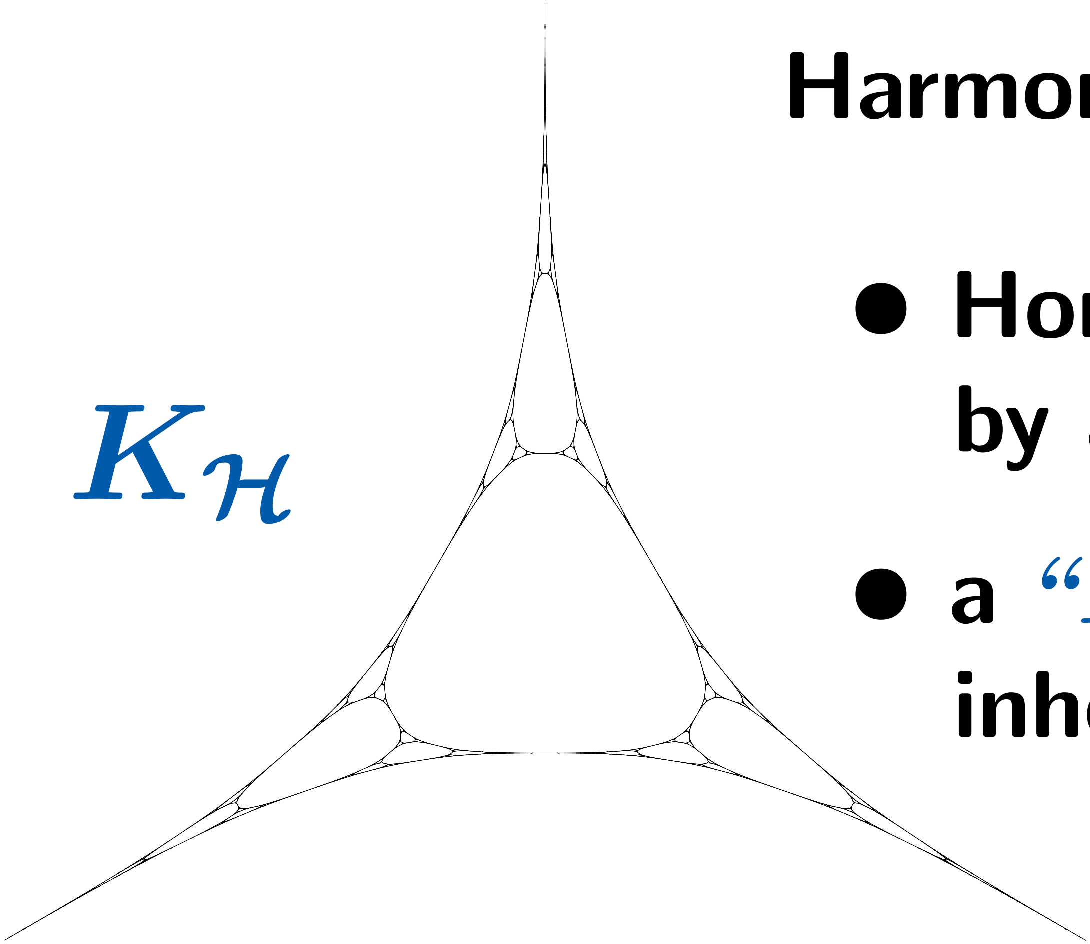

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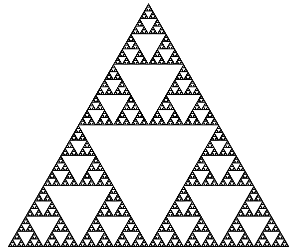
- **Homeomor. to** $K :=$  **by a harmonic map**
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cf. Weyl's Laplacian eigenvalue asymp. for $U \subset \mathbb{R}^d$

- $\{\lambda_n^U\}_{n\in\mathbb{N}}$: the eigenvalues of $-\Delta_U^{\mathrm{Dirichlet}}$
- $\mathcal{N}_U(\lambda) := \#\{n \in \mathbb{N} \mid \lambda_n^U \leq \lambda\}$,

$$\mathcal{Z}_U(t) := \sum_{n\in\mathbb{N}} e^{-t\lambda_n^U} = \int_U p_t^U(x,x)\,dx$$

Thm (Weyl 1912). $\mathcal{N}_U(\lambda) \overset{\lambda\to\infty}{\sim} c_d \mathrm{Vol}_d(U)\lambda^{d/2}$.

Equivalently, $\mathcal{Z}_U(t) \overset{t\downarrow 0}{\sim} (4\pi)^{-d/2}\mathrm{Vol}_d(U)t^{-d/2}$.

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$p_t^U(x,x) \overset{t\downarrow 0}{\sim} (4\pi)^{-d/2}t^{-d/2}$ + "$\leq$" (some uniformity)

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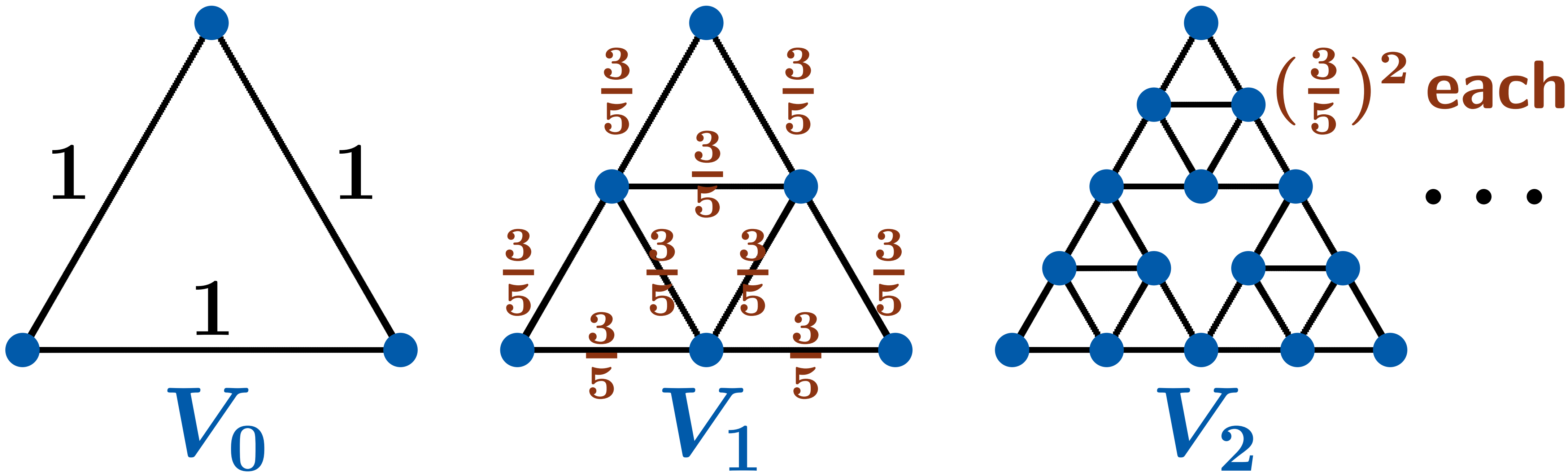
1 Measurable Riemannian structure on the S.G.

▷ $(\mathcal{E}, \mathcal{F})$: Standard Dirich. form on K ($\mathcal{F} \subset C(K)$)

"$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle dx$"



$$(\mathcal{E}_m, \mathbb{R}^{V_m}) \xrightarrow{m \to \infty} (\mathcal{E}, \mathcal{F})$$

1 Measurable Riemannian structure on the S.G.

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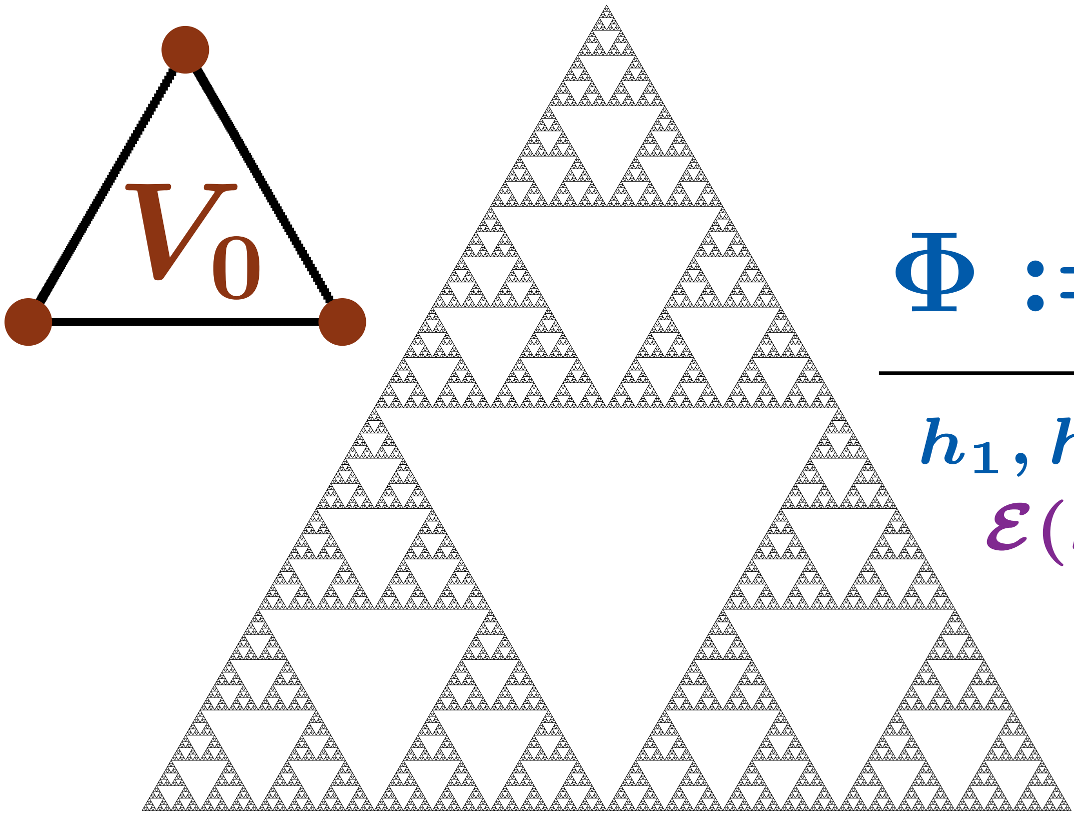
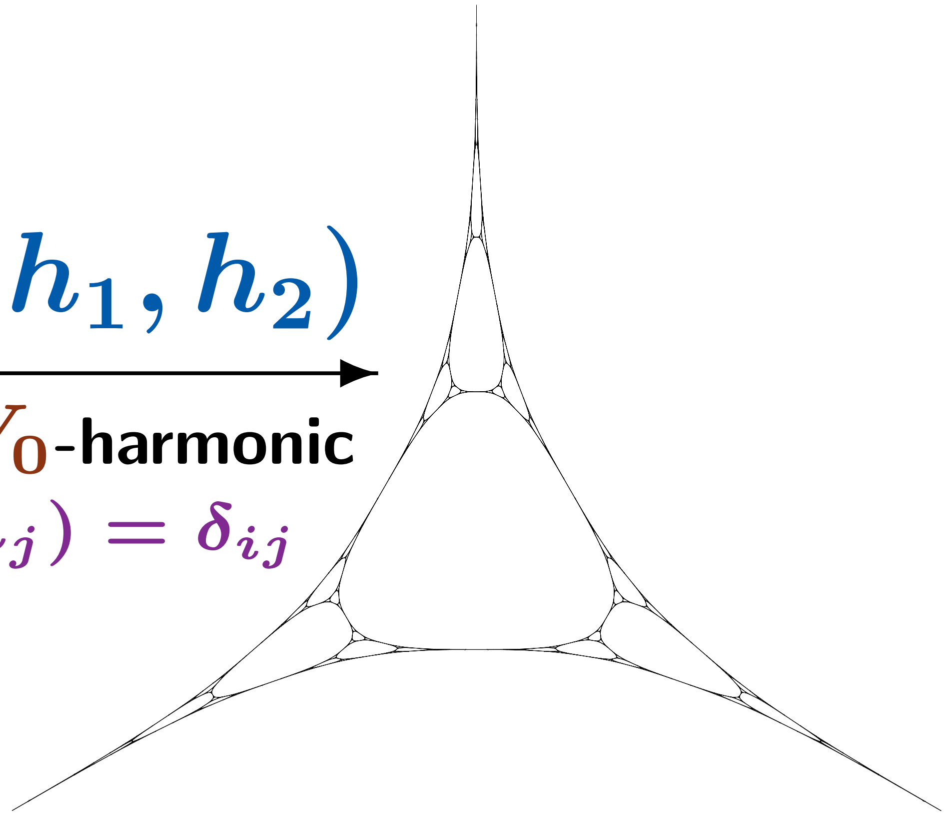
Kigami '93: Harmonic embedding $\Phi : K \to K_{\mathcal{H}}$



$$\Phi := (h_1, h_2)$$

h_1, h_2 : V_0-harmonic

$$\mathcal{E}(h_i, h_j) = \delta_{ij}$$



Sierpinski gasket K Harmonic Sier. gasket $K_{\mathcal{H}}$

Energy measures $\mu_{\langle u\rangle}$, $u \in \mathcal{F}$

$$\int_K f\,d\mu_{\langle u\rangle} = \mathcal{E}(fu,u) - \tfrac{1}{2}\mathcal{E}(f,u^2), \quad {}^{\forall} f \in \mathcal{F}.$$

"$d\mu_{\langle u\rangle} = |\nabla u|^2 dx$"

▷ $\mu := \mu_{\langle h_1\rangle} + \mu_{\langle h_2\rangle}$: Kusuoka measure
(Energy of the "embedding" Φ)

Thm (Kusuoka '89, Kigami '93).
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● $\mu \perp$ self-similar (Bernoulli) meas. (Hino-Nakahara '06)

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for $u = v \circ \Phi$, $v \in C^1(\mathbb{R}^2)$, where $\nabla u := (\nabla v) \circ \Phi$.

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- μ : "Riemannian volume measure"
- $Z\nabla u$: "gradient vector field" of $u \in C^1(K_{\mathcal{H}})$
- $(\mathcal{E}, \mathcal{F})$: assocaited Dirichlet form ("H^1-Sobolev sp.")

▷ Δ_μ: Laplacian for $(K, \mu, \mathcal{E}, \mathcal{F})$, that is,

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▷ $p_t^{\mathcal{H}}(x,y)$: fundamental solution for $\frac{\partial u}{\partial t} = \Delta_\mu u$ (heat kernel)

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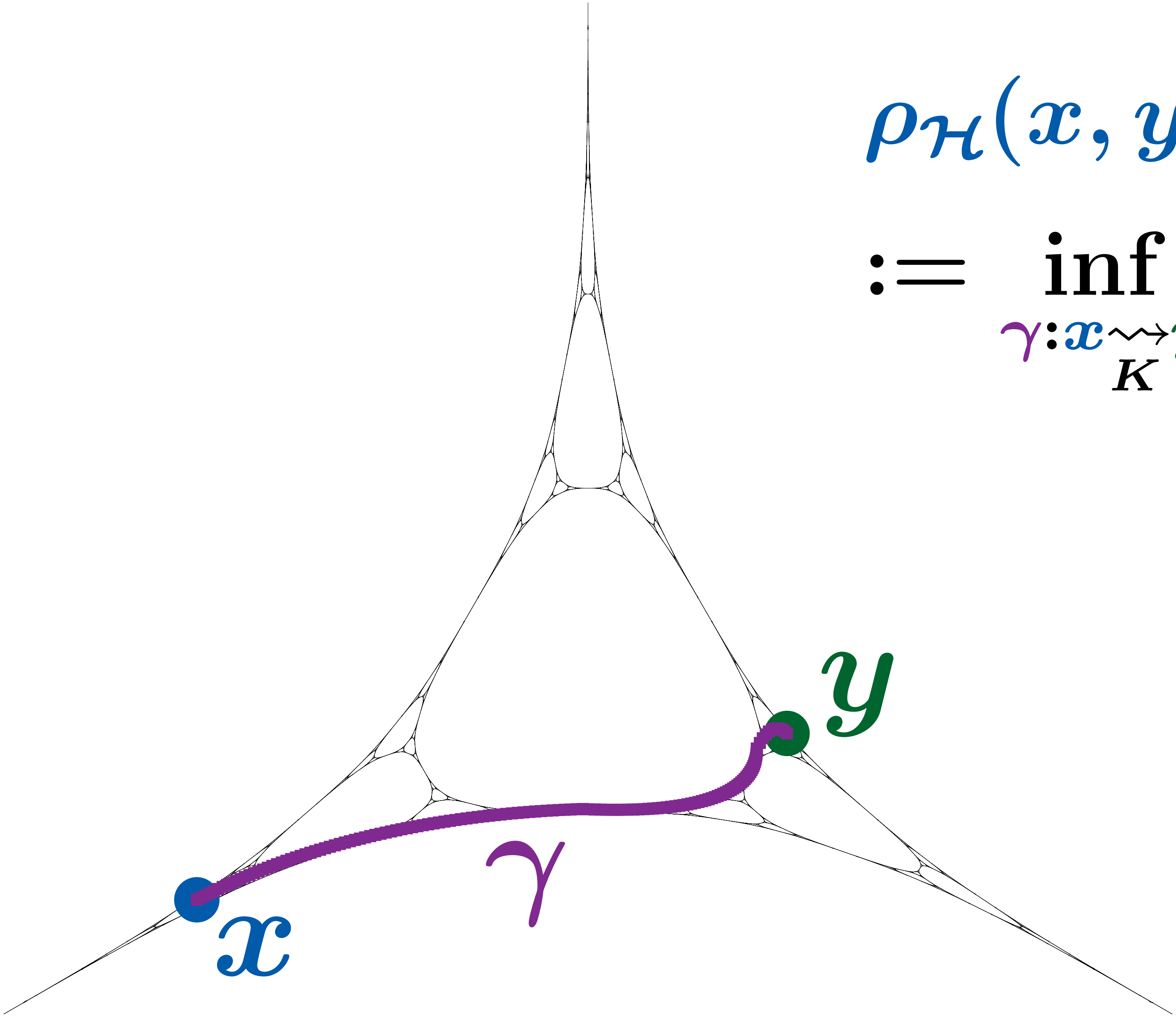
$\rho_{\mathcal{H}}(x, y)$: Geodesic metric in $K_{\mathcal{H}}$

(Kigami '08)

$$\rho_{\mathcal{H}}(x, y) := \inf_{\gamma : x \underset{K}{\rightsquigarrow} y} \ell_{\mathbb{R}^2}(\Phi \circ \gamma)$$



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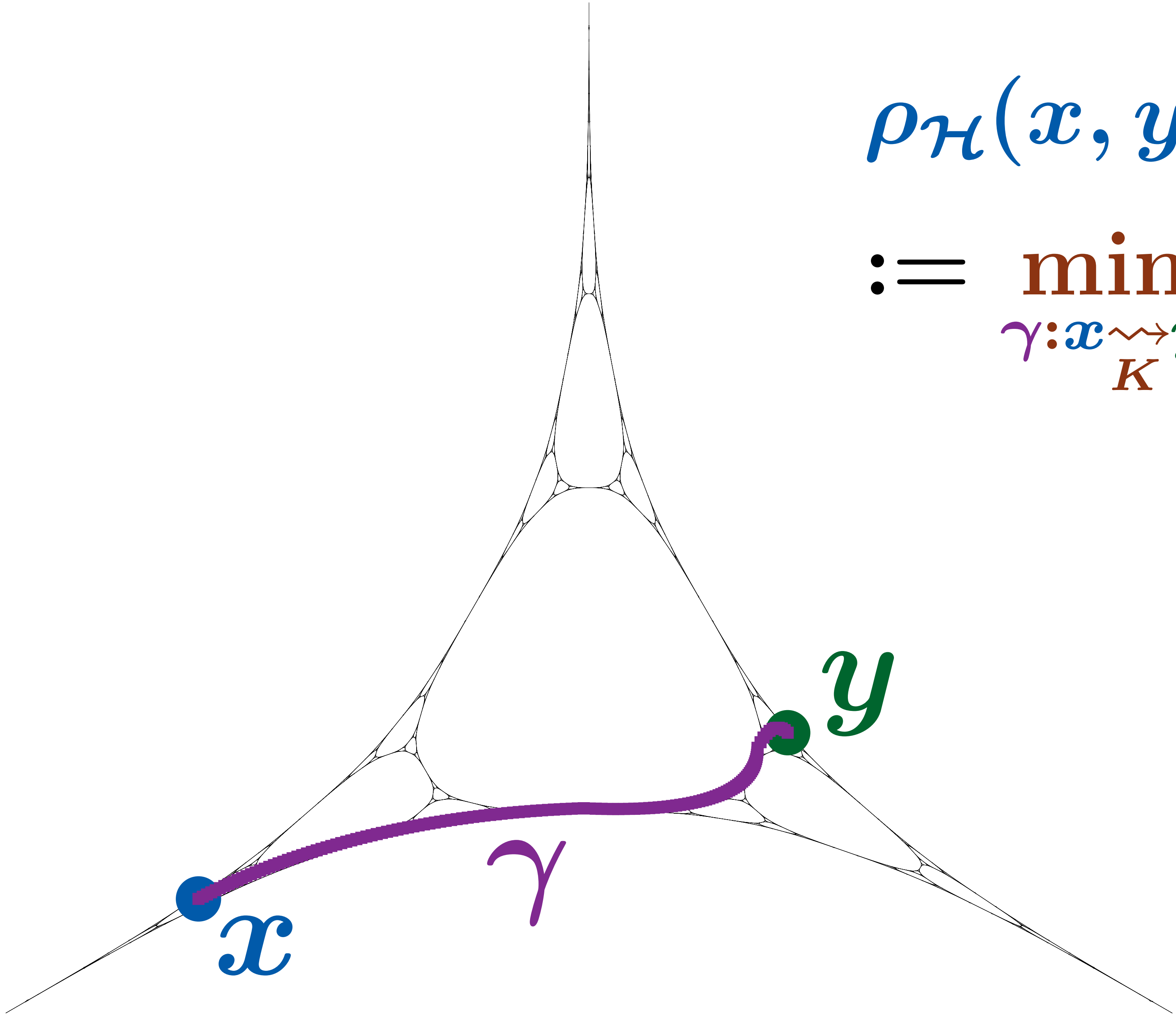

Gaussian heat kernel bound and Varadhan's asymp.

Thm (Kigami '08). For $t > 0$, $x, y \in K$,

$$p_t^{\mathcal{H}}(x,y) \asymp \frac{c_1}{\mu\bigl(B_{\sqrt{t}}(x,\rho_{\mathcal{H}})\bigr)} \exp\left(-\frac{\rho_{\mathcal{H}}(x,y)^2}{c_2 t}\right).$$

Thm (K. '12). For any $x, y \in K$,

$$\rho_{\mathcal{H}}(x,y) = \sup\{u(x) - u(y) \mid u \in \mathcal{F},\ \mu_{\langle u \rangle} \leq \mu\}.$$

Cor (Thm + Ramírez '01). For any $x, y \in K$,

(Vrd) $$\lim_{t \downarrow 0} 4t \log p_t^{\mathcal{H}}(x,y) = -\rho_{\mathcal{H}}(x,y)^2.$$

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Cor (Thm + Ramírez '01). For any $x, y \in K$,

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Gaussian heat kernel bound and Varadhan's asymp.

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2 Main Thm: Weyl's Laplacian eigenvalue asymp.

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Prop (K.). $\mathcal{H}^d_{\rho_{\mathcal{H}}}(B_r(x, \rho_{\mathcal{H}})) \asymp r^d$, $r \in (0, 1], x \in K$.

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Proof. To follow Kigami-Lapidus' method, we use Kesten's renewal thm for Markov chains [Ann. Prob. '74].

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Q. How are $\mu = \mu_{\langle h_1 \rangle} + \mu_{\langle h_2 \rangle}$ and $\mathcal{H}^d_{\rho_{\mathcal{H}}}$ related?

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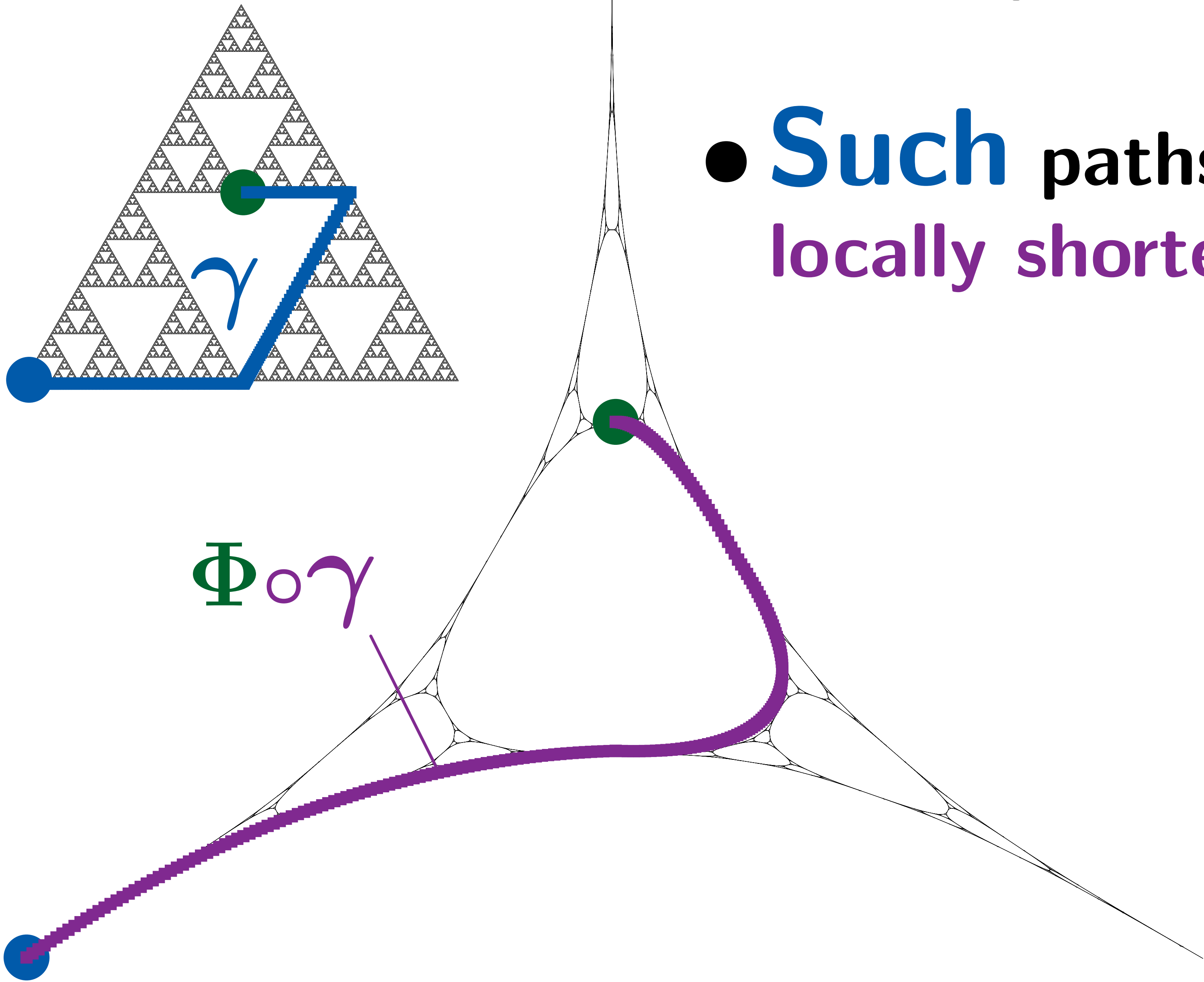
$$\because \dim_{\mathrm{H},\rho_{\mathcal{H}}}\left(\left\{ x \in K \,\middle|\, \lim_{r \downarrow 0} \frac{\log \mu(B_r(x,\rho_{\mathcal{H}}))}{\log r} = d^{\mathrm{loc}} \right\}\right) = d^{\mathrm{loc}}.$$

$\uparrow$ μ-full by Prop., $\mathcal{H}^d_{\rho_{\mathcal{H}}}$-zero by $d^{\mathrm{loc}} < d$

3 Connections to theories on metric meas. spaces

Characterization of shortest paths in $K_{\mathcal{H}}$ (K. '13)



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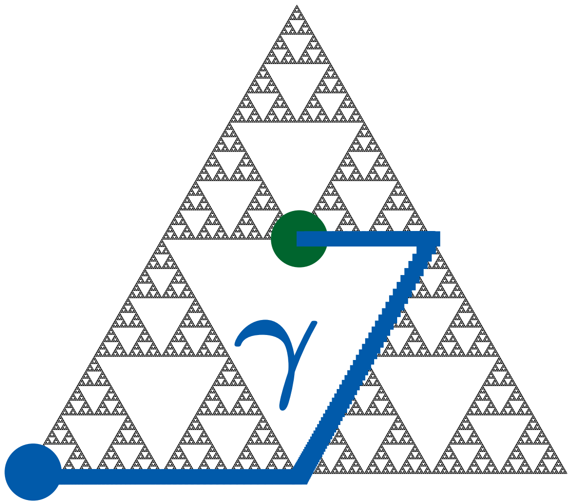

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Cor (K.). For $(K, \rho_{\mathcal{H}}, \mu)$, $k \in \mathbb{R}$, $N \in [1, \infty]$,

- CD(k, N) (Sturm 06', Lott-Villani '07, '09) fails.
- MCP(k, N) (Sturm 06', Ohta '07) fails, $N < \infty$.

▷ CD(k, N), MCP(k, N): metric-measure paraphrase of

$$\mathrm{Ric}_g \geq kg \quad \text{and} \quad \dim M \leq N.$$

Rademacher's thm for *"Riemannian structure"*

Thm (Koskela-Zhou '12, *cf.* Hino '10). Let $u \in \mathcal{F}$. Then for μ-a.e. $x \in K$, $^{\exists 1}\widetilde{\nabla} u(x) \in T_x K$ s.t.

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Moreover $d\mu_{\langle u \rangle} = |\widetilde{\nabla} u|^2 d\mu$, $\mathcal{E}(u,u) = \int_K |\widetilde{\nabla} u|^2 d\mu$.

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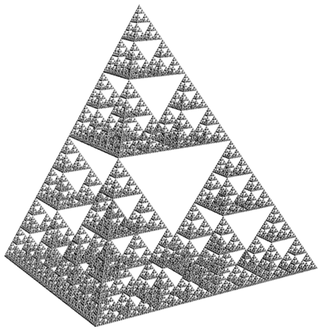
-  should be fine but not small difficulties (note that $\dim T_x K = 1$ (Kusuoka '89))

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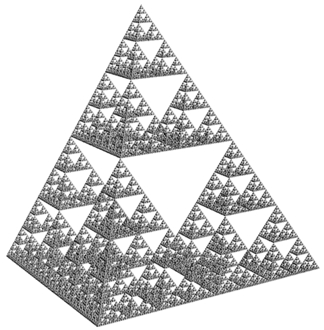
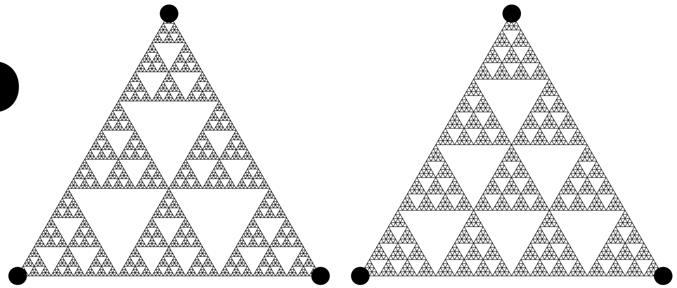
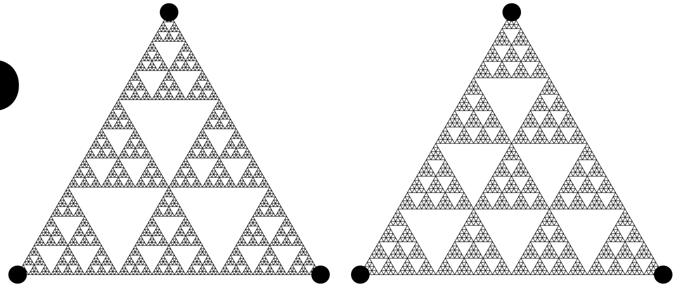
-  should be fine but not small difficulties (note that $\dim T_x K = 1$ (Kusuoka '89))

-  Gaussian HK bound NOT valid, Difficulties also for Weyl's asymp.

-  **Gaussian HK bound NOT valid, Difficulties also for Weyl's asymp.**

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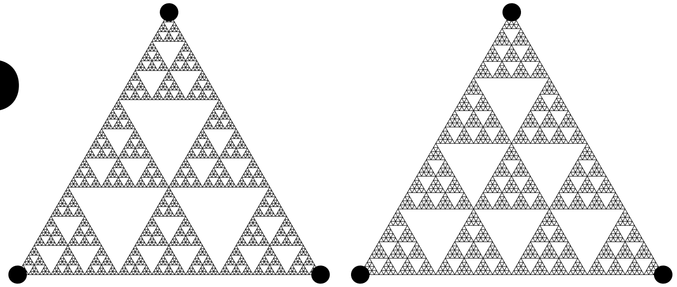
-  **Gaussian HK bound NOT valid, Difficulties also for Weyl's asymp.**

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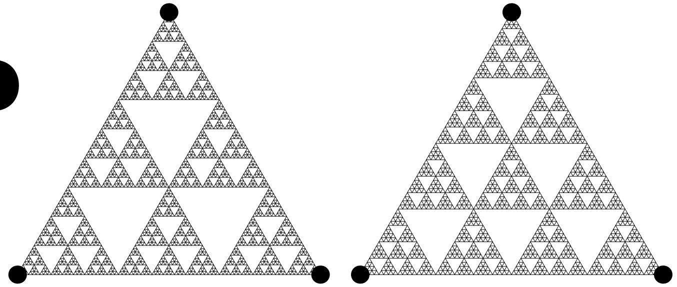
-  **Gaussian HK bound NOT valid, Difficulties also for Weyl's asymp.**

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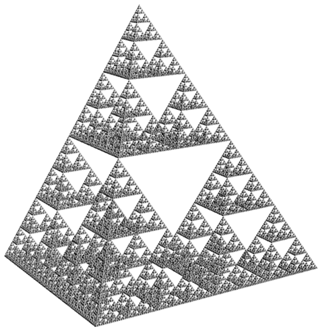
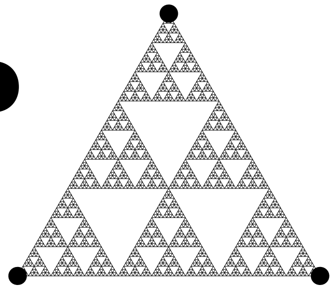
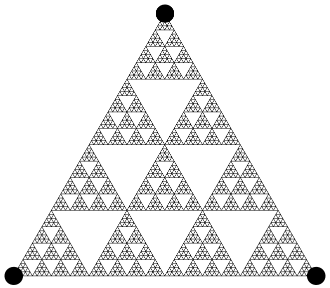
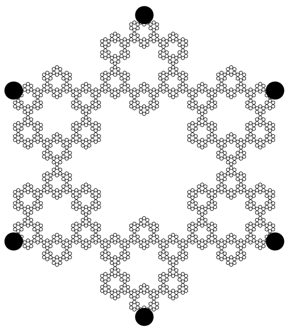
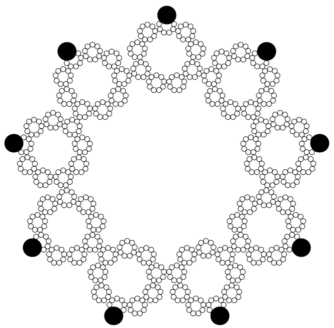
-  should be fine but not small difficulties (note that $\dim T_x K = 1$ (Kusuoka '89))
-   Gaussian HK bound NOT valid, Difficulties also for Weyl's asymp.
-   Weyl's asymp. should be possible by special reasons (ongoing work) ($h_i|_{K_w}$ can be constant!)

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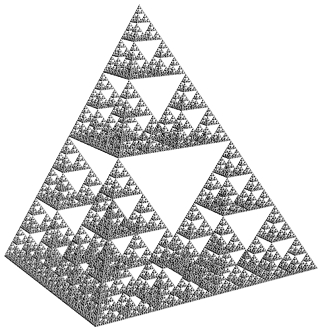
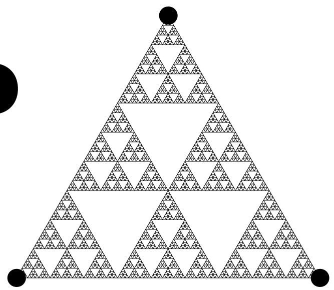
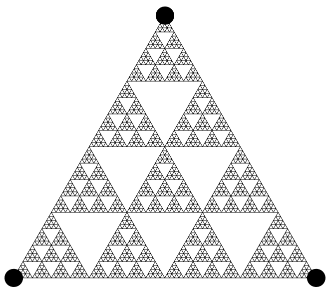
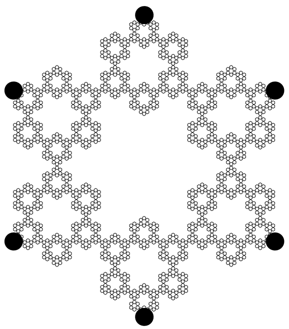
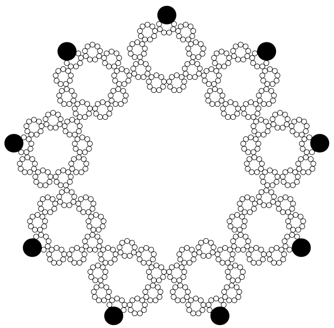
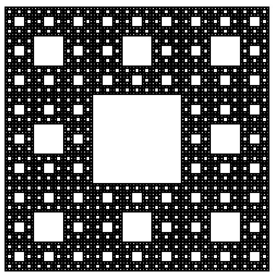
-  should be fine but not small difficulties (note that $\dim T_x K = 1$ (Kusuoka '89))
-   Gaussian HK bound NOT valid, Difficulties also for Weyl's asymp.
-   Weyl's asymp. should be possible by special reasons (ongoing work) ($h_i|_{K_w}$ can be constant!)
-  Would love to do but absolutely NO idea!