Traces of smooth functions on regular subsets

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Sobolev spaces

Let $1 \leq p \leq \infty$ and k-non-negative integer.

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  - ▹ For $0 < \alpha < 1$ Besov spaces $B^\alpha_{p,p}(\mathbb{R}^n)$ coincide with fractional Sobolev spaces

$$W^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha p}} \, dy \, dx < \infty \right\}$$

Local polynomial approximations

Let $f \in L^u_{\mathrm{loc}}(\mathbb{R}^n)$ and $1 \leq u \leq \infty$. The *normalized local best approximation* of f on a cube Q is

$$\mathcal{E}_k(f,Q)_{L^u(\mathbb{R}^n)} := \inf_{P \in \mathcal{P}_{k-1}} \left(\frac{1}{|Q|} \int_Q |f(x) - P(x)|^u \, dx \right)^{1/u},$$

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▷ Case $k = 1$

$$Pr_{1,Q} f = f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$$

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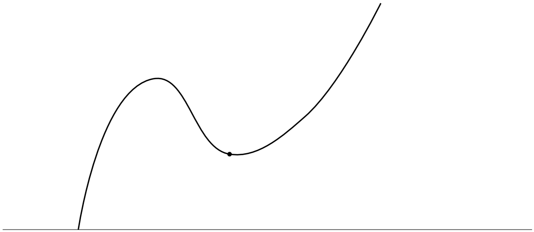

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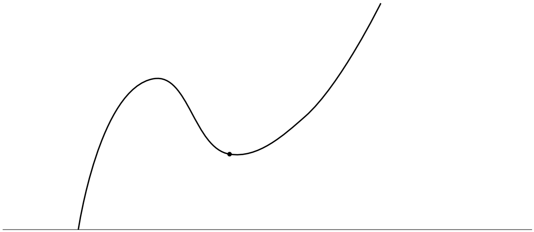

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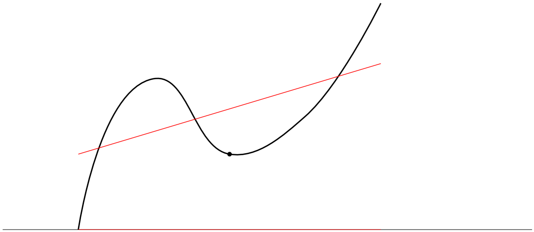

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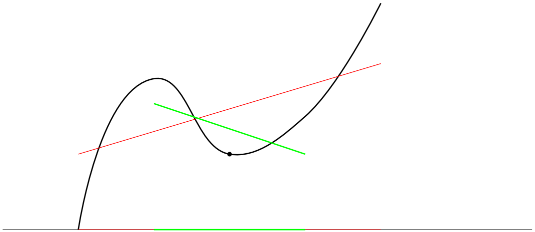

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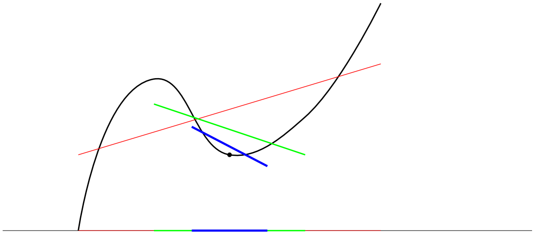

The *sharp maximal function* of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is

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Th. (Calderon, 1972)

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- A.P. Calderón, R. Scott - *Sobolev type inequalities for $p > 0$*, Studia Math., 1978.

Besov spaces

Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $1 \leq u \leq \min(p, q)$,
$\alpha > 0$, k be an integer: $k > \alpha$ (e.g. $k = [\alpha] + 1$)

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Besov space $B^{\alpha}_{p,q}(\mathbb{R}^n)$ consists of functions $f \in L^p(\mathbb{R}^n)$ such that

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- In particular, $B^{\alpha}_{p,p}(\mathbb{R}^n) = W^{\alpha,p}(\mathbb{R}^n)$, $0 < \alpha < 1$, $1 < p < \infty$.

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- $F^\alpha_{p,p}(\mathbb{R}^n) = B^\alpha_{p,p}(\mathbb{R}^n)$

Ahlfors d-regular sets

Let H^d denote d-dimensional Hausdorff measure on $\mathbb{R}^n$ and

$$Q(x, r) = \{y \in \mathbb{R}^n : \ \|x - y\|_\infty \leq r\}.$$

A subset $S \subset \mathbb{R}^n$ is called an Ahlfors d-regular (or d-set) if there are c_1, $c_2 > 0$:

$$c_1 r^d \leq H^d(Q(x, r) \cap S) \leq c_2 r^d$$

for every $x \in S$ and $0 < r \leq R$

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- Examples
  Cantor-type sets,

  self-similar sets...

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$$\mathcal{E}_k(f,Q)_{L^u(S)} := \inf_{P \in \mathcal{P}_{k-1}} \left(\frac{1}{H^d(Q \cap S)} \int_{Q \cap S} |f-P|^u \, dH^d \right)^{1/u}.$$

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The Besov space $B^{\alpha}_{p,q}(S)$, $\alpha > 0$, $1 \leq p,q \leq \infty$, is the set of those functions $f \in L^p(S)$ for which the norm $\|f\|_{B^{\alpha}_{p,q}(S)}$ is finite.

$$\|f\|_{B^{\alpha}_{p,q}(S)} := \|f\|_{L^p(S)} + \left(\int_0^1 \left(\frac{\|\mathcal{E}_k(f,Q(\cdot,\tau))_{L^u(S)}\|_{L^p(S)}}{\tau^{\alpha}} \right)^q \frac{d\tau}{\tau} \right)^{\frac{1}{q}},$$

where $1 \leq u \leq p$ and k is an integer such that $\alpha < k$.

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where $1 \leq u \leq p$ and k is an integer such that $\alpha < k$.

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Let $S \subset \mathbb{R}^n$ be a d-set with $n-1 < d \leq n$

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Trace of a function on a subset

Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $S \subset \mathbb{R}^n$. At those points $x \in S$ where exists

$$\bar{f}(x) = \lim_{r \to 0+} \fint_{Q(x,r)} f(y)\,dy = \lim_{r \to 0+} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} f(y)\,dy,$$

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- If $f \in B^\alpha_{p,q}(\mathbb{R}^n)$ or $F^\alpha_{p,q}(\mathbb{R}^n)$, then for a d-set S with

$$d > n - \alpha p$$

  the trace $f|_S$ is well defined.

Theorem*(A. Vähäkangas, L.I., 2011)

Let S be an d-set, $n-1<d<n$, $1 \leq p,q < \infty$ and $\alpha > (n-d)/p$. Then

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Remez-type inequality

Let S be a d-set, $n-1<d\leq n$.

Suppose that $Q=Q(x_Q,r_Q)$ *and* $Q'=Q(x_{Q'},r_{Q'})$ *are cubes in* $\mathbb{R}^n$ *such that* $x_{Q'}\in S$, $Q'\subset Q$, $r_Q\leq Rr_{Q'}$ *and* $r'_Q\leq R$ *for some* $R>0$.

Then, $\forall\ p\in\mathcal{P}_k$

$$\left(\frac{1}{|Q|}\int_Q|p|^r\,dx\right)^{1/r}\leq C\left(\frac{1}{\mathcal{H}^d(Q'\cap S)}\int_{Q'\cap S}|p|^u\,d\mathcal{H}^d\right)^{1/u},$$

where $1\leq u,r\leq\infty$ *and* C *depends on* S, R, n, u, r, k.

▷ A. Brudnyi and Yu. Brudnyi - *Remez type inequalities and Morrey-Campanato spaces on Ahlfors regular sets*, Contemp. Math., 2007

The construction of the extension operator is based on a modification of the Whitney extension method

Let $\mathcal{W}_S$ denote a Whitney decomposition of $\mathbb{R}^n \setminus S$ and

$\Phi := \{\varphi_Q : Q \in \mathcal{W}_S\}$ be a smooth partition of unity

To every cube $Q = Q(x_Q, r_Q) \in \mathcal{W}_S$ assign the cube $a(Q) := Q(a_Q, r_Q/2)$, where $a_Q \in S$ is such that $\|x_Q - a_Q\|_\infty = \operatorname{dist}(x_Q, S)$

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If $f \in L^1_{\mathrm{loc}}(S)$ and $k \in \mathbb{N}$, then

$$\operatorname{Ext}_{k,S} f(x) := \begin{cases} f(x), & \text{if } x \in S; \\ \sum_{Q \in \mathcal{W}_S} \varphi_Q(x) P_{k-1,Q} f(x), & \text{if } x \in \mathbb{R}^n \setminus S, \end{cases}$$

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where the projection $P_{k,Q} : L^1(a(Q) \cap S) \to \mathcal{P}_k$ are such that

$$\left(\fint_{a(Q)\cap S} |f - P_{k-1,Q} f|^u \, dH^d \right)^{1/u} \approx \mathcal{E}_k(f, a(Q))_{L^u(S)}.$$

Metric measure space

Let (X, d, μ) be a metric measure space, where (X, d) is a separable metric space and μ is a Borel regular measure such that $0 < \mu(B) < \infty$ for every ball $B \subset X$.

A measure μ on X is called doubling if there is a positive constant C_μ such that

$$\mu(B(x, 2r)) \leq C_\mu \, \mu(B(x, r)),$$

for each $x \in X$ and $r > 0$.

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Hajłasz-Sobolev spaces

Let $S \subset X$. For a measurable function u, denote by $D(u)$ the class of all measurable functions $g : S \to [0, \infty)$ such that:

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \quad \mu \,\text{a.e.}$$

Let $1 \leq p < \infty$, The space $M^{1,p}(S)$ consists of all $u \in L^p(S)$ such that $D(u) \cap L^p(S) \neq \emptyset$. For $u \in M^{1,p}(S)$, the norm is defined as follows:

$$\|u\|_{M^{1,p}(S)} := \|u\|_{L^p(S)} + \inf_{g \in D(u)} \|g\|_{L^p(S)}$$

Measure density condition

A measurable set $S \subset X$ is said to be regular if there are constants $\theta_S \geq 1$ and $\delta_S > 0$ such that for every $x \in S$ and $0 < r \leq \delta_S$

$$\mu(B(x,r)) \leq \theta_S \mu(B(x,r) \cap S). \qquad (1)$$

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- If $S \subset X$ is a closed set such that (1) holds, then there is a bounded linear extension operator of $M^{1,p}(S)$ into $M^{1,p}(X)$ for all $1 \leq p < \infty$.

Measure density condition

A measurable set $S \subset X$ is said to be regular if there are constants $\theta_S \geq 1$ and $\delta_S > 0$ such that for every $x \in S$ and $0 < r \leq \delta_S$

$$\mu(B(x,r)) \leq \theta_S \mu(B(x,r) \cap S). \tag{1}$$

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Besov spaces

Let $0 < \alpha < 1$ and $1 \leq p, q < \infty$. The Besov space $B^{\alpha}_{p,q}(X)$ is the space of all functions $u \in L^p(X)$ for which

$$\|f\|_{B^{\alpha}_{p,q}} := \left(\int_0^{\infty} \left(\int_X \fint_{B(x,t)} |f(x) - f(y)|^p \, d\mu(y) d\mu(x) \right)^{q/p} \frac{dt}{t^{\alpha q + 1}} \right)^{1/q} < \infty.$$

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▷ A. Gogatishvili, P. Koskela, N. Shanmugalingam: Interpolation properties of Besov spaces defined on metric spaces (2008)

Let $S \subset X$, $0 < \alpha < 1$ and u be a measurable function on S.

A sequence of nonnegative measurable functions $\vec{g} = \{g_k\}_{k \in \mathbb{Z}}$ on S is a fractional α-Hajłasz gradient of u if there exists $E \subset S$ with $\mu(E) = 0$:
for all $k \in \mathbb{Z}$ and $x, y \in S \setminus E$ satisfying $2^{-k-1} \le d(x, y) < 2^{-k}$,

$$|u(x) - u(y)| \le d(x, y)^{\alpha}(g_k(x) + g_k(y)).$$

Denote by $\mathbb{D}^{\alpha}(u)$ the collection of all fractional α-Hajłasz gradients of u.

Besov-type spaces

The Hajłasz-Besov space $N^{\alpha}_{p,q}(S) = N^{\alpha}_{p,q}(S,d,\mu)$ is the space of all functions $u \in L^p(S)$ for which there exists $\vec{g} \in \mathbb{D}^{\alpha}(u)$ satisfying

$$\|\vec{g}\|_{l^q(S,L^p)} := \left(\sum_{k \in \mathbb{Z}} \|g_k\|^q_{L^p(S)} \right)^{1/q} < \infty.$$

The norm in $N^{\alpha}_{p,q}(S)$ is defined as

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▷ P. Koskela, D. Yang, Y. Zhou: Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings, 2011

Extension Theorem (Heikkinen, I., Tuominen)

Suppose that S is a closed regular subset of X, $0 < \alpha < 1$, $1 \leq p, q < \infty$. Then there is a bounded linear extension operator of $N^{\alpha}_{p,q}(S)$ into $N^{\alpha}_{p,q}(X)$ for all $0 < \alpha < 1$, $1 \leq p, q < \infty$.

Spaces of Triebel-Lizorkin type

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Thank you!