Regularity for almost minimizers with free boundary

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Minimizers with free boundary

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain, $q_\pm \in L^\infty(\Omega)$ and

$$K(\Omega) = \left\{ u \in L^1_{loc}(\Omega) ;\ \nabla u \in L^2(\Omega) \right\}.$$

Minimizing problem with free boundary: Given $u_0 \in K(\Omega)$ minimize

$$J(u) = \int_\Omega |\nabla u(x)|^2 + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}}(x)$$

among all $u = u_0$ on $\partial\Omega$.

- One phase problem arises when $q_- \equiv 0$ and $u_0 \geq 0$.
- The general problem is know as the two phase problem.

Alt-Caffarelli

- Minimizers for the one phase problem exist.
- If u is a minimizer of the one phase problem, then $u \geq 0$, u is subharmonic in Ω and
$$\Delta u = 0 \text{ in } \{u > 0\}$$
- u is locally Lipschitz in Ω.
- If q_+ is bounded below away from 0, that is there exists $c_0 > 0$, such that $q_+ \geq c_0$, then:
  - for $x \in \{u > 0\}$
$$\frac{u(x)}{\delta(x)} \sim 1 \text{ where } \delta(x) = \mathrm{dist}(x, \partial\{u > 0\})$$
  - $\{u > 0\} \cap \Omega$ is a set of locally finite perimeter, thus $\partial\{u > 0\} \cap \Omega$ is (n-1)-rectifiable.

Alt-Caffarelli-Friedman

- Minimizers for the two phase problem exist.
- If u is a minimizer of the two phase problem, then $u^{\pm}$ are subharmonic and

$$\Delta u = 0 \text{ in } \{u > 0\} \cup \{u < 0\}$$

- u is locally Lipschitz in Ω.
- If $q_{\pm}$ are bounded below away from 0, then
  - for $x \in \{u^{\pm} > 0\}$

$$\frac{u^{\pm}(x)}{\delta(x)} \sim 1 \text{ where } \delta(x) = \mathrm{dist}(x, \partial\{u^{\pm} > 0\})$$

  - $\{u^{\pm} > 0\} \cap \Omega$ are sets of locally finite perimeter.

Regularity of the free boundary $\Gamma(u)$

- If u is a minimizer for the one phase problem $\Gamma(u) = \partial\{u > 0\}$
- If q_+ is Hölder continuous and $q_+ \geq c_0 > 0$ then
  - if $n = 2, 3$, $\Gamma(u)$ is a $C^{1,\beta}$ (n-1)-dimensional submanifold.
  - if $n \geq 4$, $\Gamma(u) = \mathcal{R}(u) \cup \mathcal{S}(u)$ where $\mathcal{R}(u)$ is a $C^{1,\beta}$ (n-1)-dimensional submanifold and $\mathcal{S}(u)$ is a closed set of Hausdorff dimension less than n-3.

- If u is a minimizer for the two phase problem $\Gamma(u) = \partial\{u > 0\} \cup \partial\{u < 0\}$
- If $q_\pm$ are Hölder continuous and $q_\pm \geq c_0 > 0$ then
  - if $n = 2, 3$, $\Gamma(u)$ is a $C^{1,\beta}$ (n-1)-dimensional submanifold.
  - if $n \geq 4$, $\Gamma(u) = \mathcal{R}(u) \cup \mathcal{S}(u)$ where $\mathcal{R}(u)$ is a $C^{1,\beta}$ (n-1)-dimensional submanifold and $\mathcal{S}(u)$ is a closed set of Hausdorff dimension less than n-3.

Contributions

- One phase problem:
  - n=2, Alt-Caffarelli
  - $n \geq 3$, Alt-Caffarelli, Caffarelli-Jerison-Kenig / Weiss
- Two phase problem:
  - n=2 , Alt-Caffarelli-Friedman
  - $n \geq 3$, Alt-Caffarelli-Friedman, Caffarelli-Jerison-Kenig / Weiss
- Da Silva-Jerison: There exists a non-smooth minimizer for J in $\mathbb{R}^7$ such that $\Gamma(u)$ is a cone.

Almost minimizers for the one phase problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain, $q_+ \in L^\infty(\Omega)$ and

$$K_+(\Omega) = \left\{u \in L^1_{loc}(\Omega)\,;\, u \geq 0 \text{ } a.e. \text{ in } \Omega \text{ and } \nabla u \in L^2_{loc}(\Omega)\right\}$$

- $u \in K_+(\Omega)$ is a (κ, α)-almost minimizers for J^+ in Ω if for any ball $B(x, r) \subset \Omega$

  $$J^+_{x,r}(u) \leq (1 + \kappa r^\alpha) J^+_{x,r}(v)$$

  for all $v \in K_+(\Omega)$ with $u = v$ on $\partial B(x, r)$, where

  $$J^+_{x,r}(v) = \int_{B(x,r)} |\nabla v|^2 + q_+^2\, \chi_{\{v>0\}}.$$

Almost minimizers for the two phase problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected Lipschitz domain, $q_\pm \in L^\infty(\Omega)$ and

$$K(\Omega) = \left\{u \in L^1_{loc}(\Omega)\,;\ \nabla u \in L^2_{loc}(\Omega)\right\}.$$

- $u \in K(\Omega)$ is a (κ, α)-almost minimizers for J in Ω if for any ball $B(x,r) \subset \Omega$

  $$J_{x,r}(u) \leq (1 + \kappa r^\alpha) J_{x,r}(v)$$

  for all $v \in K(\Omega)$ with $u = v$ on $\partial B(x,r)$, where

  $$J_{x,r}(v) = \int_{B(x,r)} |\nabla v|^2 + q_+^2\, \chi_{\{v>0\}} + q_-^2\, \chi_{\{v>0\}}.$$

Almost minimizers are continuous

Theorem: Almost minimizers of J are continuous in Ω. Moreover if u is an almost minimizer for J there exists a constant $C > 0$ such that if $B(x_0, 2r_0) \subset \Omega$ then for $x, y \in B(x_0, r_0)$

$$|u(x) - u(y)| \leq C|x - y| \left(1 + \log \frac{2r_0}{|x - y|}\right).$$

Remark: Since almost-minimizers do not satisfy an equation, good comparison functions are needed.

Sketch of the proof

To prove regularity of u, an almost minimizer for J, we need to control the quantity

$$\omega(x,s) = \left(\fint_{B(x,s)} |\nabla u|^2 \right)^{1/2}$$

for $s \in (0,r)$ and $B(x,r) \subset \Omega$.

Consider u_r^* satisfying $\Delta u_r^* = 0$ in $B(x,r)$ and $u_r^* = u$ on $\partial B(x,r)$. Then since $|\nabla u_r^*|^2$ is subharmonic

$$\begin{aligned} \omega(x,s) &\leq \left(\fint_{B(x,s)} |\nabla u - \nabla u_r^*|^2 \right)^{1/2} + \left(\fint_{B(x,s)} |\nabla u_r^*|^2 \right)^{1/2} \\ &\leq \left(\frac{r}{s}\right)^{n/2} \left(\fint_{B(x,r)} |\nabla u - \nabla u_r^*|^2 \right)^{1/2} + \left(\fint_{B(x,r)} |\nabla u_r^*|^2 \right)^{1/2} \end{aligned}$$

The almost minimizing property comes in

Since $\Delta u_r^* = 0$ in $B(x,r)$ and $u_r^* = u$ on $\partial B(x,r)$

$$\int_{B(x,r)} \langle \nabla u - \nabla u_r^*, \nabla u_r^* \rangle = 0.$$

Thus since $q_\pm \in L^\infty(\Omega)$

$$\begin{aligned}
\int_{B(x,r)} |\nabla u - \nabla u_r^*|^2 &= \int_{B(x,r)} |\nabla u|^2 - \int_{B(x,r)} |\nabla u_r^*|^2 \\
&\leq (1+\kappa r^\alpha) \int_{B(x,r)} |\nabla u_r^*|^2 - \int_{B(x,r)} |\nabla u_r^*|^2 + Cr^n \\
&\leq \kappa r^\alpha \int_{B(x,r)} |\nabla u_r^*|^2 + Cr^n \\
&\leq \kappa r^\alpha \int_{B(x,r)} |\nabla u|^2 + Cr^n.
\end{aligned}$$

Iteration scheme

$$\omega(x,s) \;\; \leq \;\; \left(1 + C\left(\frac{r}{s}\right)^{n/2} r^{\alpha/2}\right)\omega(x,r) + C\left(\frac{r}{s}\right)^{n/2}.$$

Set $r_j = 2^{-j}r$ for $j \geq 0$.

$$\omega(x, r_{j+1}) \leq \left(1 + C2^{n/2} r_j^{\alpha/2}\right)\omega(x, r_j) + C2^{n/2}.$$

Iteration yields

$$\omega(x, r_j) \leq C\omega(x,r) + Cj,$$

which for $s \in (0, r)$ ensures

$$\omega(x,s) \leq C\left(\omega(x,r) + \log\frac{r}{s}\right).$$

Local regularity on each phase

Theorem: Let u be an almost minimizer for J in Ω. Then u is locally Lipschitz in $\{u > 0\}$ and in $\{u < 0\}$.

Theorem: Let u be an almost minimizer for J in Ω. Then there exists $\beta \in (0, 1)$ such that u is $C^{1,\beta}$ locally in $\{u > 0\}$ and $\{u < 0\}$.

Proof: Refine the argument above.

Local regularity for minimizers

Theorem [AC], [ACF]: Let u be a minimizer for J in Ω. Then u is locally Lipschitz.

Remarks: Elements of the proof:

- $u^{\pm}$ are subharmonic in Ω,
- u harmonic on $\{u^{\pm} > 0\}$,
- the 2-phase case requires a monotonicity formula introduced by Alt-Caffarelli-Friedman [ACF], that is

$$\Phi(r) = \frac{1}{r^4}\left(\int_{B(x,r)} \frac{|\nabla u^+|^2}{|x-y|^{n-2}}\,dy\right)\left(\int_{B(x,r)} \frac{|\nabla u^-|^2}{|x-y|^{n-2}}\,dy\right)$$

  is an increasing function of $r > 0$.

Local regularity for almost minimizers

Theorem: Let u be an almost minimizer for J in Ω. Then u is locally Lipschitz.

Remarks: Elements of the proof:

- analysis of the interplay between

$$m(x,r) = \frac{1}{r} \fint_{\partial B(x,r)} u, \ \ \frac{1}{r} \fint_{\partial B(x,r)} |u| \ \text{ and } \ \omega(x,r) = \left(\fint_{B(x,s)} |\nabla u|^2 \right)^{1/2}$$

- the 2-phase case requires an almost [ACF]-monotonicity formula, i.e. we need to control the oscillation of $\Phi(r)$ on small intervals.

Sketch of the proof

For $1 \ll K$ and $0 < \gamma \ll 1$ if $B(x, 2r) \subset \Omega$ consider:

- **Case 1:**

$$\left\{ \begin{array}{rcl} \omega(x,r) & \geq & K \\ |m(x,r)| & \geq & \gamma\,(1+\omega(x,r)) \end{array} \right.$$

- **Case 2:**

$$\left\{ \begin{array}{rcl} \omega(x,r) & \geq & K \\ |m(x,r)| & < & \gamma\,(1+\omega(x,r)) \end{array} \right.$$

- **Case 3:**

$$\omega(x,r) \leq K$$

Case 1

If u is an almost minimizer for J in Ω, $B(x,2r) \subset \Omega$ and

$$\begin{cases} \omega(x,r) & \geq & K \\ |m(x,r)| & \geq & \gamma(1+\omega(x,r)) \end{cases}$$

then there exists $\theta \in (0,1)$ such that $u \in C^{1,\beta}(B(x,\theta r))$ and

$$\sup_{B(x,\theta r)} |\nabla u| \lesssim \omega(x,r).$$

Cases 2 & 3

If u is an almost minimizer for J^+ in Ω, $B(x, 2r) \subset \Omega$ and

$$\begin{cases} \omega(x,r) & \geq & K \\ m(x,r) & < & \gamma\,(1+\omega(x,r)) \end{cases}$$

then for $\theta \in (0,1)$ there exists $\beta \in (0,1)$ such that

$$\omega(x, \theta r) \leq \beta\omega(x,r).$$

If only cases 2 and 3 occur then

$$\limsup_{s\to 0} \omega(x,s) \lesssim K$$

and if x is a Lebesgue point of ∇u then

$$|\nabla u(x)| \lesssim K.$$

First steps toward understanding the free boundary for almost minimizers: blow-up

Let u be an almost minimizers for J in Ω with $q_\pm \in L^\infty(\Omega) \cap C(\Omega)$. If $J = J^+$ we assume $q_- \equiv 0$. Let

$$\Gamma(u) = \partial\{u > 0\} \cup \partial\{u < 0\}.$$

Let $x_0 \in \Gamma(u)$ and $r_k \to 0$. Consider

$$u_k(x) = \frac{u(r_k x + x_0)}{r_k}$$

Modulo passing to a subsequence:

$$\begin{aligned} u_k &\longrightarrow u_\infty \text{ uniformly on compact sets of } \mathbb{R}^n \\ u_k &\longrightarrow u_\infty \text{ in } W^{1,2}_{loc}(\mathbb{R}^n). \end{aligned}$$

Moreover u_∞ is a global minimizers of J^∞ where

$$J^\infty_{x,r}(v) = \int_{B(x,r)} |\nabla v|^2 + q_+^2(x_0)\,\chi_{\{v>0\}} + q_-^2(x_0)\,\chi_{\{v>0\}}.$$

Non-degeneracy near the free boundary

Let u be an almost minimizers for J in Ω with $q_\pm \in L^\infty(\Omega) \cap C(\Omega)$. Assume

$$q_+ \geq c_0 > 0,$$

then there exists $\eta > 0$ so that if $x_0 \in \partial\{u > 0\}$ and $B(x_0, 2r_0) \subset \Omega$ then for $r \in (0, r_0)$

$$\frac{1}{r} \fint_{\partial B(x_0,r)} u^+ \geq \eta$$

and

$$u^+(x) \geq \frac{\eta}{4}\,\delta(x) \quad \text{for} \quad x \in B(x_0, r_0) \cap \{u > 0\}.$$

Remark: In this case the blow ups of u are not identically equal to 0.

What type of information does this provide?

Let u be an almost minimizers for J in Ω with $q_\pm \in L^\infty(\Omega) \cap C(\Omega)$. Assume

$$q_+ > q_- \geq c_0 > 0.$$

Let $x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\}$ then the tangent function u_∞ of u at x_0 satisfies

$$u_\infty(x) = \mu_+ \langle x, e\rangle^+ - \mu_- \langle x, e\rangle^-$$

for some $e \in \mathbb{S}^{n-1}$.

Remarks:

- All blow-ups of $\Gamma(u)$ at x_0 are (n-1)-dimensional planes. The unit normal e to a given blow-up to $\Gamma(u)$ at x_0 determines the corresponding blow-up function u_∞.
- $\mu_\pm$ are independent of the blow up sequence.
- A priori e depends on the blow-up sequence.

Some related questions

- Under the assumptions that $q_\pm \in L^\infty(\Omega) \cap C(\Omega)$ and $q_\pm$ are bounded below by $c_0 > 0$, we expect that, for u almost minimizer of J in Ω, $\Omega \cap \{u > 0\}$ and $\Omega \cap \{u < 0\}$ be sets of locally finite perimeter.

- Similar results should be true for almost minimizers of functionals of the type:

$$J(u) = \int_\Omega |\nabla u(x)|_g^2 + q_+^2(x)\chi_{\{u>0\}}(x) + q_-^2(x)\chi_{\{u<0\}}(x),$$

  where $|\nabla u|_g$ denotes the norm of ∇u computed in the metric g and g is assumed to be Hölder continuous.