Metric spaces with unique and uniformly close tangents and their metric dimensions

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Intro

Riemannian manifolds are locally Euclidean.
Riemannian manifolds are infinitesimally Euclidean.
SubRiemannian manifolds are infinitesimally Carnot groups.

Q: What are the metric spaces with the property that there are other metric spaces that describe their infinitesimal structure?
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Other examples of infinitesimally Euclidean spaces

1. Reifenberg vanishing flat spaces ■
2. There are examples of spaces that are n-regular, admit Poincaré inequality, are a.e. infinitesimally $\mathbb{R}^n$, but have no manifold points. [Hanson-Heinonen]

Discussion on the notion of 'infinitesimal'

A differential curve is infinitesimally a line.
A differential function is infinitesimally a linear map.
Need: notion of limit of (pointed) metric space (by Gromov)
We consider a distance on the spaces of pointed metric spaces: $\mathrm{Dist}_{GH}(\,(X,x),(Y,y)\,) =$

$$= \inf\left\{ \epsilon > 0 \;\middle|\; \begin{array}{c} \exists \text{ extension of} \\ \text{distances on} \end{array} \quad Y \sqcup X : \begin{array}{c} d(x,y) \leq \epsilon \\ B_X(x, 1/\epsilon) \subseteq B_{Y\sqcup X}(Y,\epsilon) \\ B_Y(y, 1/\epsilon) \subseteq B_{Y\sqcup X}(X,\epsilon) \end{array} \right\}$$

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Tangent spaces

If $X = (X, d_X)$ is a metric space and $\lambda > 0$, then $\lambda X := (X, \lambda d_X)$ is a metric space.

Definition (Set of tangents)

$\mathrm{Tan}(X, x) := \{(Y, y) \ : \ \exists \lambda_j \to \infty$ s.t. $(\lambda_j X, x) \to (Y, y)\}$ is the *collection of tangents* of X at x.

If X is doubling, $\mathrm{Tan}(X, x)$ is nonempty [Gromov]

Definition (Unique tangents)

X admits *unique tangents* if $\#\mathrm{Tan}(X, x) = 1, \forall x \in X$. In other words, $\forall x \in X, \exists$ a metric space, which we denote by $T_x X$, such that $\mathrm{Tan}(X, x) = \{T_x X\}$.

N.B. $T_x X$ depends on x.

Example

Balloon

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General spaces with unique tangents

Fact: $\exists$ restrictions on general possible $T_x X$.

Theorem (ELD, 2011)

Let X be a metric space with unique tangents.
Then $T_x X$ admits dilations, $\forall x \in X$.

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If there is a doubling measure μ on X, then $T_x X$ is a homogeneous space, for μ-a.e. $x \in X$.

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If moreover X is geodesic, then $T_x X$ is a (subFinsler) Carnot group, for μ-a.e. $x \in X$.

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Problem

Knowing $T_x X$, what can we deduce about X?
Example of answer:

Theorem (David Herron, Xiangdong Xie, 2011)

Let X be a complete doubling geodesic metric space. Then X is annular quasiconvex if and only if, for each 'weak' tangent (Y, y), the space $Y \setminus \{y\}$ is connected.

[Bloch's principle, Zalcman Lemma, Väisälä]

Problem

More specific problem: $\exists$ relations of dimensions?
In general, no...
$\blacksquare$

Uniform tangents

Definition (Uniformly close tangents)

Let X be a metric space with unique tangents.
We say that X has *uniformly close tangents* if, for each $x \in X$, the convergence of the dilated spaces of (X, x) toward $T_x X$ is uniform.
In other words, $\forall \epsilon > 0$, $\exists \lambda_\epsilon > 0$ s.t.

$$\mathrm{Dist}_{GH}((\lambda X, x), T_x X) < \epsilon, \qquad \forall x \in X, \forall \lambda > \lambda_\epsilon.$$

Example

A metric space X is called a *Reifenberg vanishing flat* if

1. admits unique tangents,
2. has uniformly close tangents,
3. $\exists n \in \mathbb{N}$ s.t. $T_x X = \mathbb{R}^n$, $\forall x \in X$.

Fact: Reifenberg vanishing flat spaces have 'nice' parameterization [David & Toro, Cheeger & Colding]

Consequence: we know their dimension.

Novelties

New results in collaborations with Urs Lang and Tapio Rajala.

Assouad dimension and Nagata dimension for metric spaces with uniformly close tangents
and applications to the Lipschitz extension problem for subRiemannian manifolds.

Assouad dimension

Definition (Assouad dimension)

The *Assouad dimension* of a metric space X is denoted by $\dim_A(X)$ and is defined as the infimum of all numbers $\alpha > 0$ with the property that $\exists C > 1$ s.t., $\forall r, \forall R$ with $0 < r < R$, each ball $B(x, R)$ can be covered with less than $C\,(R/r)^{\alpha}$ balls of radius r.

$$X \text{ doubling} \iff \dim_A(X) < \infty$$

Theorem (ELD & T. Rajala, 2013)

X metric space with unique and uniformly closed tangents.
Assume $\dim_A T_x X = \alpha$, $\forall x \in X$.
Then $\dim_A X = \alpha$, *locally.*

Proof of the theorem.

‘Proof’ of the theorem

Want: cover $B(x,R)$ with $\simeq (R/r)^\alpha$ balls of radius r.
Take δ so small that everything will work...
Uniform close tangents $\implies \exists\lambda_0$ s.t. $\forall\lambda > \lambda_0\ \forall y \in X$

$$(\lambda X, y) \simeq T_y X, \quad \text{i.e.,} \quad d((\lambda X, y), T_y X) \lesssim \delta$$

$\implies$ in λX, roughly $(\frac{1}{\delta})^\alpha$ balls of the form $B(z,\delta)$ cover $B(y,1)$
Assume $R < 1/\lambda_0$.
Let $m \in \mathbb{N}$ s.t. $\delta^m \simeq r/R$.
$(\frac{1}{R}X, x) \simeq T_x X \implies (\frac{1}{\delta})^\alpha$ balls of the form $B(y,\delta R)$ cover $B(x,R)$
$(\frac{1}{\delta R}X, y) \simeq T_y X \implies (\frac{1}{\delta})^\alpha$ balls of form $B(y',\delta^2 R)$ cover $B(y,\delta R)$
$\circ\circ\circ$ repeat m times.
$\implies (\frac{1}{\delta})^{\alpha m} \simeq (\frac{R}{r})^\alpha$ balls of radius $\delta^m R \simeq r$ cover $B(x,R)$.
QED

Definition (Nagata dimension)

The *Nagata dimension* is the infimum of all integers n with the following property:

> $\exists c > 0$ s.t., $\forall s > 0$, the metric space admits
> an s-bounded covering of the form $\mathcal{B} = \bigcup_{k=0}^{n} \mathcal{B}_k$
> where each distinct pair of sets in $\mathcal{B}_k$ are cs-separated.

Theorem (U. Lang & T. Schlichenmaier)

Suppose that X, Y *are metric spaces,* $\dim_N X \leq n < \infty$*, and* Y *is complete. If* Y *is Lipschitz* m*-connected for* $m = 0, 1, \dots, n-1$*, then the pair* (X, Y) *has the Lipschitz extension property.*

Theorem (U. Lang & T. Schlichenmaier)

Suppose that Y *is a complete metric space with* $\dim_N Y \leq n < \infty$*, and* Y *is Lipschitz* m*-connected for* $m = 0, 1, \dots, n$*. Then* Y *is an absolute Lipschitz retract; equivalently, the pair* (X, Y) *has the Lipschitz extension property for every metric space* X*.*

Theorem (ELD & T. Rajala, 2013)

X metric space with unique and uniformly closed tangents. Assume $\dim_N X < \infty$ and $\dim_N T_x X = n$, $\forall x \in X$. Then $\dim_N X = n$, locally.

Theorem (U. Lang & ELD, 2012)

It G is a Carnot group, then $\dim_N G = \dim_{top} G$.

Corollary

The local Nagata dimension of a equiregular subRiemannian manifold equals its topological dimension.

Impossible parametrizaton by tangents

For subRiemannian manifolds the tangent space is not a local model.

Theorem (ELD & Ottazzi & Warhurst 2011)

$\exists$ *a nilpotent Lie group equipped with a left-invariant subRiemannian distance that is* **not** *locally quasiconformally equivalent to its tangent.*

THANKS