Analysises on metric-measure spaces – Cheeger energy and the measurable Riemannian structure

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Introduction

Analysises on measure-metric spaces:

(Ch) **Analogy of differentiable manifold, e.g.** $\mathbb{R}^n$

gradient, the Sobolev inequalities, differentiable strucure,....

Hajłasz: Sobolev spaces $M^{1,p}(X)$

Shanmugalingam: Newtonian spaces $N^{1,p}(X)$

Cheeger: $H^{1,p}(X)$

Measurable differential structure

(Fr) **Analysis on Fractals – Brownian motions**

Limit of Discrete objects, for example, graph laplacian, Random walks

"proper scaling limit"

Dirichlet forms and (associated) Brownian motion

Sierpinski gasket(Kusuoka, Goldstein), Siperpinski carpet(Barlow-Bass)

Measurable Riemannian structure

(Rc) **Ricci curvature lower bound, Bochner, Bakery-Emery**

Curvature-dimension lower bounds —— CD(K, N), etc

Lott-Villani, Sturm, Ambrosio, Gigli, Savaré,.....

Conjecture, obervation, impression, hocus-pocus?

(1) (Ch) $\cap$ (Fr) has positive measure.

(2) (Fr) $\cap$ (Rc) has “very small” measure.

Cheeger's result

(X, d, μ): a measure-metric space

Notation. $B(x,r) = \{y | y \in X, d(x,y) < r\}$, $V(x,r) = \mu(B(x,r))$.

Assumption 0.1. For any $x, y \in X$, there exists a rectifiable curve γ joining x and y.

Theorem 0.2 (**Cheerger's theorem**).

$$(1,p)\textbf{-Poincaré inequality}(1 \le p < \infty)$$
$$+ \textbf{ the volume doubling property of } \mu,$$
$$\Downarrow$$
$$\textbf{measurable differentiable structure on } (X, d, \mu)$$

which is a collection of pairs $\{(X_i, \xi^{(i)})\}_{i \ge 1}$ *that satisfies the conditions in the next to the next slide:*

Definition 0.3 (**Volume doubling property**). μ has the volume doubling propety $\underset{\text{def}}{\Leftrightarrow}$ $\exists C > 0$, $\forall x \in X, r > 0$

$$\mu(B(x, 2r)) \leq C\mu(B(x, r))$$

Definition 0.4 (**Lipschitz functions**). $f : X \to \mathbb{R}$: **Lipchitz** $\underset{\text{def}}{\Leftrightarrow}$

$$\sup_{x,y \in X, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < +\infty.$$

$f : X \to \mathbb{R}$: **locally Lipschitz** $\underset{\text{def}}{\Leftrightarrow}$ $\forall x \in X, r > 0$, $f|_{B(x,r)}$ is Lipschitz.
$\mathcal{L}ip(X) \underset{\text{def}}{=}$ the collection of Lipschitz functions
$\mathcal{L}ip_{\text{loc}}(X) \underset{\text{def}}{=}$ the collection of locally Lipschitz functions

For $f : X \to \mathbb{R}$, we define

$$\operatorname{lip} f(x) \underset{\text{def}}{=} \liminf_{r \downarrow 0} \sup_{y \in B(x,r)} \frac{|f(x) - f(y)|}{r}.$$

Definition 0.5 (**Poincaré inequality**)**.** Let $1 \leq q \leq p < \infty$. A metric-measure space (X, d, μ) admits a (q,p)**-Poincaré inequality** if $\exists C, L \geq 1$, $\forall f \in \mathcal{L}ip(X), x \in X, r > 0$,

$$\left(⨍_{B(x,r)} |f - f_B|^q d\mu \right)^{\frac{1}{q}} \leq Cr \left(⨍_{B(x,Lr)} (\operatorname{lip} f)^p d\mu \right)^{\frac{1}{p}}.$$

If $q' \leq q \leq p \leq p'$, then (q,p)-Poincaré $\Rightarrow$ (q', p')-Poincaré.

1 X_i *is a measurable subset of* X *and* $\mu(X_i) > 0$. $\mu(X \setminus \cup_{i\geq 1} X_i) = 0$.

2 $\forall i, \xi^{(i)} = (\xi_1^{(i)}, \ldots, \xi_{m_i}^{(i)}) \in \mathcal{L}ip(X)^{m_i}$ *such that*

$$\sup_{i\geq 1} m_i < +\infty : \textit{the dimension of } \{(X_i, \xi^{(i)})\}_{i\geq 1}$$

3 $\forall i$, $\exists$*a linear map* $d^i : \mathcal{L}ip_{\text{loc}}(X) \to \{\textit{measurable functions on } X_i\}^n$

$$\boxed{\limsup_{y\to x} \frac{|f(y) - f(x) - d^i f(x) \cdot (\xi^{(i)}(y) - \xi^{(i)}(x))|}{d(x,y)} = 0}$$

for μ*-a.e.* $x \in X_i$. *Moreover,* $d^i(fg) = f d^i g + g d^i f$.

4 $\forall i$ *and for* μ*-a.e.* $x \in X_i$, $\exists$*a norm* $|\cdot|_{i,x}$ *of* $\mathbb{R}^{m_i}$, $\forall f \in \mathcal{L}ip_{\text{loc}}(X)$, $|d^i f(x)|_{i,x}$ *is measurable,*

$$|d^i f(x)|_{i,x} = |d^j f(x)|_{j,x}$$

for μ*-a.e.* $x \in X_i \cap X_j$.

Theorem 0.6 (**Sobolev space $H^{1,p}$**). *Under the same assumptions as Cheeger's theorem, define*

$$||f||_{1,p} = ||f||_p + ||df||_p,$$

where $||df||_p = \left(\int_X (|df(x)|_x)^p \mu(dx) \right)^{1/p}$. *Then the closure of*

$$\{u|u \in \mathcal{L}ip_{\mathrm{loc}}(X), ||u||_{1,p} < +\infty\},$$

which is denoted by $H^{1,p}(X)$, *is a* **reflective Banach space**.

$||df||_p^p$ is called the Cheeger p-energy.

Example 0.7 (Heisenberg group). Define a (non-commutative) group structure on $\mathbb{R}^3$ as follows

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y))$$

Define

$$||(x, y, t)|| = ((x^2 + y^2)^2 + t^2)^{1/4}$$

and $d(a, b) = ||a^{-1} \cdot b||$, where $a, b \in \mathbb{R}$. Let μ be the Lebesgue measure of $\mathbb{R}^3$. Then $(\mathbb{R}^3, d, \mu)$ satisfies $(1, 1)$-Poincaré inequality. μ satisfies the volume doubling property.

Example 0.8. $\{M_n\}_{n\geq 1}$: a sequence of Riemannian manifold which satisfies

$$\inf_{n\geq 1} Ricci(M_n) > -\infty$$

and

$$\sup_{n\geq 1} \operatorname{diam}(M_n) < +\infty.$$

(M, d, μ): Gromov-Hausdorff limit of $\{M_n\}_{n\geq 1}$. Then (M, d, μ) satisfies 1-Poincaré inequality and the volume doubling property.

Example 0.9 (Fat Sierpinski carpet). J. Mackay, J. Tyson and K. Wildrick, Modulus and Poincaré inequalities on non-self-similar Sierpinski carpets with positive area

Reminiscence of differentiable structure..........

Cheeger 2-energy as Dirichlet form

Theorem 0.10 (Cheeger)**.** *Assume that μ has the volume doubling property and (X, d, μ) satisfies $(2, 2)$-Poincaré inequality. Then the Cheeger 2-energy is a* **local regular Dirichlet form** *on $L^2(X, \mu)$.*

$$\boxed{\text{2-Cheeger energy} \longrightarrow \text{Diffusion process on } X}$$

Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$	$\mathcal{E}(u,v) = \displaystyle\int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx$
$\mathcal{E}$: non-negative quadratic form with the Markov property	$\mathcal{F} = W^{1,2}(\mathbb{R})$: Sobolev space
$\downarrow$	$\downarrow$
$\mathcal{E}(u,v) = \int_X u(Lv)d\mu$	$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} u(-\Delta v)dx$
$-L$: Laplacian, $L \geq 0$, self-adjoint	$\Delta = \displaystyle\sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$
$\downarrow$	$\downarrow$
$\dfrac{\partial u}{\partial t} = -Lu$: Heat equation	$\dfrac{\partial u}{\partial t} = \Delta u$
$\downarrow$	$\downarrow$
$u(x,t) = e^{-tL}u_0 =$ initial condition	$u(x,t) = e^{t\Delta}u_0$
$\downarrow$	$\downarrow$
Process $(\{X_t\}_{t>0}, \{P_x\}_{x\in X})$ with $E_x(u(X_t)) = (e^{-tL}u)(x)$	The Brownian motion on $\mathbb{R}^n$

On the other hand, by Saloff-Coste, Grigor'yan

$$\text{Poincaré inequality} + \text{volume doubling}$$
$$\Downarrow$$

Gaussian heat kernel esitimate

Heat kernel $p(t,x,y)$ = the fundamental solution of the Heat equation:

$$\frac{\partial u}{\partial t} = \Delta u \Rightarrow u(t,x) = \int_X p(t,x,y) u_0(y) d\mu$$

$$p(t,x,y) = \frac{c_1}{t^{n/2}} \exp\Big(-c_2 \frac{|x-y|^2}{t}\Big) : \text{ordinary heat kernel on } \mathbb{R}^n$$

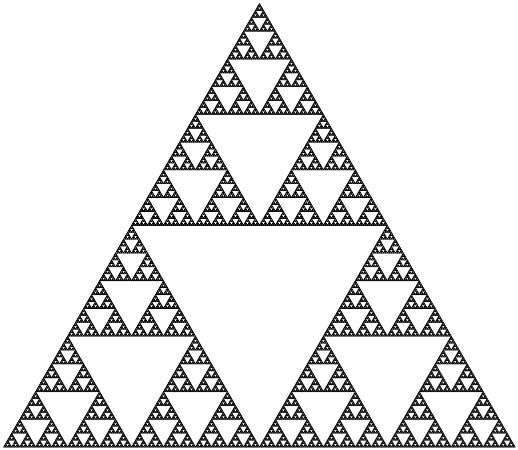
$p(t,x,y)$**: the heat kernel** associated with the 2-Cheeger energy

$$p(t,x,y) \asymp \frac{c_1}{V(x,\sqrt{t})} \exp -\left(c_2 \frac{d(x,y)^2}{t}\right)$$

while the heat kernels on many of fractals like the SG and SC satisfies the **sub-Gaussian estimate**

$$p(t,x,y) \asymp \frac{c_1}{V(x,t^{1/\beta})} \exp -\left(c_2 \left(\frac{d(x,y)^\beta}{t}\right)^{\frac{1}{\beta-1}}\right)$$

for $\beta > 2$. ($\beta = 2$ is the Gaussian.)

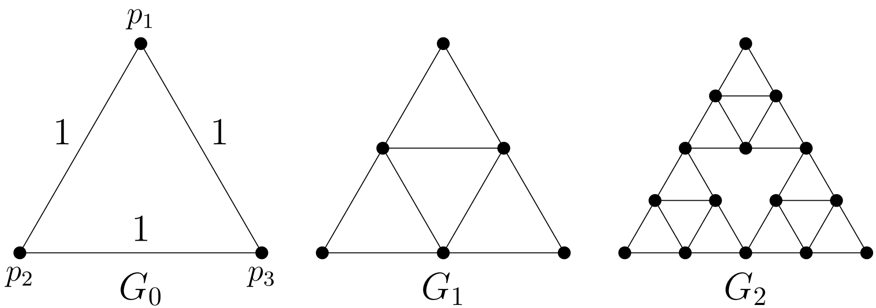


Analysis on Fractals

The Sierpinski Gasket K

$$\dim_H K = \frac{\log 3}{\log 2}$$

the Hausdorff dimension with respect to the Euclidean metric

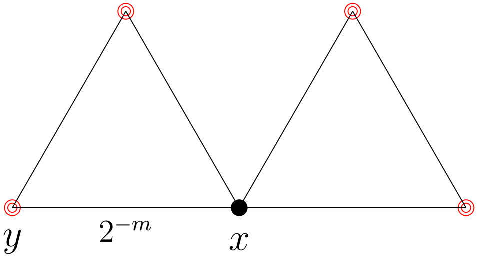


The Sierpsinski Gasket: Approximation by Graphs G_m

$$F_i(z) = (z - p_i)/2 + p_i \quad \text{for } i = 1, 2, 3$$
$$V_0 = \{p_1, p_2, p_3\}$$
$$V_{m+1} = F_1(V_m) \cup F_2(V_m) \cup F_3(V_m)$$

$K = \overline{\cup_{m \geq 0} V_m}$: the Sierpinski gasket

$$\boxed{K = F_1(K) \cup F_2(K) \cup F_3(K)}$$

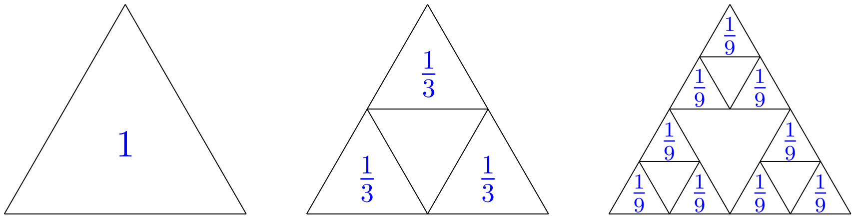


$x \in V_m$ ◎'s $= V_{m,x}$: direct neighbors of x in V_m

Define

$$\boxed{H_{m,x}u = \sum_{y \in V_{m,x}} (u(y) - u(x))} \text{ : \textbf{Graph Laplacian}}$$

$$\boxed{(\Delta_\nu u)(x) = \lim_{m \to \infty} 5^m H_{m,x} u}$$



$$\nu \underset{\text{def}}{=} \text{the normalized } \frac{\log 3}{\log 2}\text{-dim. Hausdorff measure}$$

$$= \text{the self-similar measure with weight } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

The standard resistance form on K: $(\mathcal{E},\mathcal{F})$

$$\mathcal{F}=\{u|\lim_{m\to\infty}\mathcal{E}_m(u,u)<+\infty\}$$

$$\mathcal{E}(u,v)=\lim_{m\to\infty}\mathcal{E}_m(u,v)\leftarrow \textbf{Energy}$$

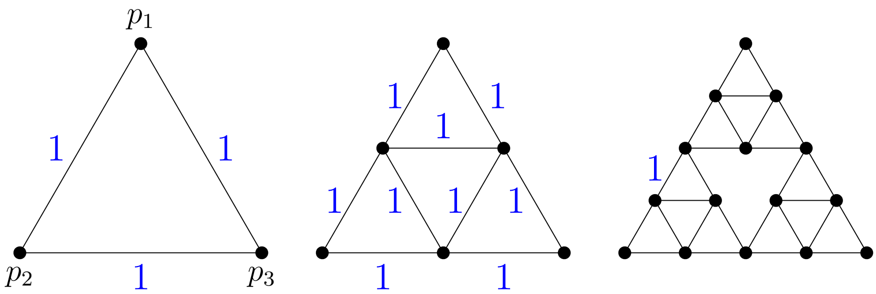
where $\boxed{\mathcal{E}_m(u,u)=\frac{1}{2}\sum_{(p,q)\text{ is an edge of the Graph }G_m}\left(\frac{5}{3}\right)^m(u(p)-u(q))^2}$.

Fact:

$$\mathcal{E}_m(u,u)\le\mathcal{E}_{m+1}(u,u)$$

Theorem 0.11. *$\mathcal{F}\subseteq C(K)$. $(\mathcal{E},\mathcal{F})$ is a* **local regular Dirichlet form** *on $L^2(K,\mu)$. In particular, $(\mathcal{E},\mathcal{F})$ is* **closed** *and*

$$\mathcal{E}(u,v)=-\int_K u\Delta_\mu v\,d\mu.$$



$$\frac{5}{3} = \textbf{Resistance scaling}$$

Attach a **resistor of resistance 1** to each edge of V_m for any m. Then

$$\text{The effective resistance between } p_1 \text{ and } p_2 = \frac{3}{2}\left(\frac{5}{3}\right)^m$$

Asymptotic behavior of the heat kernel

Theorem 0.12 (Barlow-Perkins)**.** *For* $0 < t \leq 1$,

$$\boxed{p_\nu(t,x,y) \asymp \frac{c_1}{(t^{d_H})^{1/d_w}} \exp\left(- c_2 \left(\frac{|x-y|^{d_w}}{t} \right)^{1/(d_w-1)} \right)},$$

where $d_w = \dfrac{\log 5}{\log 2}$*:* **the walk dimension**,

$d_H = \log 3/\log 2$*: the Hausdorff dimension*

sub-Gaussian heat kernel estimate

$d_w > 2$: slower than the Gaussian

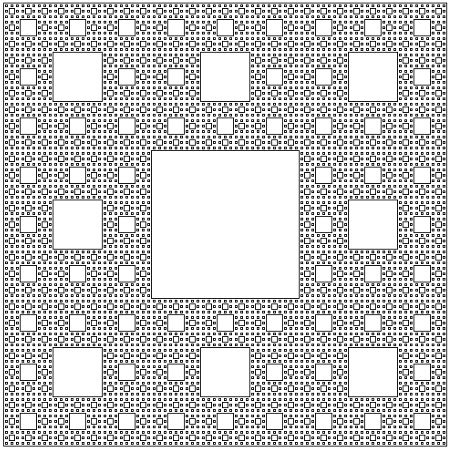


Figure 1: Sierpinski Carpet

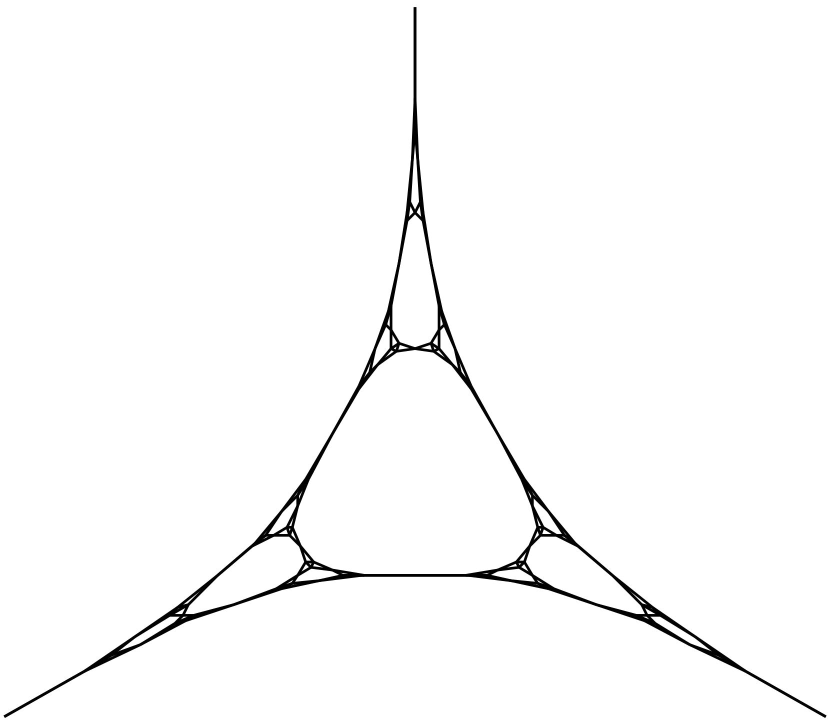
Construction: Barlow-Bass, Kusuoka-Zhou
Heat kernel estimate: Barlow-Bass
Uniquensee: Barlow-Bass-Kumagai-Teplyaev

Measurable Riemannian geometry

$\exists$a harmonic quaisymmetric map $\Phi : K \to \mathbb{R}^2$ such that the statements of the following pages are true. Let

$$K_H = \Phi(K) : \textbf{the harmonic Sierpinski gasket}$$

We are going to identify K with $K_H \subset \mathbb{R}^2$.



Riemannian volume = the Kusuoka measure

$\exists \mu_*$: Borel regular probability measure on K_H – the Kusuoka measure

$$\boxed{\mu_* \text{ is mutually singular to } \nu!}$$

Riemannian metric

For μ_*-a.e. $x \in K_H$,

$\exists Z(x)$: 2×2-matrix, $\operatorname{rank} Z(x) = 1$, $\operatorname{trace} Z(x) = 1$

the orthogonal projection to the "tangent space" of K_H at x

Gradient

$\forall u \in \mathcal{F}$, $\exists \widetilde{\nabla} u : K_H \to \mathbb{R}^2$: gradient of u

Theorem [**Measurable Riemannian structure**, Kusuoka]

$$\boxed{\mathcal{E}(u, v) = \int_{K_H} (\widetilde{\nabla} u, Z \widetilde{\nabla} v) d\mu_*} \qquad (\textbf{MRS})$$

Moreover, let

$$C^1(K_H) = \{u|_{K_H} : \exists U \supseteq K_H \text{ , an open subset of } \mathbb{R}^2, \text{ such that } u \in C^1(U)\}.$$

Then $C^1(K_H) \subseteq \mathcal{F}$ and $\forall v \in C^1(K_H)$, $\boxed{\widetilde{\nabla} v = {}^t\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)}$.

Riemannian distance: $d_*(x,y)$

For $x, y \in K_H$

$d_*(x,y) = \inf\{\text{the length of a rectifiable curve in } K_H \text{ between } x \text{ and } y\}$

Theorem 0.13 (**Gaussian heat kernel estimate**). *Let $p_*(t,x,y)$: the heat kernel associated with $(\mathcal{E},\mathcal{F})$ on $L^2(K_H,\mu_*)$. Then*

$$\boxed{p_*(t,x,y) \asymp \frac{c_1}{V_{d_*}(x,\sqrt{t})} \exp\left(-c_2 \frac{d_*(x,y)^2}{t}\right)},$$

where $B_{d_*}(x,r) = \{y | d_*(x,y) < r\}$ *and* $V_{d_*}(x,r) = \mu_*(B_{d_*}(x,r))$.

Theorem 0.14 (Kajino)**.** *$\forall K \in \mathbb{R}, N \in [1, \infty]$, $\mathrm{CD}(K,N)$ nor $\mathrm{MCP}(K,N)$ does not hold.*

Cheeger construction from $(K_H, d_*, \mu_*) \longrightarrow$

$$H^{1,2}(K_H) = \mathcal{F} \text{ and } \mathcal{E} = \text{the Cheeger 2-energy.}$$

By Hino,
Stongly local regular Dirichlet form + finite index $\Rightarrow$

measurable Riemannian structure (**MRS**)

$(\mathcal{E}, \mathcal{F})$: strongly local Dirichlet form on $L^2(X, \mu)$

Proposition 0.15 (**Energy measure**). *Let $(\mathcal{E}, \mathcal{F})$ be a strong local Dirichlet form on $L^2(X, \mu)$. $\forall f \in \mathcal{F} \cap L^\infty(X), \exists a$ Borel regular measure μ_f on X such that $\forall g \in \mathcal{F} \cap C_0(X)$,*

$$\boxed{\int_X g d\mu_f = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g)}.$$

μ_f: Energy measure of f
Note that

$$\boxed{\mu_f(X) = 2\mathcal{E}(u, u)}.$$

Theorem 0.16 (Hino: existence of measurable Riemannian structure). *$\exists\mu$: a Borel regular measure on X such that $\forall f \in \mathcal{F}$,*

$$\mathcal{E}(f,f) = \frac{1}{2}\int_X \frac{d\mu_f}{d\mu} d\mu$$

and μ is minimal among the meausres which have the above property. Moreover, if the index p of $(\mathcal{E},\mathcal{F})$ is finite, then $\exists\nabla : \mathcal{F} \to \{$measurable functions on $X\}^p$, Z_x : $p \times p$-matrix such that $\forall u, v \in \mathcal{F}$,

$$\boxed{\mathcal{E}(u,v) = \frac{1}{2}\int_X (Z_x \nabla u, \nabla v)\mu(dx)}.$$

Theorem 0.17 (Grigor'yan-Lau-Hu, Kumagai-Sturm). *Assume* $\exists \beta \geq 2, c_1, c_2, c_3, \epsilon > 0$, $\forall x, y \in X, t \in (0,1]$,

$$\frac{c_3}{V(x, t^{1/\beta})} \leq p(t,x,y) \leq \frac{c_1}{V(x, t^{1/\beta})} \exp\left(-c_2 \left(\frac{|x-y|^\beta}{t} \right)^{1/(\beta-1)} \right)$$

Lower estimate $\leq$*: for any* $x, y \in X$ *with* $d(x,y) \leq \epsilon t^\beta$*, near diagonal*
Upper estimate $\leq$*: for any* $x, y \in X$*, any* $t \in (0,1]$.
Then for any $g \in C_0(X)$,

$$\boxed{\int_X g \, d\mu_f \asymp \limsup_{r \downarrow 0} \int_X g(x) \left(⨍_{B(x,r)} \left(\frac{|f(x) - f(y)|}{r^{\beta/2}} \right)^2 \mu(dy) \right) \mu(dx)}$$

Appriximation of the density $\dfrac{d\mu_f}{d\mu}$

Gaussian heat kernel estimate: $\beta = 2 \Rightarrow (\mathcal{E}, F)$: Cheeger 2-energy (Koskela-Zhou)

sub-Gaussian heat kernel estimate: $\beta > 2$???

$(\mathcal{E}, \mathcal{F})$: a strongly local Dirichlet form $\rightarrow$ a measurable Riemannian structure

$\Downarrow$ Changing a measure μ and a distance d

measurable differentiable structure?