Notions of differential structure on metric measure spaces and applications

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Aim of the talk

On mms there is a well established notion of Sobolev space.

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2) this is useful for understanding both analysis and geometry of mms

Neither doubling nor Poincaré are assumed.

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- Preliminaries
  - Sobolev space over a metric measure space
  - Differential calculus on normed spaces
- Analysis
  - Differentials and gradients
  - Horizontal and vertical derivatives
  - Averaging out the unsmoothness
  - Distributional Laplacian
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Test plans

Let (X, d) be complete and separable and $\boldsymbol{m}$ a non-negative Radon measure on it.

Let $\boldsymbol{\pi} \in \mathscr{P}(C([0,1], X))$. We say that $\boldsymbol{\pi}$ is a test plan provided:

- for some $C > 0$ it holds

$$\mathrm{e}_{\mathrm{t}\,\sharp}\boldsymbol{\pi} \leq \mathrm{C}\boldsymbol{m}, \qquad \forall \mathrm{t} \in [0,1].$$

- it holds

$$\iint_0^1 |\dot{\gamma}_t|^2 \, dt \, d\boldsymbol{\pi} < \infty$$

The Sobolev class $S^2(X, d, \boldsymbol{m})$

We say that $f : X \to \mathbb{R}$ belongs to $S^2(X, d, \boldsymbol{m})$ provided there exists $G \in L^2(X, \boldsymbol{m})$ such that

$$\int \big|f(\gamma_1) - f(\gamma_0)\big| \, d\boldsymbol{\pi}(\gamma) \le \iint_0^1 G(\gamma_t)|\dot{\gamma}_t| \, dt \, d\boldsymbol{\pi}(\gamma)$$

for any test plan $\boldsymbol{\pi}$.

Any such G is called ‘weak upper gradient’ of f.

The minimal G in the $\boldsymbol{m}$-a.e. sense will be denoted by $|Df|$

Basic properties

Locality

$$|Df| = |Dg| \qquad \boldsymbol{m}\text{-}a.e.\ on\ \{f = g\}$$

Chain rule

$$|D(\varphi \circ f)| = |\varphi'| \circ f\, |Df|$$

for φ Lipschitz

'Leibniz rule'

$$|D(fg)| \leq |f||Dg| + |g||Df|$$

for $f, g \in S^2 \cap L^\infty$

The Sobolev space $W^{1,2}(X, d, \boldsymbol{m})$

The Sobolev space is defined as

$$W^{1,2}(X, d, \boldsymbol{m}) := L^2(X, \boldsymbol{m}) \cap S^2(X, d, \boldsymbol{m})$$

endowed with the norm

$$\|f\|_{W^{1,2}} := \sqrt{\|f\|_{L^2}^2 + \||Df|\|_{L^2}^2}$$

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Differentials

Given $f : \mathbb{R}^d \to \mathbb{R}$ smooth, its differential $Df : \mathbb{R}^d \to T^*\mathbb{R}^d$ is intrinsically defined by

$$Df(x)(v) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}, \qquad \forall x \in \mathbb{R}^d,\ v \in T_x\mathbb{R}^d$$

Gradients

To define the gradient of a smooth f we need more structure: a norm.

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A way to get it is starting from the observation that for any tangent vector w it holds

$$Df(x)(w) \leq \|Df(x)\|_* \|w\| \leq \frac{1}{2}\|Df(x)\|_*^2 + \frac{1}{2}\|w\|^2.$$

Then we can say that $v = \nabla f(x)$ provided $=$ holds, or equivalently

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Rmk.
Uniqueness holds iff the norm is strictly convex

Linearity holds iff the norm comes from a scalar product.

An important identity

$$\max_{v \in \nabla g(x)} Df(v) = \inf_{\varepsilon > 0} \frac{\|D(g+\varepsilon f)\|_*^2(x) - \|Dg\|_*^2(x)}{2\varepsilon}$$

$$\min_{v \in \nabla g(x)} Df(v) = \sup_{\varepsilon < 0} \frac{\|D(g+\varepsilon f)\|_*^2(x) - \|Dg\|_*^2(x)}{2\varepsilon}.$$

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The object $D^{\pm} f(\nabla g)$

For $f, g \in S^2$, the functions $D^{\pm} f(\nabla g) : X \to \mathbb{R}$ are defined by

$$D^{+} f(\nabla g) := \inf_{\varepsilon > 0} \frac{|D(g + \varepsilon f)|^2 - |Dg|^2}{2\varepsilon}$$

$$D^{-} f(\nabla g) := \sup_{\varepsilon < 0} \frac{|D(g + \varepsilon f)|^2 - |Dg|^2}{2\varepsilon}$$

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Notice that

$$D^- f(\nabla g) \leq D^+ f(\nabla g), \qquad \boldsymbol{m} - a.e.$$

$$|D^{\pm} f(\nabla g)| \leq |Df||Dg| \in L^1(X, \boldsymbol{m}),$$

$$D^+(-f)(\nabla g) = -D^- f(\nabla g) = D^+ f(\nabla(-g)), \qquad \boldsymbol{m} - a.e.$$

Calculus rules

Locality

$$D^{\pm} f(\nabla g) = D^{\pm} \tilde{f}(\nabla \tilde{g}), \qquad \boldsymbol{m}\text{-a.e. on } \{f = \tilde{f}\} \cap \{g = \tilde{g}\}$$

Calculus rules

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Chain rule

$$\begin{aligned} D^{\pm}(\varphi \circ f)(\nabla g) &= \varphi' \circ f \, D^{\pm \mathrm{sign}(\varphi' \circ f)} f(\nabla g), \\ D^{\pm} f(\nabla(\varphi \circ g)) &= \varphi' \circ g \, D^{\pm \mathrm{sign}(\varphi' \circ g)} f(\nabla g) \end{aligned}$$

for φ Lipschitz

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for φ Lipschitz

Leibniz rule

$$D^{+}(f_1 f_2)(\nabla g) \leq f_1\, D^{\ \mathrm{sign}(f_1)} f_2(\nabla g) + f_2\, D^{\ \mathrm{sign}(f_2)} f_1(\nabla g),$$
$$D^{-}(f_1 f_2)(\nabla g) \geq f_1\, D^{-\mathrm{sign}(f_1)} f_2(\nabla g) + f_2\, D^{-\mathrm{sign}(f_2)} f_1(\nabla g)$$

For $f_1, f_2 \in S^2 \cap L^\infty$, and $g \in S^2$.

Special situations

$(X, d, \boldsymbol{m})$ is infinitesimally strictly convex provided

$$D^+ f(\nabla g) = D^- f(\nabla g), \qquad \boldsymbol{m} - a.e.$$

for any $f, g \in S^2$. In this case the common value will be denoted by $Df(\nabla g)$. For $g \in S^2$ the map

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is linear.

$(X, d, \boldsymbol{m})$ is infinitesimally Hilbertian if $f \mapsto \int |Df|^2 \, d\boldsymbol{m}$ is a quadratic form. In this case

$$D^+ f(\nabla g) = D^- f(\nabla g) = D^+ g(\nabla f) = D^- g(\nabla f), \qquad \boldsymbol{m} - a.e.$$

and we denote these quantities by $\nabla f \cdot \nabla g$.

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Plan representing gradients: definition

For $g \in S^2$ and $\boldsymbol{\pi} \in \mathscr{P}(C([0,1],X))$ test plan it holds

$$\varlimsup_{t\downarrow 0} \int \frac{g(\gamma_t) - g(\gamma)}{t}\, d\boldsymbol{\pi} \leq \frac{1}{2} \int |Dg|^2(\gamma_0)\, d\boldsymbol{\pi} + \varlimsup_{t\downarrow 0} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2\, ds\, d\boldsymbol{\pi}$$

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We say that $\boldsymbol{\pi}$ *represents* ∇g, provided it holds

$$\varliminf_{t \downarrow 0} \int \frac{g(\gamma_t) - g(\gamma)}{t} \, d\boldsymbol{\pi} \geq \frac{1}{2} \int |Dg|^2(\gamma_0) \, d\boldsymbol{\pi} + \varlimsup_{t \downarrow 0} \frac{1}{2t} \iint_0^t |\dot{\gamma}_s|^2 \, ds \, d\boldsymbol{\pi}$$

Plan representing gradients: existence

Theorem (G. '12, Ambrosio-G.-Savaré '11).
For $g \in S^2$ and $\mu \in \mathscr{P}(X)$ such that $\mu \leq C\boldsymbol{m}$, a plan $\boldsymbol{\pi}$ representing ∇g and such that ${e_0}_\sharp \boldsymbol{\pi} = \mu$ exists.

First order differentiation formula

Let $f, g \in S^2$, and $\boldsymbol{\pi}$ which represents ∇g.
Then

$$\begin{aligned} &\overline{\lim_{t\downarrow 0}} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, d\boldsymbol{\pi} \\ \geq &\underline{\lim_{t\downarrow 0}} \int \frac{f(\gamma_t) - f(\gamma_0)}{t} \, d\boldsymbol{\pi} \end{aligned}$$

First order differentiation formula

Let $f, g \in S^2$, and $\boldsymbol{\pi}$ which represents ∇g.
Then

$$\begin{aligned}\int D^+ f(\nabla g)(\gamma_0)\, d\boldsymbol{\pi} &\geq \varlimsup_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t}\, d\boldsymbol{\pi} \\ &\geq \varliminf_{t\downarrow 0} \int \frac{f(\gamma_t) - f(\gamma_0)}{t}\, d\boldsymbol{\pi} \geq \int D^- f(\nabla g)(\gamma_0)\, d\boldsymbol{\pi}\end{aligned}$$

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Integrating a function to get improved regularity

Theorem (G. '13) Let:

- $(X, \mathsf{d}, \boldsymbol{m})$ be infinitesimally Hilbertian,
- $t \mapsto \mu_t$ a W_2-geodesic such that
  - $\cup_t \operatorname{supp}(\mu_t)$ bounded
  - $\mu_t \leq C\boldsymbol{m}$ for every $t \in [0, 1]$
  - with densities continuous in L^p for some (and thus any) $p < \infty$
- $f \in S^2(X, \mathsf{d}, \boldsymbol{m}) \cap L^1(X, \boldsymbol{m})$

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- $f \in S^2(X, \mathrm{d}, \boldsymbol{m}) \cap L^1(X, \boldsymbol{m})$

$$\text{Then the map} \qquad t \quad \mapsto \quad \int f \, d\mu_t \qquad \text{is } C^1$$

and its derivative is given by

$$\frac{d}{dt} \int f \, d\mu_t = - \int \nabla f \cdot \nabla \varphi_t \, d\mu_t$$

where $\frac{\varphi_t}{1-t}$ is any Kantorovich potential from μ_t to μ_1.

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Distributional Laplacian

Let $(X, d, \boldsymbol{m})$ be infinitesimally strictly convex, $\Omega \subset X$ open, $g \in S^2(\Omega)$

We say that $g \in D(\Delta, \Omega)$ if there exists a Radon measure μ on Ω such that

$$-\int_\Omega Df(\nabla g)\, d\boldsymbol{m} = \int_\Omega f\, d\mu,$$

holds for every f Lipschitz in $L^1(|\mu|)$ with $\boldsymbol{m}(\mathrm{supp}(f)) < \infty$ with support contained in Ω. In this case we put $\Delta g_{|_\Omega} := \mu$

Calculus rules

Chain rule

$$\Delta(\varphi \circ g) = \varphi' \circ g \, \Delta g + \varphi'' \circ g |Dg|^2 \boldsymbol{m}$$

On infinitesimally Hilbertian spaces:

Linearity

$$\Delta(g_1 + g_2) = \Delta g_1 + \Delta g_2$$

Leibniz rule

$$\Delta(g_1 g_2) = g_1 \Delta g_2 + g_2 \Delta g_1 + 2\nabla g_1 \cdot \nabla g_2 \boldsymbol{m}$$

Nonlinear potential theory ($p = 2$)

It is the study of 2-minimizers, defined as those $g \in S^2$ such that

$$\int_\Omega |Dg|^2 \, d\boldsymbol{m} \leq \int_\Omega |D(g+f)|^2 \, d\boldsymbol{m},$$

for every $f \in S^2$ with compact support contained in Ω.

Typically under the assumptions that the measure is doubling and the space supports a 2-Poincaré inequality.

Relation with nonlinear potential theory

Theorem (G. Mondino '12) Let $(X, d, \boldsymbol{m})$ be a doubling space supporting a 2-Poincaré inequality and infinitesimally strictly convex.

Then g is a 2-minimizer on Ω if and only if $g \in D(\Delta, \Omega)$ and $\Delta g_{|_\Omega} = 0$.
Similar results hold for sub/super-minimizers.

Relation with nonlinear potential theory

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Then g is a 2-minimizer on Ω if and only if $g \in D(\Delta, \Omega)$ and $\Delta g_{|_\Omega} = 0$. Similar results hold for sub/super-minimizers.

In particular, sub/super-minimizer have the sheaf property.

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Laplacian comparison

On a Riemannian manifold M with $Ric \geq 0$, $\dim \leq N$ it holds

$$\Delta \frac{1}{2} d^2(\cdot, \overline{x}) \leq N$$

in the sense of distributions.

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Does the same hold on abstract spaces?

Laplacian comparison

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Does the same hold on abstract spaces?

Theorem (G. '12) Let $(X, \mathrm{d}, \boldsymbol{m})$ be an infinitesimally strictly convex $CD(0, N)$ space and $\overline{x} \in X$. Then

$$\Delta \frac{\mathrm{d}^2(\cdot, \overline{x})}{2} \leq N\boldsymbol{m}$$

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Versions of the splitting theorem

Splitting (Cheeger-Gromoll '71)
Let M be a Riemannian manifold with $\mathrm{Ric} \geq 0$ which contains a line.
Then $M = N \times \mathbb{R}$ for some Riemannian manifold N.

Versions of the splitting theorem

Splitting (Cheeger-Gromoll '71)
Let M be a Riemannian manifold with $\mathrm{Ric} \geq 0$ which contains a line.
Then $M = N \times \mathbb{R}$ for some Riemannian manifold N.

Almost splitting (Cheeger-Colding '96)
Let M be a Riemannian manifold with $\mathrm{Ric} \geq -\varepsilon$ which contains a geodesic with length L. $\varepsilon, L^{-1} \ll 1$
Then 'a big portion of M is GH-close to a product'

An equivalent version of the almost splitting

Let (M_n) be a sequence of Riemannian manifolds with uniformly bounded dimension, $\mathrm{Ric}(M_n) \geq -\varepsilon_n$ containing lines γ_n of length L_n, with $\varepsilon_n, L_n^{-1} \to 0$.

Assume that (M_n) converges in the pGH-sense to a limit space (X, d).

Then (X, d) splits.

Non-smooth splitting

Question: Let $(X, \mathrm{d}, \boldsymbol{m})$ be a $CD(0, N)$ space containing a line.

Non-smooth splitting

Question: Let $(X, \mathsf{d}, \boldsymbol{m})$ be a $CD(0, N)$ space containing a line.

Can we say that there exists a $CD(0, N-1)$ space $(X', \mathsf{d}', \boldsymbol{m}')$ such that $(X, \mathsf{d}, \boldsymbol{m})$ is isomorphic to

$$(X' \times \mathbb{R}, \mathsf{d}' \otimes \mathsf{d}_{\mathrm{Eucl}}, \boldsymbol{m}' \times \mathcal{L}^1),$$

where

$$(\mathsf{d}' \otimes \mathsf{d}_{\mathrm{Eucl}})\big((x', t), (y', s)\big) := \sqrt{\mathsf{d}'(x', y')^2 + |t - s|^2}$$

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Answer: No. (Cordero Erasquin-Villani-Sturm '06)

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Answer: No. (Cordero Erasquin-Villani-Sturm '06)

Theorem (G. '13)
The answer become yes if we add the assumption that the space is infinitesimally Hilbertian.

Don’t forget the stability

The non-smooth splitting would be of little use if we don’t know that

$$CD(0,N) + \text{infinitesimal Hilbertianity}$$

is stable.

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is stable.

This is true and follows from the study of the heat flow
G. ’09
G.-Kuwada-Ohta ’10
Ambrosio-G.-Savaré ’11 I-II
Ambrosio-G.-Mondino-Rajala ’12
G.-Mondino-Savaré ’13

Thank you