Geometric implications of Poincaré inequalities in metric measure spaces

Estibalitz Durand Cartagena

UNED (Spain)
Dpto. de Matemática Aplicada

IPAM program “Interactions between Analysis and Geometry”
Workshop I: Analysis on metric spaces
IPAM, UCLA
• Jesús A. Jaramillo (Universidad Complutense de Madrid)
• Nages Shanmugalingam (University of Cincinnati)
• Alex Williams (Texas Tech University)
Analysis on metric spaces

- Zero Order Calculus
- Spaces of Homogeneous type: doubling measures (Coifman-Weiss 70’s)
  - Lebesgue points
  - Vitali covering
  - Maximal operator
- Doubling spaces with Poincaré inequalities (Heinonen-Koskela 98)
  - Cheeger Differentiable structure
  - First order Sobolev-type spaces
- First order calculus
Lipschitz function spaces

$(X, d)$ metric space

**Definition**
A function $f : X \to \mathbb{R}$ is **Lipschitz** if there is a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C \, d(x, y) \quad \forall x, y \in X.$$
Lipschitz function spaces

$(X, d)$ metric space

**Definition**
A function $f : X \to \mathbb{R}$ is **Lipschitz** if there is a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C d(x, y) \quad \forall x, y \in X.$$  

★ $\text{LIP}(X) = \{f : X \to \mathbb{R} : f \text{ is Lipschitz}\}$

★ $\text{LIP}^\infty(X) = \{f : X \to \mathbb{R} : f \text{ is Lipschitz and bounded}\}$

$$\|f\|_{\text{LIP}^\infty} = \|f\|_{\infty} + \text{LIP}(f)$$
Pointwise Lipschitz function spaces

Definition
Given a function \( f : X \to \mathbb{R} \) the pointwise Lipschitz constant of \( f \) at \( x \in X \) is defined as

\[
\text{Lip} f(x) = \limsup_{y \to x, y \neq x} \frac{|f(x) - f(y)|}{d(x, y)}.
\]
Pointwise Lipschitz function spaces

**Definition**
Given a function $f : X \to \mathbb{R}$ the pointwise Lipschitz constant of $f$ at $x \in X$ is defined as

$$\text{Lip}_f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

**Example**
If $f \in C^1(\Omega)$, $\Omega^{\text{op}} \subset \mathbb{R}^n$ (or of a Riemannian manifold), then

$$\text{Lip}_f(x) = |\nabla f(x)| \quad \forall x \in \Omega.$$
Doubling measures

\((X, d, \mu)\) metric measure space, \(\mu\) Borel regular measure
Doubling measures

$(X, d, \mu)$ metric measure space, $\mu$ Borel regular measure

**Definition**

$\mu$ is **doubling** if $\exists C > 0$ constant such that

$$0 < \mu(B(x, 2r)) \leq C \mu(B(x, r)) < \infty \quad \forall x \in X, r > 0.$$ 

- $X$ complete + $\mu$ doubling $\implies X$ proper
Doubling measures

\((X, d, \mu)\) metric measure space, \(\mu\) Borel regular measure

**Definition**
\(\mu\) is **doubling** if \(\exists \, C > 0\) constant such that

\[0 < \mu(B(x, 2r)) \leq C \mu(B(x, r)) < \infty \quad \forall \, x \in X, r > 0.\]

- \(X\) complete + \(\mu\) doubling \(\implies\) \(X\) proper

**Definition**
A **curve** in \(X\) is a continuous mapping \(\gamma : [a, b] \to X\).
A **rectifiable curve** is a curve with finite length.
Examples

\begin{itemize}
  \item \((\mathbb{R}^n, | \cdot |, \mathcal{L}^n) \quad C = 2^n\)
  \item \((C, | \cdot |, \mathcal{H}^{\log_2^3})\)
  \item \([[0, 1], |x - y|^{1/2}, \mathcal{H}^2)\)
\end{itemize}

\[f(x) = x\]

\[\frac{|f(x) - f(y)|}{|x - y|^{1/2}} = |x - y|^{1/2} \xrightarrow{y \to x} 0\]
Sierpiński carpet

\[ Q_0 = [0, 1]^2 \]
Sierpiński carpet

$Q_1$
Sierpiński carpet $Q_2$
Sierpiński carpet

$Q_3$
Sierpiński carpet

$Q_4$
Sierpiński carpet
Sierpiński carpet: $S_3 = (X, d, \mu)$

$$d = d_{e|X}$$

Equally distributing unit mass over $Q_n$ leads to a natural probability doubling measure $\mu$ on $S_3$. ($\mu$ is comparable to $\mathcal{H}^s$, $s = \frac{\log 8}{\log 3}$).
Classical Poincaré inequality

One way to view the Fundamental Theorem of Calculus is:

\[
\text{infinitesimal data} \leadsto \text{local control}
\]
Classical Poincaré inequality

One way to view the Fundamental Theorem of Calculus is:

\[
\text{infinitesimal data } \rightsquigarrow \text{ local control}
\]

This principle can apply in very general situation in the form of a Poincaré inequality:

\[
\exists C = C(n) > 0: \forall B \equiv B(x, r) \subset \mathbb{R}^n \forall f \in W^{1,p}(\mathbb{R}^n) \int_B |f - f_B| \, dL^n \leq C(n) r \left( \int_B |\nabla f| \, dL^n \right)^{1/p}
\]
Classical Poincaré inequality

One way to view the Fundamental Theorem of Calculus is:

\[ \text{infinitesimal data} \leadsto \text{local control} \]

This principle can apply in very general situation in the form of a Poincaré inequality:

\[ \exists C = C(n) > 0: \forall B \equiv B(x, r) \subset \mathbb{R}^n \forall f \in W^{1,p}(\mathbb{R}^n) \]

\[ \int_B |f - f_B| d\mathcal{L}^n \leq C(n) r \left( \int_B |\nabla f|^p d\mathcal{L}^n \right)^{1/p} \]

Notation:

\[ \int_B f d\mathcal{L}^n = f_B = \frac{1}{\mathcal{L}^n(B)} \int_B f d\mathcal{L}^n \]
Poincaré inequalities in metric measure spaces

\((X, d, \mu)\) metric measure space
Poincaré inequalities in metric measure spaces

$(X, d, \mu)$ metric measure space

**Definition (Heinonen-Koskela 98)**

A non-negative Borel function $g$ on $X$ is an upper gradient for $f : X \to \mathbb{R} \cup \{\pm \infty\}$ if

$$|f(x) - f(y)| \leq \int_{\gamma} g,$$

$\forall x, y \in X$ and every rectifiable curve $\gamma_{xy}$.

**Examples**

- $g \equiv \infty$ is an upper gradient of every function on $X$.
- If there are no rectifiable curves in $X$ then $g \equiv 0$ is an upper gradient of every function.
- If $f \in \text{LIP}(X)$ then $g \equiv \text{LIP}(f)$ and $g(x) = \text{Lip} f(x)$ are upper gradients for $f$. 
**p-Poincaré inequality**

**Definition (Heinonen-Koskela 98)**

Let $1 \leq p < \infty$. We say that $(X, d, \mu)$ supports a **weak $p$-Poincaré inequality** if there exist constants $C_p > 0$ and $\lambda \geq 1$ such that for every Borel measurable function $f : X \rightarrow \mathbb{R}$ and every upper gradient $g : X \rightarrow [0, \infty]$ of $f$, the pair $(f, g)$ satisfies the inequality

$$\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C_p \, r \left( \int_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p}$$

\forall B(x, r) \subset X.

**Notation:**

$$\int_B f \, d\mu = f_B = \frac{1}{\mu(B)} \int_B f \, d\mu$$
Examples

- $(\mathbb{R}^n, | \cdot |, \mathcal{L}^n)$
- Riemannian manifolds with non-negative Ricci curvature
- Heisenberg group with its Carnot-Carathéodory metric and Haar measure $\rightsquigarrow$ Subriemannian geometry
- Boundaries of certain hyperbolic buildings: Bourdon-Pajot spaces $\rightsquigarrow$ Geometric group theory
- Laakso spaces, …
Geometric implications of $p$-Poincaré inequalities

- $X$ is connected
Geometric implications of $p$-Poincaré inequalities

- $X$ is connected
- **Semmes 98** $p < \infty$

\[
\begin{align*}
X \text{ complete } &p\text{-PI} \\
\mu \text{ doubling} &
\end{align*}
\implies X \text{ is quasiconvex}
\]
Geometric implications of $p$-Poincaré inequalities

- $X$ is connected
- Semmes 98 $p < \infty$

\[
\begin{aligned}
X \text{ complete } p\text{-PI} & \\
\mu \text{ doubling} & \\
\implies \quad X \text{ is quasiconvex}
\end{aligned}
\]

**Definition**
A metric space $(X, d)$ is **quasiconvex** if there exists a constant $C \geq 1$ such that for each pair of points $x, y \in X$, there exists a curve $\gamma$ connecting $x$ and $y$ with

\[\ell(\gamma) \leq Cd(x, y).\]
Geometric implications of $p$-Poincaré inequalities

- $X$ is connected
- Semmes 98 $p < \infty$

\[
\begin{align*}
X \text{ complete } p\text{-PI} & \quad \mu \text{ doubling} \\
\implies X \text{ is quasiconvex}
\end{align*}
\]

\(\nexists\) $(S_3, d, \mu)$ is quasiconvex but does not admit any $p$-PI
Geometric implications of $p$-Poincaré inequalities

- $X$ is connected
- *Semmes 98* $p < \infty$

\[
\left\{ \begin{array}{c}
X \text{ complete } p\text{-PI} \\
\mu \text{ doubling}
\end{array} \right\} \implies X \text{ is quasiconvex}
\]

\[
\nexists (S_3, d, \mu) \text{ is quasiconvex but does not admit any } p\text{-PI}
\]

- *Heinonen-Koskela 98, Kinnunen-Latvala 02, Saloff-Coste 02, Keith 03, Miranda 03, Korte 07, ....*
Counterexample

- \((S_3, d, \mu)\) does not admit a 1-PI
Counterexample

- \((S_3, d, \mu)\) does not admit a 1-PI

Let \(T_n\) be the vertical strip of width \(3^{-n}\).
Counterexample

$T_1$
Counterexample

\[ T_2 \]
Counterexample

$T_3$
Counterexample

\[
\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C r \left( \int_{B(x,r)} g^p \, d\mu \right)^{1/p}
\]

Define \( f_n \in \text{LIP}(S_3) \) such that \( \int_{S_3} |f_n - (f_n)_{S_3}| \, d\mu > C \) but

\[
\int_{S_3} \text{lip}(f_n) \, d\mu = 3^n \cdot \mu(T_n) = 3^n \cdot \frac{2^n}{8^n} \to 0 \quad (n \to \infty)
\]
Counterexample

- \((S_3, d, \mu)\) does not admit any \(p\)-PI

**Bourdon-Pajot 02** Let \((X, d, \mu)\) be a bounded metric measure space with \(\mu\) doubling and \(p\)-PI, and let \(f : X \to I\) be a surjective Lipschitz function from \(X\) onto an interval \(I \subset \mathbb{R}\). Then, \(\mathcal{L}_I^1 \ll f#\mu\). Here \(f#\mu\) denotes the push-forward measure of \(\mu\) under \(f\).

**Proof.**

Let \(f\) be the projection on the horizontal axis. It can be checked that \(f#\mu \perp \mathcal{L}^1\).

**Question** Higher dimensions?
Generalized Sierpinski carpets: $S_a$

$$a = (a_1^{-1}, a_2^{-1}, \ldots) \in \left\{ \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \right\}^\mathbb{N}$$
Generalized Sierpinski carpets: $S_a$

$$a = (a_1^{-1}, a_2^{-1}, \ldots) \in \left\{ \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \right\}^\mathbb{N}$$

For $a = \left(\frac{1}{a}, \frac{1}{a}, \frac{1}{a}, \ldots\right)$ $a$ odd,

- $S_a$ does not admit any $p$–PI
Generalized Sierpinski carpets: $S_a$

\[ a = (a_1^{-1}, a_2^{-1}, \ldots) \in \left\{ \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots \right\}^\mathbb{N} \]

For \( a = \left( \frac{1}{a}, \frac{1}{a}, \frac{1}{a}, \ldots \right) a \text{ odd}, \)

- $S_a$ does not admit any $p$-PI

Mackay, Tyson, Wildrick (To appear)

- $(S_a, d, \mu)$ supports a 1-PI if and only if $a \in \ell^1$
- $(S_a, d, \mu)$ supports a $p$-PI for some $p > 1$ if and only if $a \in \ell^2$
Which is the role of the exponent $p$?

\[
\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq Cr \left( \int_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p}
\]

Hölder inequality: $p$-PI $\Rightarrow$ $q$-PI for $q \geq p$
Which is the role of the exponent \( p \)?

\[
\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq Cr \left( \int_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p}
\]

Hölder inequality: \( p\)-PI \( \implies \) \( q\)-PI for \( q \geq p \)

Federer-Fleming, Maz'ya 60 Miranda 03

\((\mathbb{R}^n)\) \( p = 1 \iff \) Isoperimetric inequality
Which is the role of the exponent $p$?

$$\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq Cr \left( \int_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p}$$

**Hölder inequality**: $p$-PI $\implies$ $q$-PI for $q \geq p$

Federer-Fleming, Maz’ya 60 Miranda 03

$(\mathbb{R}^n)$ $p = 1$ $\iff$ Isoperimetric inequality

**Example**

$X := \{(x, y) \in \mathbb{R}^2 : x \geq 0, 0 \leq y \leq x^m\}$

$(X, | \cdot |, \mathcal{L}^2|_X)$ $X$ has $p$–PI $\iff$

$p > m + 1$
Glueing spaces together

Heinonen-Koskela 98

Suppose $X$ and $Y$ are locally compact $Q$–regular metric measure spaces and that $A$ is a closed subset of $X$ that has an isometric copy inside $Y$. Suppose there are numbers $Q \geq s > Q − p$ and $C \geq 1$ so that $H_s(\mathcal{E})(A \cap B_R) \geq C^{-1}R^s$ for all balls $B_R$ either in $X$ or in $Y$ that are centered at $A$ with radius $0 < R < \min\{\text{diam}X, \text{diam}Y\}$.

If $X$ and $Y$ admit a $p$- PI $\implies X \cup_A Y$ admits a $p$- PI.
What happens when $p \to \infty$?

\[
\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq Cr \left( \int_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p}
\]

Hölder inequality: $p$-PI $\implies$ $q$-PI for $q \geq p$
What happens when $p \to \infty$?

\[
\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq Cr \left( \int_{B(x,\lambda r)} g^p \, d\mu \right)^{1/p}
\]

Hölder inequality: $p$-PI $\implies q$-PI for $q \geq p$

Definition

$(X, d, \mu)$ supports a **weak $\infty$-Poincaré inequality** if there exist constants $C > 0$ and $\lambda \geq 1$ such that for every function $f : X \to \mathbb{R}$ and every upper gradient $g$ of $f$, the pair $(f, g)$ satisfies

\[
\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq Cr \|g\|_{L^\infty(B(x,\lambda r))}
\]

$\forall B(x, r) \subset X.$
$X$ complete and $\infty$-PI\n$\mu$ doubling\n\\implies\quad X$ is quasiconvex
\\iff

- $d = d_e|_X, \mu = \mathcal{H}^s, s = \frac{\log 8}{\log 3}$
- $(X, d)$ is quasiconvex
- $(X, d, \mu)$ does not admit any $p$-PI, $1 \leq p \leq \infty$

Sierpiński carpet
Modulus of a family of curves

Definition
Let $\Gamma \subset \Upsilon = \{\text{non constant rectifiable curves of } X\}$ and $1 \leq p \leq \infty$. For $\Gamma \subset \Upsilon$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_{\gamma} \rho \geq 1 \quad \text{for all } \gamma \in \Gamma.$$ 

$$\operatorname{Mod}_p(\Gamma) = \begin{cases} \inf_{\rho \in F(\Gamma)} \int_X \rho^p \, d\mu, & \text{if } p < \infty \\ \inf_{\rho \in F(\Gamma)} \|\rho\|_{L^\infty}, & \text{if } p = \infty \end{cases}$$

If some property holds for all curves $\gamma \in \Upsilon \setminus \Gamma$, where $\operatorname{Mod}_p \Gamma = 0$, then we say that the property holds for $p$–a.e. curve.

Remark
$\operatorname{Mod}_p$ is an outer measure
Lemma
Let $\Gamma \subset \Upsilon$ and $1 \leq p \leq \infty$. The following conditions are equivalent:
(a) $\mathrm{Mod}_p \Gamma = 0$.
(b) There exists a Borel function $0 \leq \rho \in L^p(X)$ such that $\int_{\gamma} \rho = +\infty$, for each $\gamma \in \Gamma$ and $\|\rho\|_{L^\infty} = 0$.

Examples
$\mathbb{R}^n, n \geq 2$
**$p$-“thick” quasiconvexity**

**Definition**

$(X, d, \mu)$ is a $p$-“thick” quasiconvex space if there exists $C \geq 1$ such that $\forall x, y \in X$, $0 < \varepsilon < \frac{1}{4}d(x, y)$,

$$\text{Mod}_p(\Gamma(B(x, \varepsilon), B(y, \varepsilon), C)) > 0,$$

where $\Gamma(B(x, \varepsilon), B(y, \varepsilon), C)$ denotes the set of curves $\gamma_{p,q}$ connecting $p \in B(x, \varepsilon)$ and $q \in B(y, \varepsilon)$ with $\ell(\gamma_{p,q}) \leq Cd(p, q)$. 

![Diagram of quasiconvex space](image)
Geometric characterization: $p = \infty$

D-C, Jaramillo, Shanmugalingam 11
Let $(X, d, \mu)$ be a complete metric space with $\mu$ doubling. Then,

\[
X \text{ is } \infty\text{-"thick" quasi-convex } \iff X \text{ admits } \infty\text{-PI}
\]

Remark
If $\mu \sim \lambda \implies \text{Mod}_\infty(\Gamma, \mu) = \text{Mod}_\infty(\Gamma, \lambda)$
\[
\implies (X, d, \mu) \text{admits } \infty\text{-PI if and only if } (X, d, \lambda) \text{admits } \infty\text{-PI}
\]
Let $(X, d, \mu)$ be a connected complete metric space supporting a doubling Borel measure $\mu$. Then

$$\text{LIP}^\infty(X) = N_1^\infty(X) \text{ with c.e.s. } \iff X \text{ admits } \infty\text{-PI}$$
Geometric implications of $p$-PI

$X$ complete and $p$-PI
\[ \mu \text{ doubling} \]
\[ \implies X \text{ is } p-\text{"thick" quasiconvex} \]

Remarks

- $p$-"thick" quasiconvex $\implies$ quasiconvex
- The characterization is no longer true for $p < \infty$

Question Are there $\infty$-thick qc spaces which are not $p$-thick qc for any $p < \infty$?
A counterexample

\[ \mu = \sum_j \chi_{Q_j} \cdot \mu_j \text{ doubling measure} \]

- \( X \) is \( p \)-thick quasi-convex \( 1 \leq p \leq \infty \) \( \implies \) \( \infty \)-PI
- \( X \) admits an \( \infty \)-PI but does **not** admit any \( p \)-PI \( (1 \leq p < \infty) \)
Persistence of $p$-PI under GH-limits

Cheeger 99
If $\{X_n, d_n, \mu_n\}_n$ with $\mu_n$ doubling measures supporting a $p$-PI $p < \infty$ (with constants uniformly bounded), and $\{X_n, d_n, \mu_n\}_n \xrightarrow{G-H} (X, d, \mu)$, then $(X, d, \mu)$ has $\mu$ doubling and supports a $p$-PI.

Corollary

The $\infty$-PI is non-stable under measured Gromov-Hausdorff limits.
Keith-Zhong 08 If $X$ is a complete metric space equipped with a doubling measure satisfying a $p$-Poincaré inequality for some $1 < p < \infty$, then there exists $\varepsilon > 0$ such that $X$ supports a $q$-Poincaré inequality for all $q > p - \varepsilon$. 
∞-admissible weights

Definition

$w \geq 0, w \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a $p$-admissible weight with $p \geq 1$ if the measure $\mu$ given by $d\mu = wd\mathcal{L}^n$ is doubling, and $(\mathbb{R}^n, | \cdot |, \mu)$ admits a weak $p$-Poincaré inequality.

Muckenhoupt-Wheeden 74

$1 \leq p < \infty \quad w \in A_p \implies w$ is $p$-admissible

Remark

$A_\infty$ weights are $\infty$-admissible

$$A_\infty = \bigcup_{p>1} A_p$$
A counterexample

$(\mathbb{R}, | \cdot |, \mu)$ admits an $\infty$-PI but \textbf{no} $p$-PI

In $\mathbb{R}$, the Riesz product

$$d\nu(x) = \prod_{k=1}^{\infty} (1 + a \cos(3^k \cdot 2\pi x)) d\mathcal{L}^1(x) \quad |a| < 1$$

is a doubling measure and $\nu \perp \mathcal{L}^1$.

Idea construct a sequence of weights $w_k, k \geq 1$ such that $w_k d\mathcal{L}^1$ “approximates” $d\nu$ better as $k \to \infty$:

$$w(x) = w_1(x) \chi_{(-\infty,2]}(x) + \sum_{k=2}^{\infty} w_k(x-k) \chi_{[k,k+1]}(x),$$

and $d\mu = w d\mathcal{L}^1$. 
Measured differentiable structures

$X$ complete, $\mu$ doubling

Cheeger 99

$X$ supports $p$-PI

$1 \leq p < \infty \implies X$ admits a “differentiable structure”

Keith 04

$X$ satisfies Lip-lip $\implies X$ admits a “differentiable structure”
Lip-lip condition

“The infinitesimal behaviour at a generic point is essentially independent of the scales used for the blow-up at that point”

Definition

$X$ satisfies Lip-lip if $\exists C > 0$ such that $\forall f \in \text{LIP}(X)$,

$$\text{Lip} f(x) \leq C \text{lip} f(x) \quad \mu\text{-a.e.} x$$

Here,

$$\text{Lip} f(x) := \limsup_{r \to 0} \sup_{0 < d(y, x) < r} \frac{|f(y) - f(x)|}{r},$$

and

$$\text{lip} f(x) := \liminf_{r \to 0} \sup_{0 < d(y, x) < r} \frac{|f(y) - f(x)|}{r}.$$ 

Remark

If $\mu \sim \lambda \implies (X, d, \mu)$ has Lip-lip iff $(X, d, \lambda)$ has Lip-lip
Lip-lip condition

Keith 02

\[ X \text{ complete and } p\text{-PI } \]
\[ \mu \text{ doubling } \]
\[ \implies X \text{ has the Lip-lip condition} \]
Lip-lip condition

Keith 02

\[ X \text{ complete and } p\text{-PI } \mu \text{ doubling } \]
\[ \implies X \text{ has the Lip-lip condition} \]
Lip-lip condition

Keith 02

$X$ complete and $p$-PI $\mu$ doubling $\implies X$ has the Lip-lip condition

Proof.

For $\mu$-a.e. $x$,

$$\frac{1}{C} \text{Lip} f(x) \leq \limsup_{r \to 0} \frac{1}{r} \int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu$$

$$\leq L \limsup_{r \to 0} \left( \int_{B(x,r)} \text{lip} f(x)^p \, d\mu \right)^{\frac{1}{p}} = L \text{lip} f(x)$$
Question

\[ X \text{ complete and } \infty\text{-PI } \mu \text{ doubling } \Rightarrow X \text{ has the Lip-lip condition } \]

Bate (Preprint 12), Gong (Preprint 12)

\[ X \text{ satisfies } \sigma-\text{Lip-lip } \mu \text{ pointwise doubling } \iff X \text{ admits a “differentiable structure”} \]
Thank you for your attention!