Geometric implications of Poincaré inequalities in metric measure spaces

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IPAM program "Interactions between Analysis and Geometry"
Workshop I: Analysis on metric spaces

IPAM, UCLA

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- Nages Shanmugalingam (University of Cincinnati)
- Alex Williams (Texas Tech University)

Analysis on metric spaces

- Zero Order Calculus
  - **Spaces of Homogeneous type: doubling measures (Coifman-Weiss 70’s)**
    - Lebesgue points
    - Vitali covering
    - Maximal operator
- First order calculus
  - **Doubling spaces with Poincaré inequalities (Heinonen-Koskela 98)**
    - Cheeger Differentiable structure
    - First order Sobolev-type spaces

Lipschitz function spaces

(X, d) metric space

Definition

A function $f : X \longrightarrow \mathbb{R}$ is Lipschitz if there is a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C\, d(x, y) \quad \forall x, y \in X.$$

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- ⋆ $\mathrm{LIP}(X) = \{f : X \longrightarrow \mathbb{R} : f \text{ is Lipschitz}\}$
- ⋆ $\mathrm{LIP}^{\infty}(X) = \{f : X \longrightarrow \mathbb{R} : f \text{ is Lipschitz and bounded}\}$

$$\|f\|_{\mathrm{LIP}^{\infty}} = \|f\|_{\infty} + \mathrm{LIP}(f)$$

Pointwise Lipschitz function spaces

Definition

Given a function $f : X \to \mathbb{R}$ the pointwise Lipschitz constant of f at $x \in X$ is defined as

$$\operatorname{Lip} f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x,y)}.$$

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Example

If $f \in C^1(\Omega)$, $\Omega \overset{\text{op}}{\subset} \mathbb{R}^n$ (or of a Riemannian manifold), then

$$\operatorname{Lip} f(x) = |\nabla f(x)| \quad \forall x \in \Omega.$$

Doubling measures

(X, d, μ) metric measure space, μ Borel regular measure

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$$0 < \mu(B(x, 2r)) \leq C\, \mu(B(x, r)) < \infty \quad \forall\, x \in X, r > 0.$$

- X complete + μ doubling $\Longrightarrow$ X proper

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Definition

A curve in X is a continuous mapping $\gamma : [a, b] \to X$.
A rectifiable curve is a curve with finite length.

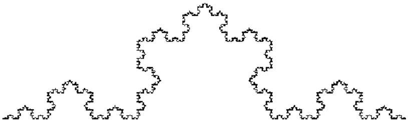
Examples

- $(\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$ $C = 2^n$
- $(C, |\cdot|, \mathscr{H}^{\frac{log2}{log3}})$

- $([0,1], |x-y|^{1/2}, \mathscr{H}^{2})$

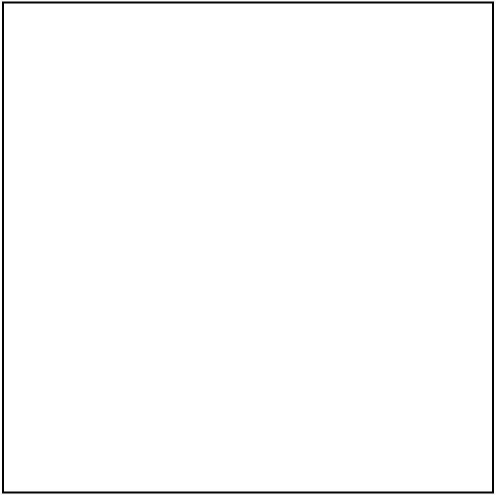
$f(x) = x$

$$\frac{|f(x)-f(y)|}{|x-y|^{1/2}} = |x-y|^{1/2} \xrightarrow{y \to x} 0$$



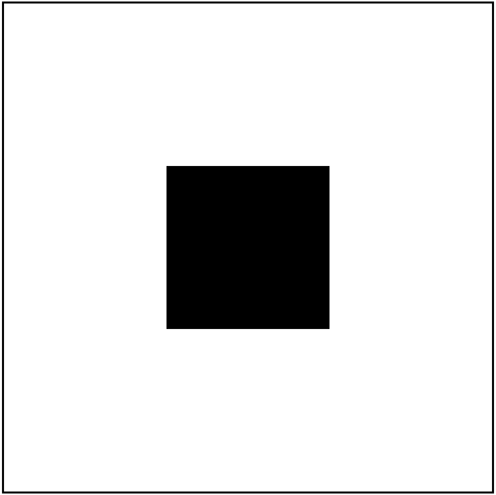
Sierpiński carpet

$$Q_0 = [0,1]^2$$



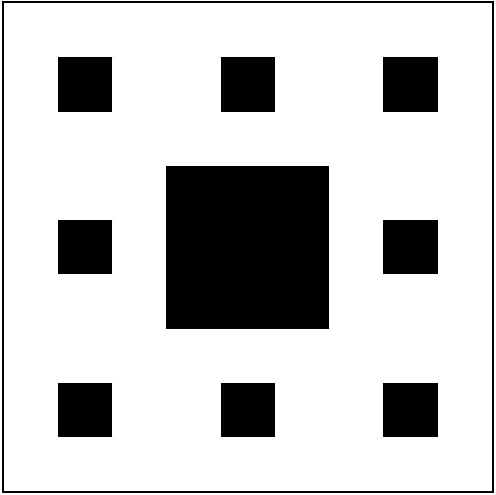
Sierpiński carpet

Q_1



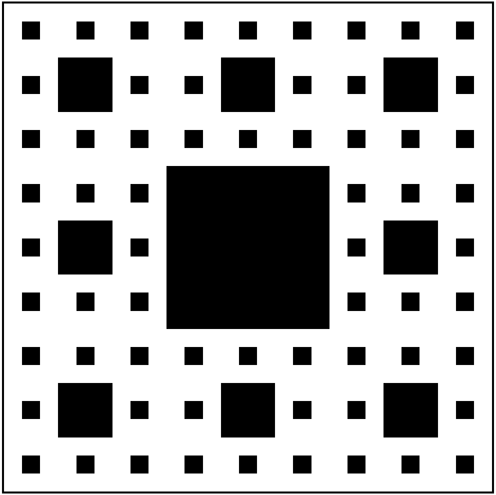
Sierpiński carpet

Q_2



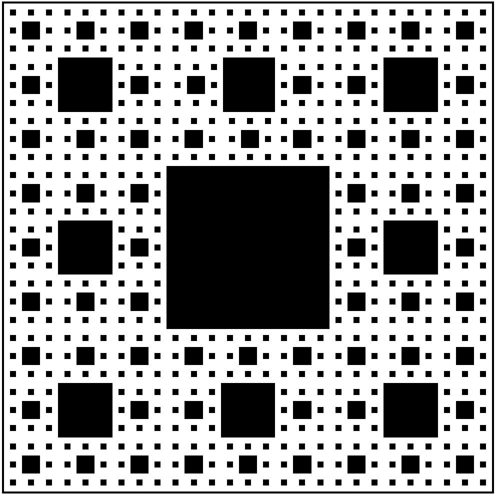
Sierpiński carpet

Q_3



Sierpiński carpet

Q_4

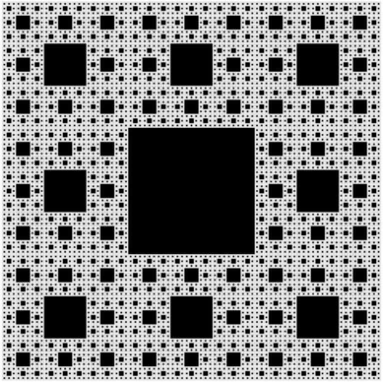


Sierpiński carpet

...

Sierpiński carpet: $S_3 = (X, d, \mu)$

$d = d_{e|X}$



Equally distributing unit mass over Q_n leads to a natural probability doubling measure μ on S_3.
(μ is comparable to $\mathcal{H}^s, s = \frac{\log 8}{\log 3}$).

Classical Poincaré inequality

One way to view the Fundamental Theorem of Calculus is:

$$\boxed{\text{infinitesimal data} \rightsquigarrow \text{local control}}$$

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This principle can apply in very general situation in the form of a Poincaré inequality:

$$\exists\, C = C(n) > 0\colon \forall B \equiv B(x,r) \subset \mathbb{R}^n\ \forall f \in W^{1,p}(\mathbb{R}^n)$$

$$\fint_B |f - f_B|\, d\mathscr{L}^n \leq C(n)\, r \left(\fint_B |\nabla f|^p\, d\mathscr{L}^n \right)^{1/p}$$

<u>Notation</u>:

$$\fint_B f\, d\mathscr{L}^n = f_B = \frac{1}{\mathscr{L}^n(B)} \int_B f\, d\mathscr{L}^n$$

Poincaré inequalities in metric measure spaces

(X, d, μ) metric measure space

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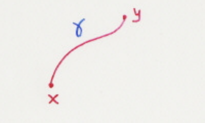
Definition (*Heinonen-Koskela 98*)

A non-negative Borel function g on X is an upper gradient for $f : X \to \mathbb{R} \cup \{\pm\infty\}$ if

$$|f(x) - f(y)| \leq \int_\gamma g,$$



$\forall x, y \in X$ and every rectifiable curve γ_{xy}.

Examples

- $g \equiv \infty$ is an upper gradient of every function on X.
- If there are no rectifiable curves in X then $g \equiv 0$ is an upper gradient of every function.
- If $f \in \mathrm{LIP}(X)$ then $g \equiv \mathrm{LIP}(f)$ and $g(x) = \mathrm{Lip}\, f(x)$ are upper gradients for f.

p-Poincaré inequality

Definition (*Heinonen-Koskela 98*)

Let $1 \le p < \infty$.We say that (X, d, μ) supports a weak p-Poincaré inequality if there exist constants $C_p > 0$ and $\lambda \ge 1$ such that for every Borel measurable function $f : X \to \overline{\mathbb{R}}$ and every upper gradient $g : X \to [0, \infty]$ of f, the pair (f, g) satisfies the inequality

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \le C_p \, r \Big(\fint_{B(x,\lambda r)} g^p \, d\mu \Big)^{1/p}$$

$\forall \, B(x, r) \subset X$.

Notation:

$$\fint_B f \, d\mu = f_B = \frac{1}{\mu(B)} \int_B f \, d\mu$$

Examples

- $(\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$
- Riemannian manifolds with non-negative Ricci curvature
- Heisenberg group with its Carnot-Carathéodory metric and Haar measure $\rightsquigarrow$ Subriemannian geometry
- Boundaries of certain hyperbolic buildings: Bourdon-Pajot spaces$\rightsquigarrow$ Geometric group theory
- Laakso spaces, ...

Geometric implications of p-Poincaré inequalities

- X is connected

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- X is connected
- Semmes 98 $\quad p < \infty$

$$\left.\begin{array}{r} X \text{ complete } p\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ is quasiconvex}$$

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Definition

A metric space (X, d) is quasiconvex if there exists a constant $C \geq 1$ such that for each pair of points $x, y \in X$, there exists a curve γ connecting x and y with

$$\ell(\gamma) \leq Cd(x, y).$$

Geometric implications of p-Poincaré inequalities

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$\boxed{\not\Leftarrow}$ (S_3, d, μ) is quasiconvex but does not admit any p-PI

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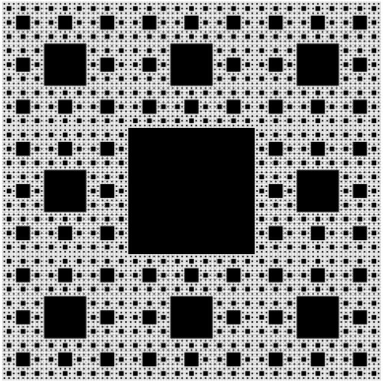
- Heinonen-Koskela 98, Kinnunen-Latvala 02, Saloff-Coste 02, Keith 03, Miranda 03, Korte 07, ...

Counterexample

- (S_3, d, μ) does not admit a 1-PI



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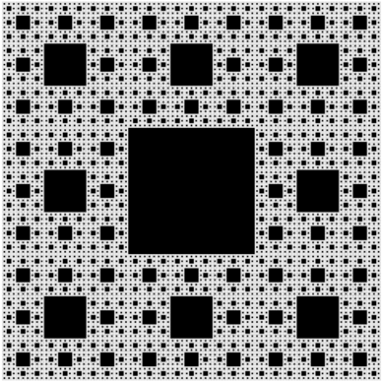
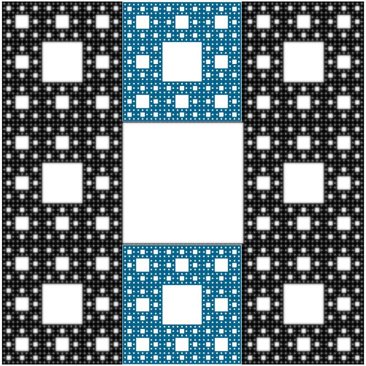


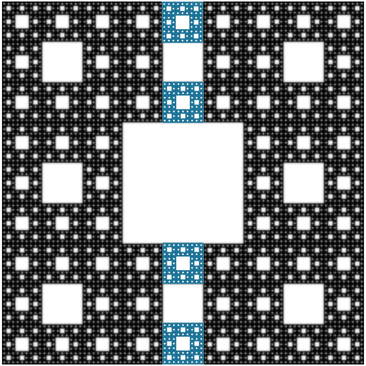
Let T_n be the vertical strip of width 3^{-n}.

Counterexample



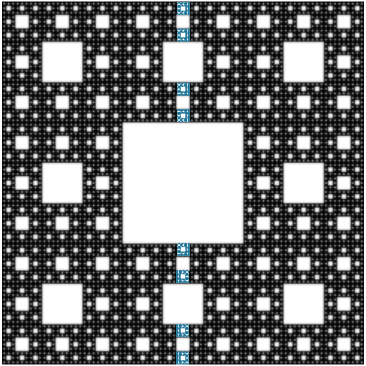
T_1

Counterexample



T_2

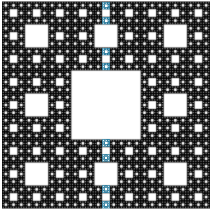
Counterexample



T_3

Counterexample

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \le C\, r \Big(\fint_{B(x,r)} g^p d\mu \Big)^{1/p}$$



T_n

Define $f_n \in \mathrm{LIP}(S_3)$ such that $\fint_{S_3} |f_n - (f_n)_{S_3}| d\mu > C$ but

$$\int_{S_3} \mathrm{lip}(f_n) d\mu = 3^n \cdot \mu(T_n) = 3^n \cdot \frac{2^n}{8^n} \to 0 \ (n \to \infty)$$

Counterexample

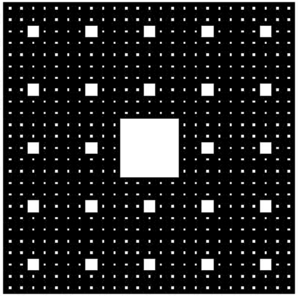
- (S_3, d, μ) does not admit any p-PI

Bourdon-Pajot 02 Let (X, d, μ) be a bounded metric measure space with μ doubling and p-PI, and let $f : X \longrightarrow I$ be a surjective Lipschitz function from X onto an interval $I \subset \mathbb{R}$. Then, $\mathscr{L}^1_{|I} \ll f_\# \mu$. Here $f_\# \mu$ denotes the push-forward measure of μ under f.

Proof.
Let f be the projection on the horizontal axis. It can be checked that $f_\# \mu \perp \mathscr{L}^1$. $\square$

Question Higher dimensions?

Generalized Sierpinski carpets: $S_{\mathbf{a}}$



$$\mathbf{a} = (a_1^{-1}, a_2^{-1}, \ldots) \in \left\{ \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{7}, \ldots \right\}^{\mathbb{N}}$$

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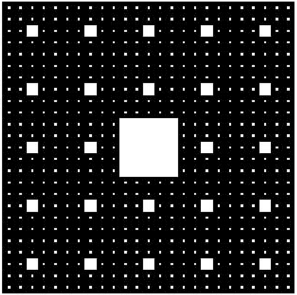

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For $\quad \mathbf{a} = \left(\dfrac{1}{a}, \dfrac{1}{a}, \dfrac{1}{a}, \ldots \right)$ a odd,

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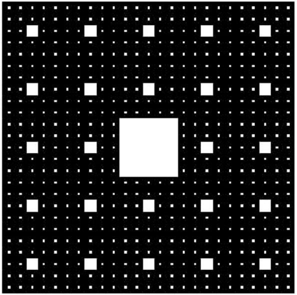
Generalized Sierpinski carpets: $S_{\mathbf{a}}$



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- $S_{\mathbf{a}}$ does not admit any $p-$PI

Mackay, Tyson, Wildrick (To appear)

- $(S_{\mathbf{a}}, d, \mu)$ supports a 1-PI if and only if $\mathbf{a} \in \ell^1$
- $(S_{\mathbf{a}}, d, \mu)$ supports a p-PI for some $p > 1$ if and only if $\mathbf{a} \in \ell^2$

Which is the role of the exponent p?

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \le Cr \Big(\fint_{B(x,\lambda r)} g^{p} \, d\mu \Big)^{1/p}$$

Hölder inequality: p-PI$\Longrightarrow q$-PI for $q \ge p$

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Federer-Fleming, Mazýa 60 Miranda 03
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Federer-Fleming, Mazýa 60 Miranda 03

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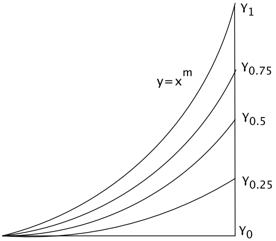

Example

$$X := \{(x, y) \in \mathbb{R}^2 \,:\, x \ge 0, 0 \le y \le x^m\}$$

$(X, |\cdot|, \mathscr{L}^2{}_{|X})$ X has p−PI $\iff$ $p > m + 1$

Glueing spaces together

Heinonen-Koskela 98

Suppose X and Y are locally compact $Q-$regular metric measure spaces and that A is a closed subset of X that has an isometric copy inside Y. Suppose there are numbers $Q \geq s > Q - p$ and $C \geq 1$ so that $\mathcal{H}_s^\infty(A \cap B_R) \geq C^{-1}R^s$ for all balls B_R either in X or in Y that are centered at A with radius $0 < R < \min\{\mathrm{diam} X, \mathrm{diam} Y\}$.

$$\text{If } X \text{ and } Y \text{ admit a } p\text{- PI} \Longrightarrow X \cup_A Y \text{ admits a } p\text{- PI}.$$

What happens when $p \to \infty$?

$$⨍_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \le Cr \left(⨍_{B(x,\lambda r)} g^{p} d\mu \right)^{1/p}$$

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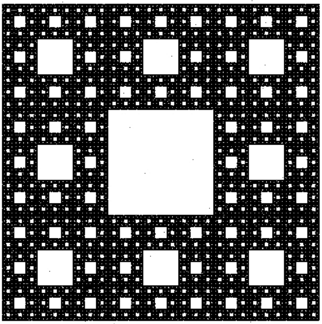
Definition
(X, d, μ) supports a weak ∞-Poincaré inequality if there exist constants $C > 0$ and $\lambda \ge 1$ such that for every function $f : X \to \mathbb{R}$ and every upper gradient g of f, the pair (f, g) satisfies

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \le Cr \|g\|_{L^\infty(B(x,\lambda r))}$$

$\forall \, B(x, r) \subset X$.

$$\left.\begin{array}{r} X \text{ complete and } \infty\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ is quasiconvex}$$

$\boxed{\not\Leftarrow}$



Sierpiński carpet

- $d = d_{e|X} \quad \mu = \mathcal{H}^s, s = \frac{\log 8}{\log 3}$
- (X, d) is quasiconvex
- (X, d, μ) does $\boxed{\text{not}}$ admit any p-PI, $1 \le p \le \infty$

Modulus of a family of curves

Definition

Let $\Gamma \subset \Upsilon = \{$non constant rectifiable curves of $X\}$ and $1 \le p \le \infty$. For $\Gamma \subset \Upsilon$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho \ge 1 \quad \text{for all } \gamma \in \Gamma.$$

$$\mathrm{Mod}_p(\Gamma) = \begin{cases} \inf_{\rho \in F(\Gamma)} \int_X \rho^p \, d\mu, & \text{if } p < \infty \\ \inf_{\rho \in F(\Gamma)} \|\rho\|_{L^\infty}, & \text{if } \boxed{p = \infty} \end{cases}$$

If some property holds for all curves $\gamma \in \Upsilon \backslash \Gamma$, where $\mathrm{Mod}_p \Gamma = 0$, then we say that the property holds for $p-$a.e. curve.

Remark

Mod_p is an outer measure

Lemma

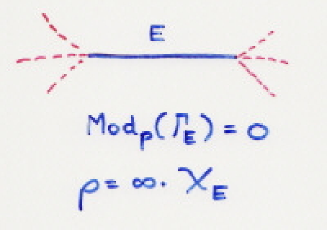
Let $\Gamma \subset \Upsilon$ *and* $1 \le p \le \infty$. *The following conditions are equivalent:*

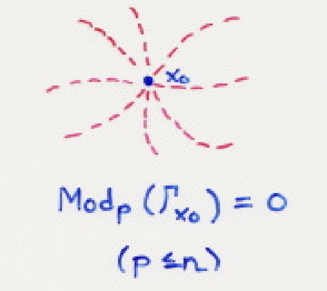
(a) $\operatorname{Mod}_p \Gamma = 0$.

(b) *There exists a Borel function* $0 \le \rho \in L^p(X)$ *such that* $\int_\gamma \rho = +\infty$, *for each* $\gamma \in \Gamma$ *and* $\|\rho\|_{L^\infty} = 0$.

Examples

$\mathbb{R}^n, n \ge 2$





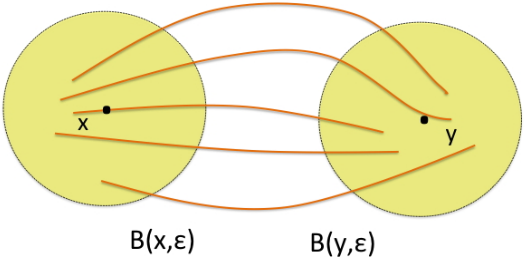
p-"thick" quasiconvexity

Definition

(X, d, μ) is a p-"thick" quasiconvex space if there exists $C \geq 1$ such that $\forall\, x, y \in X$, $0 < \varepsilon < \frac{1}{4}d(x, y)$,

$$\mathrm{Mod}_p(\Gamma(B(x, \varepsilon), B(y, \varepsilon), C)) > 0,$$

where $\Gamma(B(x, \varepsilon), B(y, \varepsilon), C)$ denotes the set of curves $\gamma_{p,q}$ connecting $p \in B(x, \varepsilon)$ and $q \in B(y, \varepsilon)$ with $\ell(\gamma_{p,q}) \leq Cd(p, q)$.



Geometric characterization: $p = \infty$

D-C, Jaramillo, Shanmugalingam 11

Let (X, d, μ) be a complete metric space with μ doubling. Then,

$$\boxed{X \text{ is } \infty\text{-“thick” quasi-convex} \iff X \text{ admits } \infty\text{-PI}}$$

Remark

If $\mu \sim \lambda \Longrightarrow \mathrm{Mod}_\infty(\Gamma, \mu) = \mathrm{Mod}_\infty(\Gamma, \lambda)$

$\Longrightarrow (X, d, \mu)$admits ∞-PI if and only if (X, d, λ)admits ∞-PI

Analytic characterization

D-C, Jaramillo, Shanmugalingam 11 Let (X, d, μ) be a connected complete metric space supporting a doubling Borel measure μ. Then

$$\boxed{\mathrm{LIP}^\infty(X) = N^{1,\infty}(X) \text{ with c.e.s.} \iff X \text{ admits } \infty\text{-PI}}$$

Geometric implications of p-PI

$$\left.\begin{array}{r} X \text{ complete and } p\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ is } p\text{-“thick” quasiconvex}$$

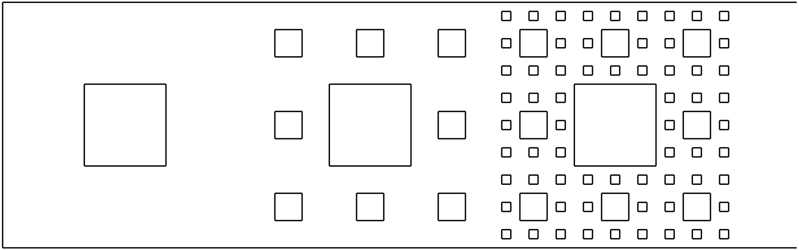
Remarks

- p-“thick” quasiconvex $\Longrightarrow$ quasiconvex
- The characterization is no longer true for $p < \infty$ $\boxed{\not\Leftarrow}$

Question Are there ∞-thick qc spaces which are not p-thick qc for any $p < \infty$?

A counterexample

$\mu = \sum_j \chi_{Q_j} \cdot \mu_j$ doubling measure



- X is p-thick quasi-convex $1 \leq p \leq \infty \implies \infty$-PI
- X admits an ∞-PI but does not admit any p-PI ($1 \leq p < \infty$)

Persistence of p-PI under GH-limits

Cheeger 99

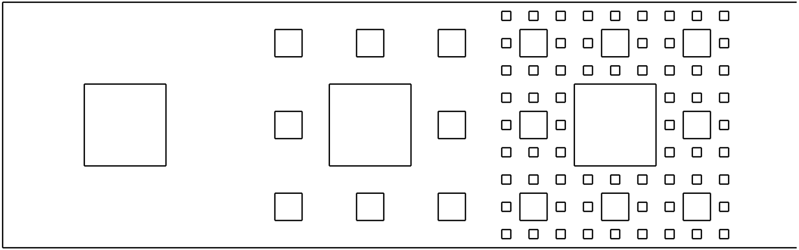
If $\{X_n, d_n, \mu_n\}_n$ with μ_n doubling measures supporting a p-PI $\boxed{p < \infty}$ (with constants uniformly bounded), and $\{X_n, d_n, \mu_n\}_n \xrightarrow{G-H} (X, d, \mu)$, then (X, d, μ) has μ doubling and supports a p-PI.

Corollary

The ∞-PI is non-stable under measured Gromov-Hausdorff limits.

Not Self-improvement of ∞-PI

Keith-Zhong 08 If X is a complete metric space equipped with a doubling measure satisfying a p-Poincaré inequality for some $1 < p < \infty$, then there exists $\varepsilon > 0$ such that X supports a q-Poincaré inequality for all $q > p - \varepsilon$.



∞-admissible weights

Definition

$w \geq 0$, $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a p-admissible weight with $p \geq 1$ if the measure μ given by $d\mu = w d\mathscr{L}^n$ is doubling, and $(\mathbb{R}^n, |\cdot|, \mu)$ admits a weak p-Poincaré inequality.

Muckenhoupt-Wheeden 74

$1 \leq p < \infty \quad w \in A_p \Longrightarrow w$ is p-admissible

Remark

$\mathcal{A}_\infty$ weights are ∞-admissible

$$\mathcal{A}_\infty = \bigcup_{p>1} A_p \qquad \boxed{\not\Leftarrow}$$

A counterexample

$(\mathbb{R}, |\cdot|, \mu)$ admits an ∞-PI but no p-PI

In $\mathbb{R}$, the *Riesz product*

$$d\nu(x) = \prod_{k=1}^{\infty} (1 + a\cos(3^k \cdot 2\pi x)) d\mathscr{L}^1(x) \quad |a| < 1$$

is a doubling measure and $\nu \perp \mathscr{L}^1$.

Idea construct a sequence of weights w_k, $k \geq 1$ such that $w_k d\mathscr{L}^1$ "approximates" $d\nu$ better as $k \to \infty$:

$$w(x) = w_1(x)\chi_{(-\infty,2]}(x) + \sum_{k=2}^{\infty} w_k(x-k)\,\chi_{[k,k+1]}(x),$$

and $d\mu = w\, d\mathscr{L}^1$.

Measured differentiable structures

X complete, μ doubling

Cheeger 99

$$\left.\begin{array}{r} X \text{ supports } p\text{-PI} \\ 1 \le p < \infty \end{array}\right\} \Longrightarrow X \text{ admits a ``differentiable structure''}$$

Keith 04

X satisfies Lip-lip $\Longrightarrow$ X admits a"differentiable structure"

Lip-lip condition

"The infinitesimal behaviour at a generic point is essentially independent of the scales used for the blow-up at that point"

Definition

X satisfies Lip-lip if $\exists\, C > 0$ such that $\forall f \in \mathrm{LIP}(X)$,

$$\mathrm{Lip}\, f(x) \leq C\, \mathrm{lip}\, f(x) \quad \mu\text{-}a.e.x$$

Here,

$$\mathrm{Lip}\, f(x) := \limsup_{r\to 0} \sup_{0<d(y,x)<r} \frac{|f(y)-f(x)|}{r},$$

and

$$\mathrm{lip}\, f(x) := \liminf_{r\to 0} \sup_{0<d(y,x)<r} \frac{|f(y)-f(x)|}{r}.$$

Remark

If $\mu \sim \lambda \implies (X, d, \mu)$ has Lip-lip iff (X, d, λ) has Lip-lip

Lip-lip condition

Keith 02

$$\left.\begin{array}{r} X \text{ complete and } p\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \Longrightarrow X \text{ has the Lip-lip condition}$$

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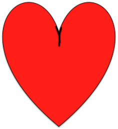
$\nLeftarrow$



Lip-lip condition

Keith 02

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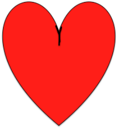

Proof.

For μ-a.e.x,

$$\frac{1}{C}\operatorname{Lip} f(x) \leq \limsup_{r\to 0} \frac{1}{r} \fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu$$

$$\leq L \limsup_{r\to 0} \left(\fint_{B(x,r)} \operatorname{lip} f(x)^p d\mu \right)^{\frac{1}{p}} = L \operatorname{lip} f(x)$$

$\square$

Question

$$\left.\begin{array}{r} X \text{ complete and } \infty\text{-PI} \\ \mu \text{ doubling} \end{array}\right\} \overset{?}{\Longrightarrow} X \text{ has the Lip-lip condition}$$

Bate (Preprint 12), Gong (Preprint 12)

$$\left.\begin{array}{r} X \text{ satisfies } \sigma-\text{Lip-lip} \\ \mu \text{ pointwise doubling} \end{array}\right\} \iff X \text{ admits a“differentiable structure”}$$

Thank you for your attention!