A stability theorem for elliptic Harnack inequalities

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The classical Harnack inequality

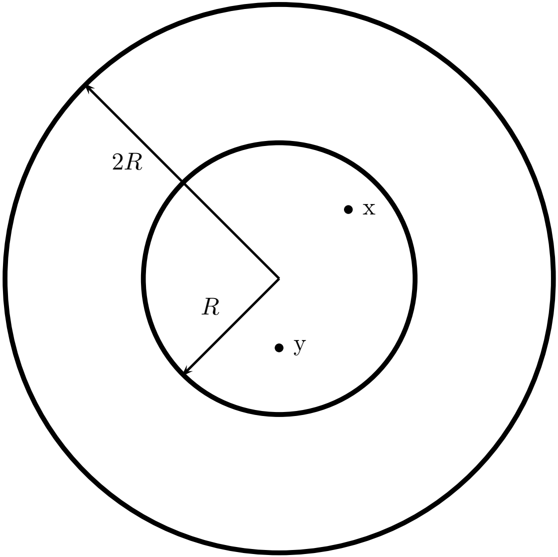
A function h is harmonic in a domain D if

$$\Delta h(x) = \sum_{i=1}^{d} \frac{\partial^2 h}{\partial x_i^2}(x) = 0, \qquad x \in D.$$

The (elliptic) Harnack inequality says that if h is harmonic and non-negative in $B(x_0, 2R)$, then all the values of h in $B(x_0, R)$ are comparable:

There exists c not depending on h, R, or x_0 such that

$$h(x) \le c h(y), \qquad x, y \in B(x_0, R).$$


2R
x
R
y

If we are in the case where the state space is $\mathbb{R}^2 = \mathbb{C}$ and if f is analytic in a domain D, then $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are harmonic there.

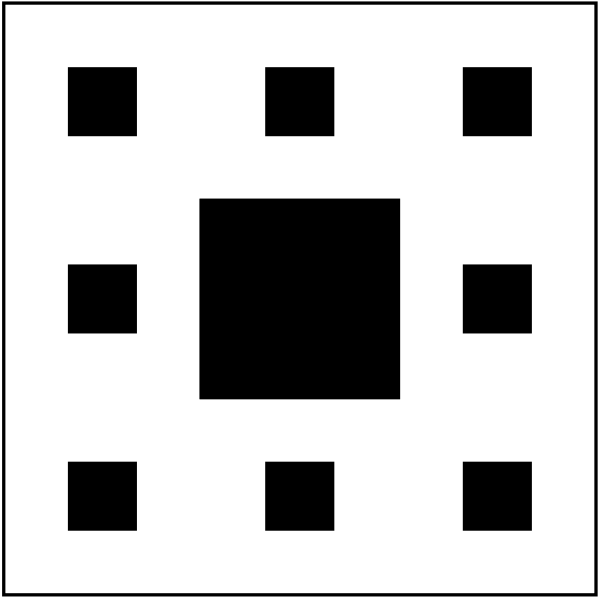
The subject of this talk

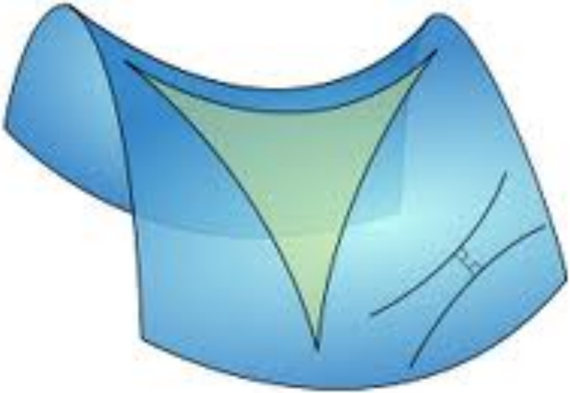
The general question we consider is:

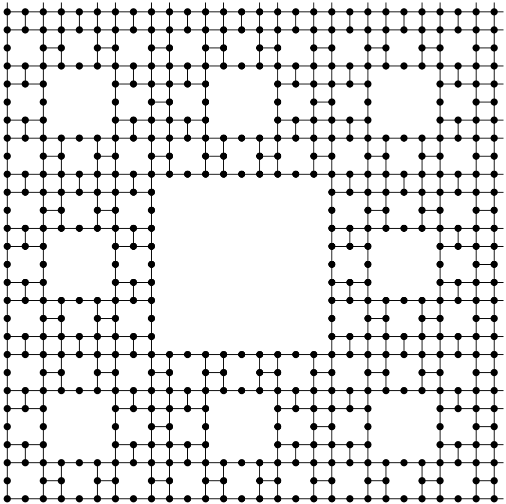
When does the Harnack inequality hold?

We might want to consider other elliptic operators than the Laplacian. And we might want to consider other state spaces than $\mathbb{R}^d$, such as fractals, manifolds, infinite graphs.

This has applications to PDE, geometry, analysis, mathematical physics, and probability.







If we have a Markov process X_t and we let

$$h(x) = \mathbb{E}^x f(X_{\tau_D}),$$

then h is harmonic, provided we replace the Laplacian by the infinitesimal generator of X. Here f is a function on the boundary of a domain D, τ_D is the first exit time from D, and $\mathbb{E}^x$ is the expectation starting at x.

For example, suppose $f = 1_A$ for A a subset of the boundary of D. If there is positive probability that X leaves D through A when started at x, the Harnack inequality guarantees that there is positive probability of leaving D through A when the process is started at points near x.


A

The Harnack inequality implies that harmonic functions are continuous, so in fact $\mathbb{P}^x(X_{\tau_D} \in A)$ will usually be a continuous function.

Moser's theorem

A landmark paper in studying Harnack inequalities is that of Moser (1961). He considered functions that were harmonic with respect to the operator given by

$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(\cdot)\frac{\partial f}{\partial x_j}(\cdot)\Big)(x),$$

where the a_{ij} are uniformly elliptic (i.e., positive definite uniformly in x) and bounded.

A function h is harmonic if $\mathcal{L}h(x) = 0$ for $x \in D$.

Moser's theorem says that the Harnack inequality holds for functions that are non-negative and harmonic in a domain in $\mathbb{R}^d$.

Dirichlet forms

When the a_{ij} are only bounded and measurable, it is not even clear how to make sense of the operator $\mathcal{L}$. We have that

$$\mathcal{L}f = g$$

if

$$\int (\mathcal{L}f)\varphi = \int g\varphi$$

for all nice φ.

Note that

$$\int \varphi(\mathcal{L}f) = \int \sum_{i,j=1}^{d} \varphi \, \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial f}{\partial x_j}\Big)$$

$$= -\int \sum_{i,j=1}^{d} a_{ij}\frac{\partial f}{\partial x_j}\frac{\partial \varphi}{\partial x_i}.$$

We define $\mathcal{L}f = g$ in the weak sense by requiring

$$\int \sum_{i,j=1}^{d} \frac{\partial f}{\partial x_j}(x) a_{ij}(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx = -\int g(x)\varphi(x)\, dx$$

for all nice φ, say $\varphi \in C^1$ with compact support in D.

$$\mathcal{E}(f,h) = \int \sum_{i,j=1}^{d} \frac{\partial f}{\partial x_j}(x) a_{ij}(x) \frac{\partial h}{\partial x_i}(x)\, dx$$

is the Dirichlet form associated to $\mathcal{L}$.

A function is harmonic if

$$\mathcal{E}(f,\varphi) = 0$$

for all nice φ.

Note that

$$\mathcal{E}_{\mathcal{L}}(f,f) = \int \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x)\, dx$$

is equivalent to $\mathcal{E}_{\Delta}(f,f)$, which means

$$c_1 \mathcal{E}_{\Delta}(f,f) \le \mathcal{E}_{\mathcal{L}}(f,f) \le c_2 \mathcal{E}_{\Delta}(f,f)$$

for all f in the domain of $\mathcal{E}$. This is a consequence of the uniform ellipticity and boundedness of the matrix $a_{ij}(x)$.

Observe that

$$\mathcal{E}_{\Delta}(f,f) = \int |\nabla f(x)|^2\, dx.$$

Rephrasing the basic question

Our question about when the Harnack inequality holds can be split into two.

(1) What are conditions on the space and operator such that the Harnack inequality holds?

(2) If the Harnack inequality holds for $\mathcal{E}$ and $\mathcal{E}'$ is equivalent to $\mathcal{E}$, does the Harnack inequality hold for $\mathcal{E}'$?

The parabolic Harnack inequality

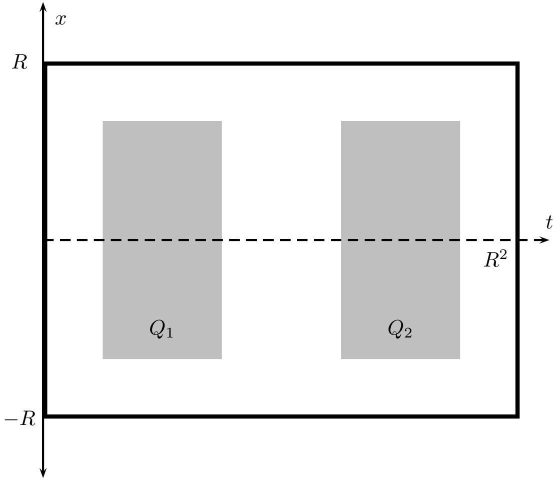
Moser (1964) also proved a parabolic Harnack inequality. If $u(x,t)$ is parabolic, which means that

$$\frac{\partial u}{\partial t}(x,t) - \mathcal{L}u(x,t) = 0$$

and u is non-negative in a larger domain, then we can say something about the values in a smaller domain.

We have

$$\sup_{Q_2} u \le c \inf_{Q_1} u.$$



x
R
t
R^2
Q_1
Q_2
$-R$

Hebisch and Saloff-Coste (2001) have studied the difference between the elliptic and parabolic Harnack inequality and discovered that it is quite narrow.

But the difference does exist. We will talk a bit more about this later.

Symmetric Markov chains

Now let's look at some other state spaces and operators to which we can address these questions.

Consider symmetric Markov chains on infinite graphs. It turns out that if we can answer our question on infinite graphs, then we also have our answer for all sorts of other domains, such as manifolds, fractals, metric measure spaces, and so on.

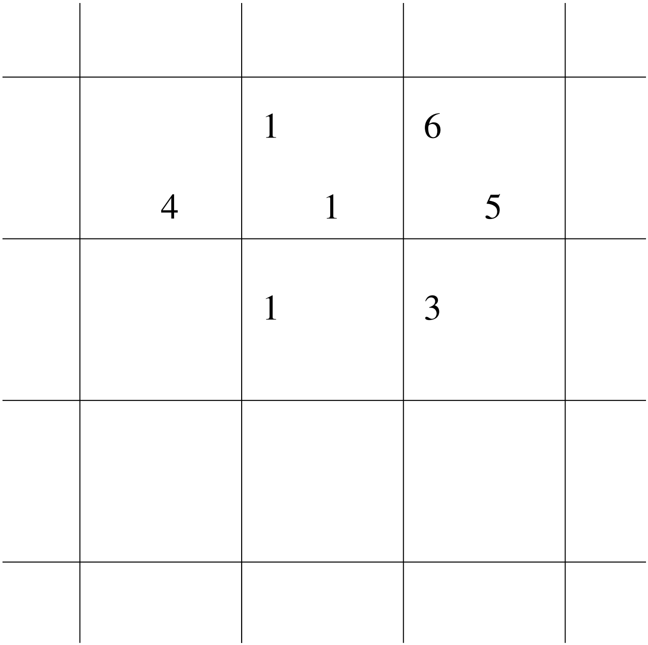
The infinite graph consists of infinitely many vertices and a collection of edges. We say $x \sim y$ if x and y are vertices and there is an edge connecting x and y. Given a point x, there is a conductance C_{xy} associated to every pair x, y with $x \sim y$, with $C_{xy} = C_{yx}$. Let $C_{xy} = 0$ if x and y do not form an edge.

Let

$$\mu_x = \sum_{\{z : z \sim x\}} C_{xz}.$$

Starting at x, we look at a Markov chain X that waits at x an exponential length of time and then jumps to a point y with probability

$$\mathbb{P}^x(X \text{ jumps to } y) = \frac{C_{xy}}{\mu_x}.$$


1
6
4
1
5
1
3

These are called symmetric Markov chains because their transition densities $p(x, y)$ with respect to $\mu(A) = \sum_{x \in A} \mu_x$ are symmetric:

$$p(x, y) = p(y, x).$$

The infinitesimal generator corresponding to the process X is

$$\mathcal{L}f(x) = \sum_{\{y:x\sim y\}} (f(y) - f(x))C_{xy},$$

and the associated Dirichlet form is

$$\mathcal{E}(f,g) = \sum_{y}\sum_{x}(f(y) - f(x))(g(y) - g(x))C_{xy}.$$

If we have two graphs consisting of the same vertices and same edges, but the conductances C_{xy}, C'_{xy} are different, then a sufficient condition for the corresponding Dirichlet forms to be equivalent is if there exist constants c_1, c_2 such that

$$c_1 C_{xy} \leq C'_{xy} \leq c_2 C_{xy}$$

for all x and y.

The main theorem of the present talk is that modulo some mild regularity conditions, if an elliptic Harnack inequality holds for $\mathcal{E}$ and $\mathcal{E}'$ is equivalent to $\mathcal{E}$, then the elliptic Harnack inequality holds for $\mathcal{E}'$.

A property is said to be stable if when it holds for one Dirichlet form, it holds for all equivalent Dirichlet forms.

We can thus say that the elliptic Harnack inequality is stable.

More on the parabolic Harnack inequality

Let's look at the parabolic Harnack inequality some more. Saloff-Coste (1992) and Grigor'yan (1992) independently proved that, provided the space satisfies some regularity conditions, then the parabolic Harnack inequality holds if and only if volume doubling and the Poincaré inequality hold.

A Dirichlet form $\mathcal{E}$ has associated with it a measure μ.

Volume doubling is a geometric condition:

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)),$$

where $B(x, r)$ is the ball of radius r about x and c does not depend on x or r.

The Poincaré inequality is an analytic inequality. In $\mathbb{R}^d$ it reads:

$$\int_{B(x_0,r)} |f(x) - \overline{f}|^2 \, dx \leq cr^2 \int_{B(x_0,r)} |\nabla f|^2 \, dx,$$

where

$$\overline{f} = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x) \, dx$$

and m is Lebesgue measure.

In general spaces, the Poincaré inequality reads

$$\int_{B(x_0,r)} |f(x) - \overline{f}|^2 \, \mu(dx) \leq cr^2 \mathcal{E}_{B(x_0,r)}(f,f),$$

where $\mathcal{E}_A$ is the analogue of $\int_A |\nabla f|^2$.

$\mathcal{E}_A$ can be shown to be the Dirichlet form of the process X with normal reflection on the boundary of A.

For symmetric chains, the Poincaré inequality reads

$$\sum_{x \in B(x_0,r)} |f(x) - \overline{f}|^2 \mu_x \leq cr^2 \sum_{x \in B(x_0,r)} \sum_{y \sim x} (f(y) - f(x))^2 C_{xy}.$$

Note that the Poincaré inequality is stable.

The regularity conditions needed on the space are that there are enough cut-off functions that are nice. More specifically, for each x and r there exists $\varphi = \varphi_{x,r}$ such that φ is 1 on $B(x, r)$, 0 on $B(x, 2r)^c$, and the L^∞ norm of $|\nabla\varphi|$ is comparable to the L^2 norm.

A nice consequence of the parabolic Harnack inequality is that one gets Gaussian type estimates. If $p(t,x,y)$ are the transition probability densities, which are the same thing as the fundamental solution to the heat equation $\partial_t u = \mathcal{L}u$, then

$$c_1 t^{-d/2} e^{-d(x,y)^2/c_2 t} \le p(t,x,y) \le c_3 t^{-d/2} e^{-d(x,y)^2/c_4 t}.$$

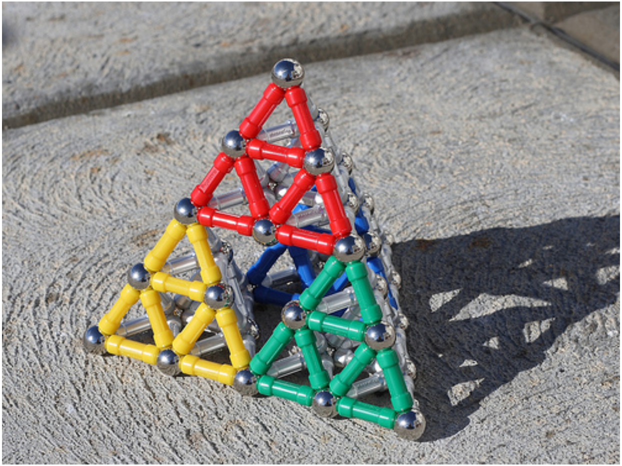
If the parabolic Harnack inequality holds, then the elliptic Harnack inequality holds. (Any harmonic function is parabolic.)

In Barlow-Bass (2004) it was shown that the converse does not hold. Part of the issue is that the r^2 scaling need not hold.

A more general type of Poincaré inequality is that

$$\int_{B(x_0,r)} |f(x) - \overline{f}|^2 \, \mu(dx) \leq cr^{\beta} \mathcal{E}_{B(x_0,r)}(f, f).$$

This shows up when studying fractals or when studying manifolds with many larger and larger obstructions.



It is not true that the parabolic Harnack inequality is equivalent to volume doubling and the Poincaré inequality in the case where not enough cut-off functions exist. Barlow-Bass showed that one needs an additional condition, the cut-off inequality.

For each x_0 and R there exists a function φ such that φ is 1 on $B(x, R)$, 0 on $B(x_0, 2R)^c$, is Hölder continuous:

$$|\varphi(x) - \varphi(y)| \le c\Big(\frac{d(x,y)}{R}\Big)^{\alpha},$$

and

$$\begin{aligned}\sum_{x,y\in B(x_0,s)} & f(x)^2|\varphi(x) - \varphi(y)|^2 C_{xy} \\ & \le c\Big(\frac{s}{R}\Big)^{\theta}\Big(\sum_{x,y\in B(x_0,2s)} |f(x) - f(y)|^2 C_{xy} \\ & \qquad + s^{-\beta} \sum_{x\in B(x_0,2s)} f(x)^2 \mu_x\Big)\end{aligned}$$

for all $s \le R$.

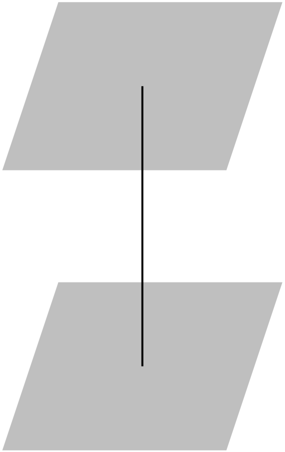
Note that this is stable.

We proved that the parabolic Harnack inequality (with parameter β) holds if and only if the Poincaré inequality (with parameter β), volume doubling, and the cut-off inequality hold.

One again gets Gaussian type estimates, but instead of $d(x, y)^2$ in the exponent, one gets $d(x, y)^{\beta/(\beta-1)}$.

A consequence of the characterization of the parabolic Harnack inequality in Barlow-Bass is that the parabolic Harnack inequality is stable. If it holds for $\mathcal{E}$ and $\mathcal{E}'$ is equivalent to $\mathcal{E}$, then the parabolic Harnack inequality holds for $\mathcal{E}'$.

For an example where the elliptic Harnack inequality does not hold, look at two copies of $\mathbb{R}^3$ with the origins of both copies connected by a one-dimensional line segment.



Adjusted Poincaré inequality

If $B(x, r)$ is the ball of radius r about x, let $V(x, r) = \mu(B(x, r))$ be the volume.

Let $C(x, r)$ be the capacity of the ball. In the case when our process is transient, this can be defined by

$$C(x, r) = \inf\{\mathcal{E}(f, f) : f = 1 \text{ on } B(x, r), f \text{ is } 0 \text{ at } \infty\}.$$

This is the same as the notion of capacity in electrical network theory.

When the process is not transient, there is a substitute we can use.

Let

$$E(x, r) = \frac{V(x, r)}{C(x, r)}.$$

When the state space is $\mathbb{R}^d$, then $V(x,r) \approx r^d$, $C(x,r) \approx r^{d-2}$, and so $E(x,r) \approx r^2$.

If the Harnack inequality does hold, it turns out that $E(x,r)$ is comparable to the expected amount of time for X_t to leave $B(x,r)$ when started at x.

We introduce the adjusted Poincaré inequality:

$$\int_{B(x_0,r)} |f(x) - \overline{f}|^2 \, \mu(dx) \leq c_1 E(x_0, r) \mathcal{E}_{B(x_0,2r)}(f, f)$$

for all x_0 and all r.

In the infinite graph case, it reads

$$\sum |f(x) - \overline{f}|^2 \mu_x \leq c_1 E(x_0, r) \sum_{x,y \in B(x_0,2r)} |f(y) - f(x)|^2 C_{xy}.$$

The adjusted Poincaré inequality is stable.

Assume the process is transient. (This assumption can be removed.) Assume also that some mild regularity holds. (More about this is a moment.)

The main theorems

Theorem 1. *The elliptic Harnack inequality holds if and only if the cut-off inequality and the adjusted Poincaré inequality hold.*

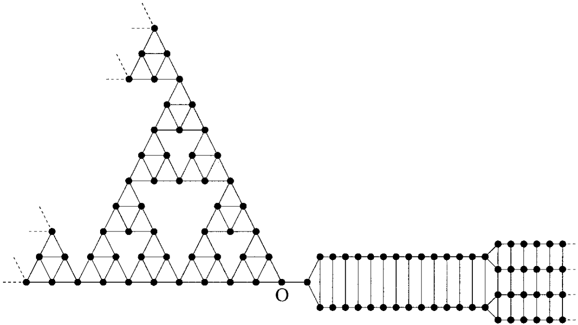
Theorem 2. *If the elliptic Harnack inequality holds for $\mathcal{E}$ and $\mathcal{E}'$ is equivalent to $\mathcal{E}$, then the elliptic Harnack inequality holds for $\mathcal{E}'$.*

About the regularity

All known proofs of any kind of Harnack inequality rely on volume doubling:

$$V(x, 2r) \leq cV(x, r).$$

Yet Delmotte (2002) has an example of a graph where the elliptic Harnack inequality holds but volume doubling does not.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We assume volume doubling holds.

We assume capacity grows at a certain minimal rate:

$$C(x, r) \le \rho C(x, 2r)$$

for some $\rho < 1$.

And we assume $E(x, r)$, the expected exit times, grow at a certain minimal rate:

$$E(x, r) \leq \rho E(x, 2r)$$

for some $\rho < 1$.

The last one has a probabilistic interpretation:

Of the time spent in $B(x, 2r)$ before exiting, not all of the time is spent in $B(x, r)$ but at least some percentage of the time is spent in $B(x, 2r) - B(x, r)$.

The purpose of these assumptions is to show that $E(x, r)$ and $E(y, r)$ are comparable if $d(x, y) \approx r$. No useful information can be obtained when $d(x, y) \gg r$.

When the process is not transient, one does the following. One lets $\widetilde{C}(x, r)$ be the capacity of $B(x, r)$ relative to $B(x, 2r)$.

$$\widetilde{C}(x, r) = \inf\{f : f = 1 \text{ on } B(x, r), f = 0 \text{ on } B(x, 2r)^c\}.$$

Our theorem applies to processes on manifolds, processes on fractals, processes on infinite graphs (here the assumptions only have to hold for $r \geq 1$), and fairly general state spaces. We need to have a measure, a metric, and a Dirichlet form. Spaces with these three features are called metric measure Dirichlet spaces.

A few comments about the proof

Theorem 2 follows from Theorem 1 and the observation that the adjusted Poincaré inequality and the cut-off inequality are stable.

Proving that the adjusted Poincaré inequality plus the cut-off inequality implies the elliptic Harnack inequality is hard, but most of the work has already been done in Moser as modified by Barlow-Bass. One proves that it is enough to use the adjusted Poincaré inequality instead of the usual Poincaré inequality.

To go the other direction, one needs to get estimates on the Green function.

Supposing the elliptic Harnack inequality holds, one has, example, that if ν is the distribution of charge on the boundary of $B(x,s)$ when the potential of $B(x,s)$ is one, and $G(x,y)$ is the Green function, then

$$\begin{aligned} 1 = G\nu(x) &= \int_{\partial B(x,s)} G(x,z)\,\nu(dz) \\ &\approx G(x,y) \int_{\partial B(x,s)} \nu(dz) \\ &= G(x,y)C(x,s), \end{aligned}$$

with $s = d(x,y)$. Hence

$$G(x,y) \approx \frac{c}{C(x,s)}, \qquad s = d(x,y).$$

Integrating this over $B(x, r)$, we get

$$\int_{B(x,r)} G(x, y)\, \mu(dy) \approx \frac{cV(x, r)}{C(x, r)} = cE(x, r).$$

In other words, the amount of time spent in $B(x, r)$ is comparable to $E(x, r)$.

Let G_D is the Green function for the process killed on exiting a domain D. Using

$$G_D(x,y) = G(x,y) - \mathbb{E}^x G(X_{\tau_D}, y),$$

where τ_D is the first exit time of the domain D, we get an estimate for $G_D(x,y)$ ball when D is a ball such that x, y are in D and are far from the boundary.

If G_D^α is the α-resolvent density:

$$G_D^\alpha f(x) = \mathbb{E}^x \int_0^{\tau_D} e^{-\alpha t} f(X_t)\, dt$$

and

$$G_D^\alpha f(x) = \int G_D^\alpha(x,y) f(y)\, \mu(dy),$$

then

$$\begin{aligned} G_D^\alpha(x,y) &= G_D(x,y) - \alpha G_D G_D^\alpha(x,y) \\ &\geq G_D(x,y) - \alpha (G_D)^2(x,y), \end{aligned}$$

by the resolvent density.

When $\alpha = 1/E(x_0, r)$ and $D = B(x_0, cr)$, we then get a lower bound for $G_D^\alpha(x, y)$.

Let $B = B(x_0, r)$, $D = B(x_0, cr)$.

From the lower bound we derive the inequality

$$\int_B (f(y) - f_B)^2 \, \mu(dy) \le c(\|f\|^2_{L^2(D)} - \|\alpha G^\alpha_D f\|_{L^2(D)}).$$

Some work with the spectral theorem shows that this is bounded by

$$cE(x_0, r)\mathcal{E}_D(f, f),$$

where $D = B(x_0, cr)$.