Sobolev and *BV* functions on metric measure spaces and weak gradients[1]

L. Ambrosio

Scuola Normale Superiore, Pisa
http://cvgmt.sns.it

[1] Papers in collaboration with N.Gigli, G.Savaré, M.Colombo, S.Di Marino

Plan

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2. Reflexivity of Sobolev spaces
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4. A dual representation for the p-modulus of curves
5. Null sets for parametric and non-parametric sets of curves

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Weak definitions of gradient

Standing assumptions. (X, d) complete and separable. $\boldsymbol{m}$ Borel nonnegative measure on (X, d), finite on bounded sets.

Notation. The slope $|Df|$, or local Lipschitz constant, is defined by

$$|Df|(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathrm{d}(y, x)}.$$

We need also the so-called asymptotic Lipschitz constant:

$$\mathrm{Lip}_a(f, x) := \inf_{r>0} \mathrm{Lip}\big(f, B_r(x)\big).$$

It is easy to see that $\mathrm{Lip}_a(f, \cdot) \geq |Df|^*$, with equality if (X, d) is length.

The metric derivative $|\dot{\gamma}| \in L^p(a, b)$ of $\gamma \in AC^p([a, b]; X)$ is defined by

$$|\dot{\gamma}_t| := \lim_{h \to 0} \frac{\mathrm{d}(\gamma_{t+h}, \gamma_t)}{|h|}.$$

The strongest definition: relaxation in L^q of Lip_a^q

Let $q \in (1, \infty)$ and $f \in L^q(X, \boldsymbol{m})$. We say that $f \in H^{1,q}(X, \mathrm{d}, \boldsymbol{m})$ if there exists Lipschitz functions $f_n \in L^q(X, \boldsymbol{m})$ satisfying $f_n \to f$ in L^q and

$$\limsup_{n\to\infty} \int_X \mathrm{Lip}_a^q(f_n, x)\, d\boldsymbol{m}(x) < \infty.$$

Remarks. A minor (but technically important) variant of Cheeger's definition, who used pairs (f, g) with g upper gradient of f. Recall (Heinonen-Koskela) that a Borel function $g : X \to [0, \infty]$ is an upper gradient of f if

$$|f(\gamma_b) - f(\gamma_a)| \le \int_a^b g(\gamma_s)|\dot\gamma_s|\, ds \quad \text{for all } \gamma \in AC([a, b]; X).$$

Another intermediate definition would be $(f, |Df|)$, because $|Df|$ is an upper gradient of f. All lead, as we will see, to the same result.

The strongest definition: relaxation in L^q of Lip_a^q

Along with the definition of $H^{1,q}(X,\mathrm{d},\boldsymbol{m})$ we have the notion of *relaxed slope*, i.e. a weak $L^q(X,\boldsymbol{m})$ limit of $\mathrm{Lip}_a(f_n,\cdot)$ for some $f_n \to f$ in L^q.

The *minimal relaxed slope* $|Df|_{*,q}$ is the one having smallest L^q norm, and it can be used to define Cheeger's energy:

$$\mathrm{C}(f) := \frac{1}{q}\int_X |Df|_{*,q}^q \, d\boldsymbol{m} \qquad\qquad f \in H^{1,q}(X,\mathrm{d},\boldsymbol{m}).$$

One can then show that $|Df|_{*,q}$ enjoys a *pointwise* minimality property

$$|Df|_{*,q} \le g \quad \boldsymbol{m}\text{-a.e. in } X\text{, for all relaxed slope } g \text{ of } f$$

and develop reasonable calculus rules (chain rule, even integration by parts) with this gradient.

The weakest definition: weak upper gradients

Idea: we can define a weak Sobolev space $W^{1,q}(X, \mathrm{d}, \boldsymbol{m})$ by requiring the validity of the upper gradient property along "almost every curve".

History: Levi 1901 (in $\mathbb{R}^2$, with horizontal and vertical lines), Fuglede 1957 (in $\mathbb{R}^n$, using the notion of q-modulus), Koskela-MacManus, Shanmugalingham 1998-99 (in metric measure spaces), AGS 2011 (in metric measure spaces, using test plans).

Test plan. For $1 \leq p \leq \infty$, we say that $\boldsymbol{\pi} \in \mathscr{P}(C([0,1];X))$ is a p-test plan if:
(a) $\boldsymbol{\pi}$ is concentrated on $AC^p([0,1];X)$;
(b) for some $C = C(\boldsymbol{\pi}) \geq 0$, $(\mathrm{e}_t)_\sharp\boldsymbol{\pi} \leq C\boldsymbol{m}$ for all $t \in [0,1]$.

Here, for $t \in [0,1]$, $\mathrm{e}_t : C([0,1];X) \to X$ is the evaluation map at time t, namely $\mathrm{e}_t(\gamma) := \gamma_t$.

***p*-null sets:** We say that $\Gamma \subset AC^p([0,1];X)$ is p-null if $\boldsymbol{\pi}^*(\Gamma) = 0$ for all p-test plans $\boldsymbol{\pi}$.

The weakest definition: weak upper gradients

Definition. We say that $g \in L^q(X, \boldsymbol{m})$ is a *q-weak upper gradient* of f, and write $g \in WUG_q(f)$, if the u.g. inequality

$$|f(\gamma_1) - f(\gamma_0)| \le \int_0^1 g(\gamma_s)|\dot\gamma_s|\,dx \quad \text{for all } \gamma \in AC([0,1];X).$$

holds for p-almost every curve, with $p = q' = q/(q-1)$.

Remark. Because of the non-concentration condition (b) imposed on test plans, the notion "$g \in WUG_q(f)$" is invariant in the $\boldsymbol{m}$-equivalence classes of f and g (the p-modulus notion, instead, has only invariance w.r.t g).

Adapting an argument of Hajlasz, one can show that $WUG_q(f)$ is a lattice and define a *minimal* q-wug, in the $\boldsymbol{m}$-almost everywhere sense, that we shall denote by $|Df|_{w,q}$.

The identification theorem

Easy inclusion and inequality. $H^{1,q}(X, \mathrm{d}, \boldsymbol{m}) \subset W^{1,q}(X, \mathrm{d}, \boldsymbol{m})$ and $|Df|_{w,q} \leq |Df|_{*,q}$. This is a consequence of the fact that $\mathrm{Lip}_a(f, \cdot)$ is an upper gradient and the stability (Fuglede) of weak upper gradients:

$f_n \to f$ in L^q, $g_n \to g$ weakly in L^q, $g_n \in WUG_q(f_n) \implies g \in WUG_q(f)$.

Theorem. [AGS, 2011] *$H^{1,q} = W^{1,q}$ and $|Df|_{w,q} = |Df|_{*,q}$ $\boldsymbol{m}$-a.e. in X.*

Sketch of proof. We need to build Lipschitz functions $f_n \to f$ in $L^q(X, \boldsymbol{m})$ satisfying

$$\limsup_{n\to\infty} \int_X \mathrm{Lip}_a^q(f_n, x)\, d\boldsymbol{m}(x) \leq \int_X |Df|_{w,q}^q(x)\, d\boldsymbol{m}(x).$$

The identification theorem

This is achieved by looking at the dissipation rate, along the gradient flow of C_q, of a suitable energy. In the simple case $q = p = 2$, after a reduction of the problem to the approximation of $\sqrt{f}$, with f probability density, the energy is the Boltzmann-Shannon entropy

$$E(f) := \int_X f \log f \, d\boldsymbol{m}.$$

This provides a family f_t of probability densities satisfying

$$\limsup_{t \downarrow 0} \frac{1}{t} \int_0^t \frac{1}{2} \int_X |D\sqrt{f_s}|^2_{*,2} \, ds \leq \frac{1}{2} \int_X |D\sqrt{f}|^2_{w,2} \, d\boldsymbol{m}.$$

Since (by definition of C_2) $\sqrt{f_s}$ can be well approximated by Lipschitz functions, a diagonal argument gives the result.

Comparison with the other previous approaches

Let $\mathscr{C}(X) = AC([0,1];X)/\sim$ be the collection of non-parametric continuous curves with finite length. Recall that the q-modulus (Buerling-Ahlfors) $\mathrm{Mod}_q(\Sigma)$ of a family $\Sigma \subset \mathscr{C}(X)$ is given by

$$\min\left\{\int_X g^q\,d\boldsymbol{m}:\ g \geq 0 \text{ Borel}, \int_\gamma g \geq 1 \text{ for all } \gamma \in \Sigma\right\}.$$

According to Koskela-MacManus, Shanmugalingham, we may define weak upper gradients $|Df|_{q,S}$ by requiring the validity of the upper gradient inequality along Mod_q-almost every curve.

Lemma. *If Σ is Mod_q-negligible, then any corresponding set $\Gamma \subset AC^q([0,1];X)$ is q-negligible.*

Proof. For any p-test plan π with $\int\int_0^1 |\dot\gamma|^p\,ds\,s\pi(\gamma) \leq L$ and any admissible g, Hölder's inequality gives

$$\pi(\Gamma) \leq \int_0^1 \int g(\gamma_t)|\dot\gamma_t|\,d\pi(\gamma)\,dt \leq L^{1/p}[C(\pi)]^{1/q}\|g\|_{L^q(X,\boldsymbol{m})}.$$

Comparison with the other previous approaches

By minimization, we obtain that $\pi(\Gamma) \leq L^{1/p}[C(\pi)]^{1/q}\big[\mathrm{Mod}_q(\Sigma)\big]^{1/q}$, proving the lemma.

It follows that $|Df|_{w,q} \leq |Df|_{S,q} \leq |Df|_{*,q}$ and the identification theorem shows that they all these notions of gradient coincide.

This is somehow surprising, since the converse in the previous lemma fails: namely, $\Gamma \subset AC^q([0,1];X)$ q-negligible *does not* imply in general that its projection on $\mathscr{C}(X)$ is Mod_q-negligible.

Example. Let $X = [0,1]^2$, $\boldsymbol{m} = \mathscr{L}^2$ and $\Gamma = \{\gamma_x\}_{x\in[0,1]}$, with $\gamma_x(t) := (x,t)$, $t \in [0,1]$. Then obviously $\mathrm{Mod}_2(\Gamma) = 1 > 0$, on the other hand $\pi(\Gamma) = 0$ for any q-test plan π simply because, for all $t \in [0,1]$, $e_t(\gamma_x) = (x,t)$ is $\boldsymbol{m}$-negligible (and one t would be enough to get the same conclusion).

This shows that, to hope for a reverse inclusion, convenient parameterizations and curve decompositions should be chosen.

Reflexivity of Sobolev spaces

Theorem. [Cheeger, '2000] *If* $(X, \mathrm{d}, \boldsymbol{m})$ *is doubling and supports a* $(1, q)$*-Poincaré inequality, then* $W^{1,q}(X, \mathrm{d}, \boldsymbol{m})$ *is reflexive for all* $q \in (1, \infty)$.

Part of the more general program developed by Cheeger on Rademacher theorem and the existence of a cotangent bundle. The next result holds without Poincaré inequality and replacing the doubling property on $(X, \mathrm{d}, \boldsymbol{m})$ with the weaker metric doubling property.

Theorem. [A.-Colombo-Di Marino] *If* (X, d) *is doubling, then* $W^{1,q}(X, \mathrm{d}, \boldsymbol{m})$ *is reflexive.*

Sketch of proof. We consider a Borel decomposition of X in sets A_i^δ on scale δ:

$$B\big(z_i^\delta, \frac{\delta}{3}\big) \subset A_i^\delta \subset B\big(z_i^\delta, \frac{5}{4}\delta\big).$$

Reflexivity of Sobolev spaces

Then, doubling ensures that for all i the set of "neighbours" A_j^δ of A_i^δ has controlled cardinality. As in the literature on q-energy and Dirichlet forms on fractals (Sturm 1998, Herman-Peirone-Strichartz 2004, see also Korevaar-Schoen), we define

$$\mathrm{C}_q^\delta(f) := \sum_i \mu(A_i^\delta) \sum_{j\sim i} \frac{|f_{A_i^\delta} - f_{A_j^\delta}|^q}{\delta^q} \quad \text{with} \quad f_{A_k^\delta} := \frac{1}{\boldsymbol{m}(A_i^\delta)} \int_{A_i^\delta} f.$$

Since $|Du|_{*,q}$ is obtained by the relaxation of $\mathrm{Lip}_a(f,\cdot)$, it is fairly easy to show that

$$\Gamma - \limsup_{\delta\downarrow 0} \mathrm{C}_q^\delta(f) \le C \int_X |Df|_{*,q}^q \, d\boldsymbol{m},$$

where C depends only on q and on the maximal cardinality of neighbours.

Reflexivity of Sobolev spaces

On the other hand, the weak upper gradient property has a nice discrete counterpart: it is not too hard to prove that

$$4\sum_i \frac{|f_{A_i^\delta} - f_{A_j^\delta}|}{\delta}\mathbf{1}_{A_i^\delta}$$

is a q-weak upper gradient, up to scale δ, for the piecewise constant approximation f_δ of f, namely $\sum_i f_{A_i^\delta}\mathbf{1}_{A_i^\delta}$. Hence, stability of weak gradients on progressively smaller scales gives

$$\frac{1}{4^q}\int_X |Df|_{w,q}^q\,d\boldsymbol{m} \leq \Gamma - \liminf_{\delta\downarrow 0} \mathrm{C}_q^\delta(f).$$

Eventually we can use the equivalence theorem to say that any Γ-limit of C_q^δ is equivalent to C_q; in the quadratic case $p = 2$, since Γ-limits of quadratic forms are quadratic forms, reflexivity is immediate. In general one can use the Clarkson inequalities and the Milman-Pettis theorem.

The space $W^{1,1}$

The proof of the identification theorem fails in the case $q = 1$, by the lack of semicontinuity of $f \mapsto \int_X |Df| \, d\boldsymbol{m}$ even in nice spaces.

4 possible definitions:

• One could define $W^{1,1}(X, \mathrm{d}, \boldsymbol{m})$ via weak upper gradients (using 1-test plans or 1-modulus)
• One could define $H^{1,1}(X, \mathrm{d}, \boldsymbol{m})$ considering approximating sequences for which $\mathrm{Lip}_a(f_n, \cdot)$ are $\boldsymbol{m}$-equiintegrable;
• One could define $H^{1,1}(X, \mathrm{d}, \boldsymbol{m})$ as the subspace of all $f \in BV(X, \mathrm{d}, \boldsymbol{m})$ such that $|\mathbf{D}f| \ll \boldsymbol{m}$.

In general, the equivalence between these definitions is an open problem (they coincide under doubling & Poincaré).

The space BV

Definition. (Miranda Jr, 1996) Let $f \in L^1(X, \mathrm{d}, \boldsymbol{m})$. We say that $f \in BV(X, \mathrm{d}, \boldsymbol{m})$ if

$$|\mathbf{D}f|(X) := \inf\left\{\liminf_{n\to\infty}\int_X |Df_n|\,d\boldsymbol{m}:\ f_n \to f \text{ in } L^1(X),\ f_n \in \mathrm{Lip}_{\mathrm{loc}}(X)\right\} < \infty.$$

Miranda proved that, for $f \in L^1_{\mathrm{loc}}(X)$, in locally compact spaces the set function $A \mapsto |\mathbf{D}f|(A)$ is always the restriction to open sets of X of a Borel (possibly infinite) measure, the so-called total variation measure.

For instance, under doubling and Poincaré and for characteristic functions $f = \mathbf{1}_E$ we have (A, 2001)

$$c\mathcal{S}^h \llcorner \partial^* E \le |\mathbf{D}\mathbf{1}_E| \le \frac{1}{c}\mathcal{S}^h \llcorner \partial^* E,$$

where $c > 0$ and the "spherical Hausdorff measure" $\mathcal{S}^h$ is built, with Carathédory's construction, out of $h(B_\rho(x)) := \boldsymbol{m}(B_\rho(x))/\rho$.

The space *BV*

Is there an equivalent definition of *BV* and of $|\mathbf{D}f|$ based on (measure) weak upper gradients?

Does it work also in non locally compact situations?

Having in mind the *BV* 1-dimensional estimate (for nice functions f)

$$
\begin{aligned}
|f\circ\gamma(1)-f\circ\gamma(0)| &\leq |\mathbf{D}(f\circ\gamma)|(0,1)=\int_0^1 |Df|(\gamma_t)|\gamma_t'|\,dt \\
&\leq \mathrm{Lip}(\gamma)\int_0^1 |Df|(\gamma_t)|\,dt
\end{aligned}
$$

we may average the inequality w.r.t. γ and give the following "weak upper gradient" definition of the space *BV*.

The space BV

Definition. Let $f \in L^1(X, \boldsymbol{m})$, we say that $f \in BV_w(X, \mathrm{d}, \boldsymbol{m})$ if there exists a positive finite measure $\boldsymbol{\mu}$ in X satisfying

$$\int \gamma_\sharp |\mathbf{D}(f \circ \gamma)| \, d\boldsymbol{\pi}(\gamma) \leq C(\boldsymbol{\pi}) \|\mathrm{Lip}(\gamma)\|_{L^\infty(\boldsymbol{\pi})} \, \boldsymbol{\mu}$$

for all ∞-test plans $\boldsymbol{\pi}$. The minimal measure $\boldsymbol{\mu}$ with this property will be denoted by $|\mathbf{D}f|_w$.

Theorem. (A-Di Marino, 2012) *$BV_w(X, \mathrm{d}, \boldsymbol{m}) = BV(X, \mathrm{d}, \boldsymbol{m})$ and $|\mathbf{D}f|_w = |\mathbf{D}f|$.*

Thanks to this result, one can also prove a non-obvious fact, namely that Miranda's definition of the set function $A \mapsto |\mathbf{D}f|(A)$ provides a σ-additive measure even without local compactness assumptions. The proof involves the gradient flow of $f \mapsto |\mathbf{D}f|(X)$ and suitable limiting versions as $p \to \infty$ of the tools (Hamilton-Jacobi equations, superposition principle) used in [AGS] in the Sobolev case.

A dual representation for the p-modulus of curves

We would like to understand whether also the notion of p-modulus can be rephrased, as in the axiomatization of test plans, in terms of suitable probability measures in the space $\mathscr{C}(X)$ of non-parametric curves.

In order to gain a convex/linear structure, it is natural to embed $\mathscr{C}(X)$ into $\mathscr{M}_+(X)$ via the map

$$J\gamma := \gamma_\sharp(|\dot\gamma|\mathscr{L}^1 \llcorner [0,1]) = N(\gamma,\cdot)\,\mathscr{H}^1,$$

where $N(\gamma,x) = \mathrm{card}(\gamma^{-1}(x))$ is the multiplicity function of γ and $\mathscr{H}^1$ is 1-dimensional Hausdorff measure.

In this way we can rephrase the upper gradient inequality in the form

$$|f(\gamma_{fin}) - f(\gamma_{ini})| \leq \int_X g\,dJ\gamma \qquad \forall \gamma \in \mathscr{C}(X).$$

A dual representation for the p-modulus of curves

Definition. [Plans with baycenter in $L^p(X,\boldsymbol{m})$] Let $\boldsymbol{\eta} \in \mathscr{P}(\mathscr{M}_+(X))$. We say that $\boldsymbol{\eta}$ has barycenter in $L^p(X,\boldsymbol{m})$ if the expected measure

$$\bar{\boldsymbol{\eta}} := \int_{\mathscr{M}_+(X)} \mu \, d\boldsymbol{\eta}(\mu)$$

has the form $b\boldsymbol{m}$ for some $b \in L^p(X,\boldsymbol{m})$.

There exists a general duality formula between plans of measures $\boldsymbol{\eta}$ with barycenter in $L^p(X,\boldsymbol{m})$ and Mod_q. The latter can now defined even for families Σ of positive finite measures in X:

$$\min\left\{\int_X g^q \, d\boldsymbol{m} : \; g \geq 0 \text{ Borel}, \int_X g \, d\mu \geq 1 \text{ for all } \mu \in \Sigma\right\}.$$

We state duality just in the classical case of families $\Sigma \subset \mathscr{C}(X)$.

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A dual formula for $\mathrm{Mod}_q(\Sigma)$

An easy duality inequality coming from Holder's inequality is:

$$\Big[\frac{\boldsymbol{\eta}(\Sigma)}{\|b\|_{L^p(X,\boldsymbol{m})}}\Big]^q \le \int_X g^q\, d\boldsymbol{m}$$

for all $\boldsymbol{\eta} \in \mathscr{P}(\mathscr{C}(X))$ with barycenter $\int J\gamma\, d\boldsymbol{\eta}(\gamma) = b\boldsymbol{m}$ in $L^p(X,\boldsymbol{m})$ and all g satisfying $\int g\, dJ\gamma \ge 1$ for all $\gamma \in \Sigma$. The following result shows that there is no duality gap:

Theorem. [A.-Di Marino-Savaré, 2013] *For $\Sigma \subset \mathscr{C}(X)$, define*

$$\mathrm{Cap}_q(\Sigma) := \max\left\{\Big[\frac{\boldsymbol{\eta}(\Sigma)}{\|b\|_{L^p(X,\boldsymbol{m})}}\Big]^q\right\},$$

where the maximization runs among $\boldsymbol{\eta} \in \mathscr{P}(\mathscr{C}(X))$ with barycenter $\int J\gamma\, d\boldsymbol{\eta}(\gamma) = b\boldsymbol{m}$ in $L^p(X,\boldsymbol{m})$, $p = q/(q-1)$. Then

$$\mathrm{Cap}_q(\Sigma) = \mathrm{Mod}_q(\Sigma) \qquad \textit{for all Suslin sets } \Sigma \subset \mathscr{C}(X).$$

A dual formula for $\mathrm{Mod}_q(\Sigma)$

Remark. The optimal η_{opt}, which is concentrated on Σ, provides somehow a more precise information, compared to the optimal g_{opt}: for instance in general it need not be $\int_\gamma g_{\rm opt} = 1$ Mod_q-a.e. on Σ, just take

$$X = [0,1], \qquad \Sigma = \left\{[0,\frac{1}{2}], \quad [\frac{1}{2},1], \quad [0,1]\right\}, \qquad g_{opt} = 2.$$

On the other hand, the absence of a duality gap gives $\int_\gamma g_{\rm opt} = 1$ η_{opt}-a.e.; this way one can canonically find a subset $\Sigma' \subset \Sigma$ with the same q-modulus on which the constraint is saturated, i.e. $\int_\gamma g_{\rm opt} = 1$ on Σ'. In the previous example

$$\Sigma = \left\{[0,\frac{1}{2}], \quad [\frac{1}{2},1]\right\}.$$

An application of the duality formula

As an application of the duality formula, we can prove a partial converse to the implication that Mod_q-negligible sets are q-negligible. Remember that the "Levi" example shows the necessity to consider subcurves and convenient reparameterizations of them.

Theorem. *Let* $\Sigma \subset \mathscr{C}(X)$ *with* $\mathrm{Mod}_q(\Sigma) > 0$. *Then there exist subfamilies* Σ_i *which exhaust* Σ, *families* $\Gamma_i \subset AC^q([0,1];X)$, $\boldsymbol{\pi}_i \in \mathscr{P}(C([0,1];X))$ *concentrated on* Γ_i *and* $g_i \in L^p(X,\boldsymbol{m})$ *such that*

$$(\mathrm{e}_t)_\sharp \boldsymbol{\pi}_i \le g_i \boldsymbol{m} \qquad \forall t \in [0,1], \ \ i \in \mathbb{N}.$$

The result is not yet optimal, because we would like $g_i \in L^\infty(X,\boldsymbol{m})$ to achieve that $\boldsymbol{\pi}_i$ are indeed test plans.

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