The IPESD Multiscale Model for the MJO: the effects of meridional momentum and temperature flux

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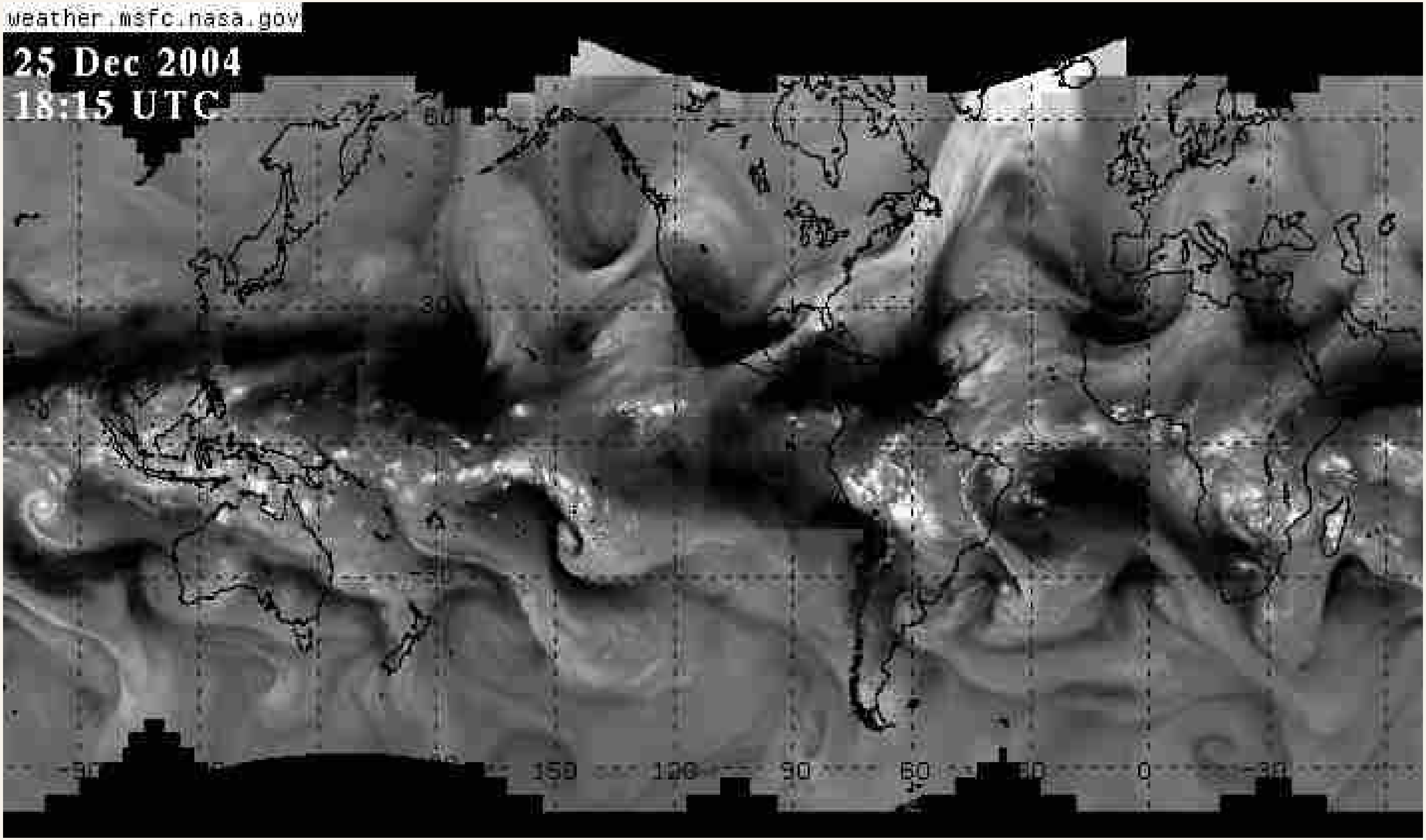
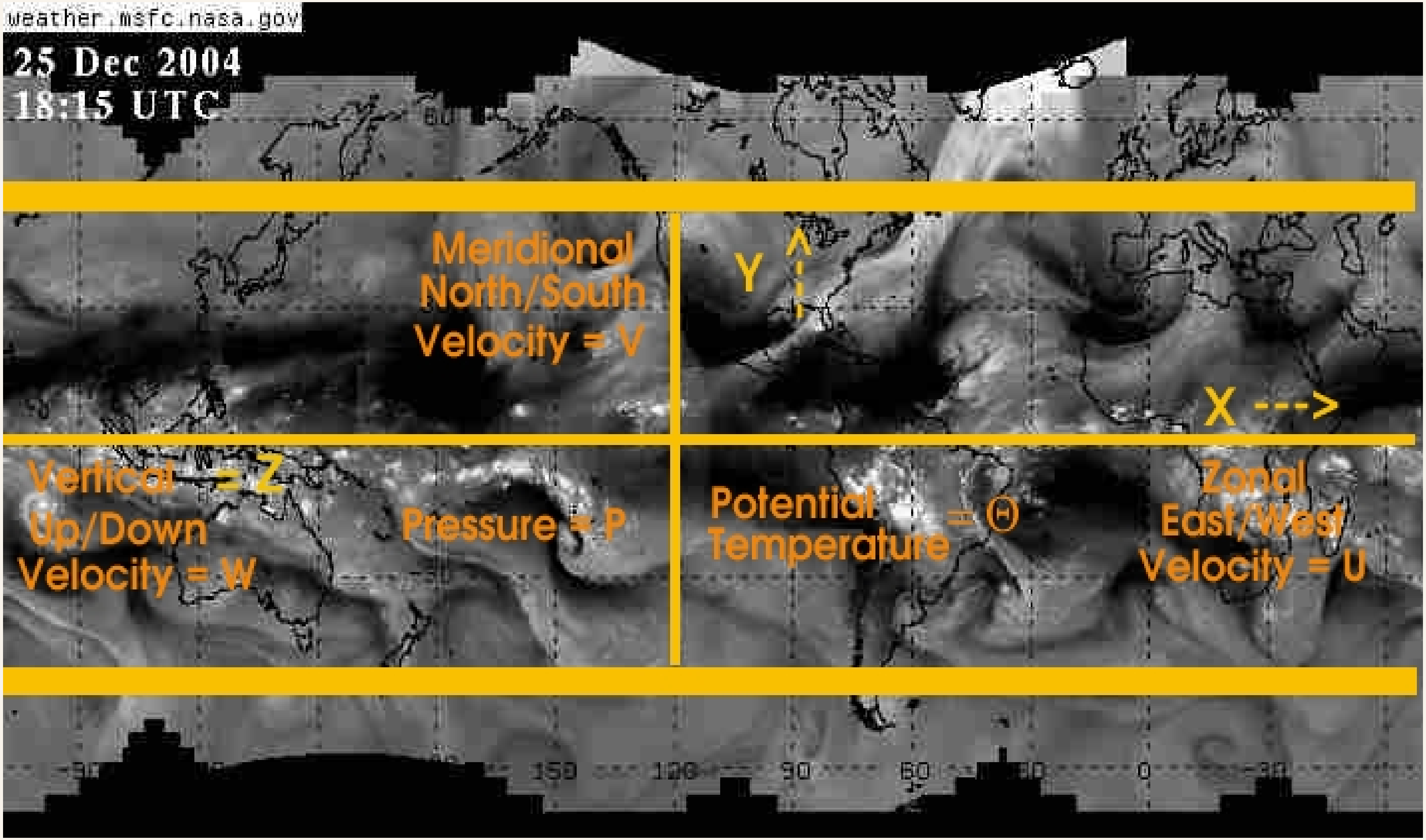
- Derivation of the IPESD multiscale models
- Explicit formulae for upscale flux convergences
- Meridional momentum flux convergence for multiple scales
- Meridional tilted heating in the MJO models: Mitch Moncrieff
- Transformed Eulerian mean: removing the meridional temperature flux convergence
- Active moisture using Majda/Khouider 3 cloud models

Water Vapor



Coordinate System



Anelastic, Hydrostatic, Equatorial β-plane

$$
\begin{aligned}
\frac{D}{D\tau}u - y\,v &= -p_x + \epsilon\, S_u \\
\frac{D}{D\tau}v + y\,u &= -p_y + \epsilon\, S_v \\
\frac{D}{D\tau}\theta + N^2\,w &= \epsilon\, S_\theta \\
p_z &= \theta \\
(\rho\,u)_x + (\rho\,v)_y + (\rho\,w)_z &= 0
\end{aligned}
$$

- $\tau \sim 8.3\,\mathrm{hrs}$, equatorial deformation time
- Horizontal Lengths, $x, y \sim 1500$ km equatorial deformation radius
- Horizontal Velocity, $u, v \sim 50\,\mathrm{ms}^{-1}$
- Vertical scale, $z \sim 5\,\mathrm{km}$
- Vertical velocity, $w \sim .25\,\mathrm{ms}^{-1}$
- Heating Rate, $S_\theta \sim 100\,\mathrm{K/day}$
- Temperature fluctuation, $\theta \sim 33\,\mathrm{K}$

- Typical tropic synoptic scale heating rates $\sim 10\,\mathrm{K/day}$
- Momentum dissipation time (Lin & Mapes, 2005) $\sim 5\,\mathrm{days}$
- Thermal dissipation time $\sim 15\,\mathrm{days}$
- Motivates the ϵ scaling of the forcing terms on the right hand side $\epsilon \sim 0.1$

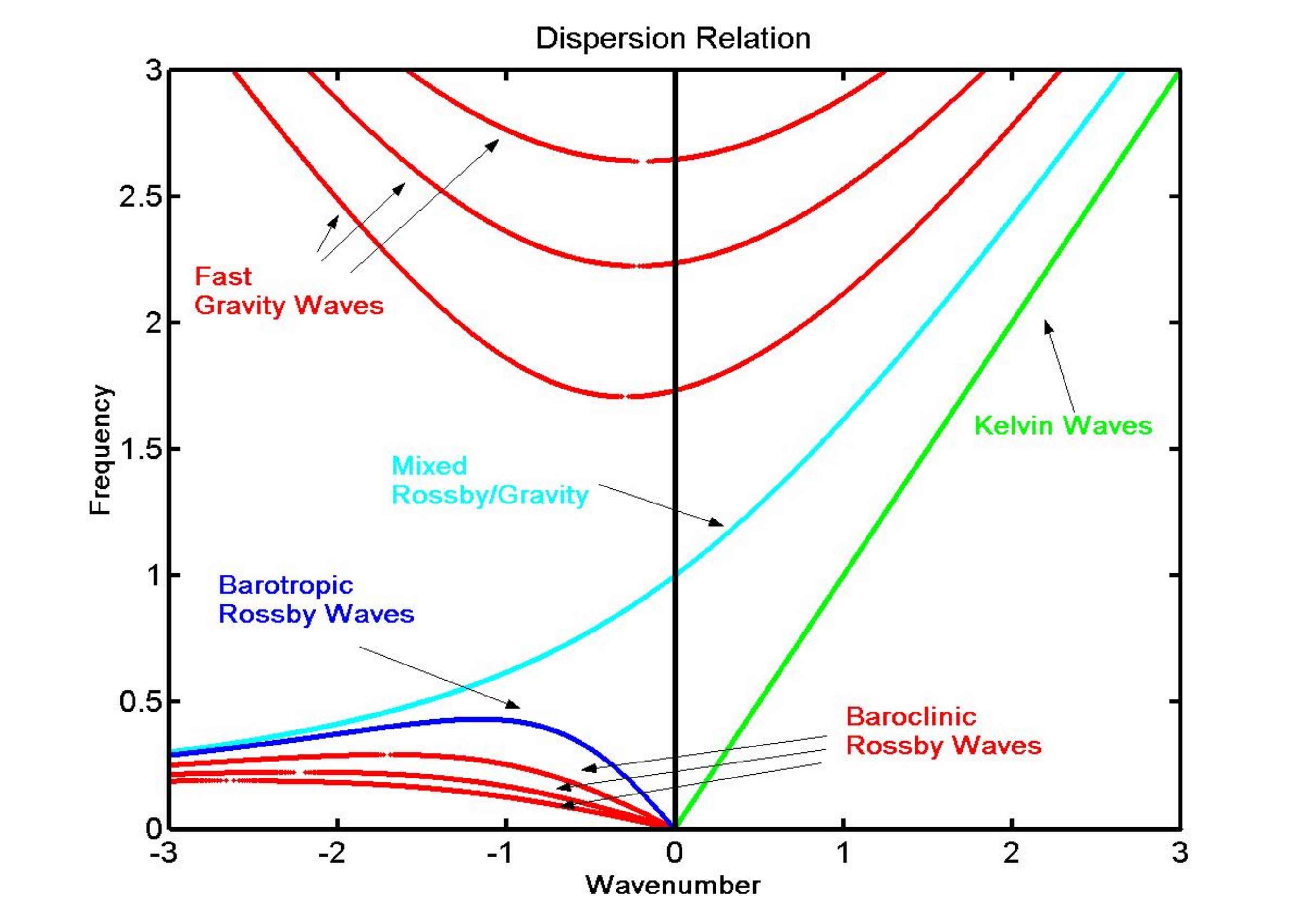
Multiple Time and Zonal Scales: Majda & Klein (2003)

- Response to $O(\epsilon)$ forcing, $\Longrightarrow |u| \sim \epsilon \sim 5\mathrm{ms}^{-1}$
- Long time scale $t \equiv \epsilon\tau \sim 3.5\,\mathrm{days}$
- Long length scale $X \equiv \epsilon x \sim 15000\,\mathrm{km}$
- Rescale fields by ϵ to reveal dominant balance
  $\Rightarrow$ Forced Linear Dynamics

$$
\begin{aligned}
u_\tau - y\,v + p_x &= S_u \quad + \epsilon\left[u_t + \vec{u}\cdot\nabla u - p_X\right] + O(\epsilon^2)\\
v_\tau + y\,u + p_y &= S_v \quad + \epsilon\left[v_t + \vec{u}\cdot\nabla v\right] + O(\epsilon^2)\\
\theta_\tau + \ w &= S_\theta \quad + \epsilon\left[\theta_t + \vec{u}\cdot\nabla \theta\right] + O(\epsilon^2)\\
p_z &= \theta\\
u_x + v_y + w_z + \epsilon\,u_X &= 0
\end{aligned}
$$

Linear Theory of Equatorial Waves

$\mathbf{L}$ is skew self adjoint $\Longrightarrow$ traveling waves



Asymptotic Expansion of Variables

- consider forcing which depends on slow time only
- separate variables into planetary scale means and synoptic scale fluctuations
- expect synoptic scale dynamics have horizontal aspect ratio $1:1$
- expect slowly evolving planetary scale dynamics, aspect ratio $x:y:z = 1:\delta:\delta$ from linear theory of long Kelvin and equatorial Rossby waves

$$
\begin{aligned}
\theta &= \theta'(X,x,y,z,t) + \overline{\Theta}(X,y,z,t) + O(\epsilon) \\
p &= p'(X,x,y,z,t) + \overline{P}(X,y,z,t) + O(\epsilon) \\
u &= u'(X,x,y,z,t) + \overline{U}(X,y,z,t) + O(\epsilon) \\
v &= v'(X,x,y,z,t) + \epsilon\overline{V}(X,y,z,t) \\
w &= w'(X,x,y,z,t) + \epsilon\overline{W}(X,y,z,t)
\end{aligned}
$$

Forcing

- all fluctuations are mean zero, i.e. $\overline{f'} = 0$
- Forcing due to latent heat release
- Consistent with observations, synoptic scale fluctuations exceed the planetary scale means

$$S_\theta = S'_\theta(X, x, y, z, t) + \epsilon \overline{S}_\theta(X, y, z, t)$$

- momentum and temperature dissipation follow linear damping

$$S_u = -\epsilon\, d_0\, u \quad \text{and} \quad S_\theta = -\epsilon\, d_\theta\, \theta$$

Synoptic Scale Balanced Dynamics: SEWTG

Well posedness of asymptotic expansion requires that the zonal mean of RHS of primitive equations be non-resonant with the $k \to 0$ nondispersive branches of the linear theory

Lowest order, requiring means of fluctuations to vanish

$$\begin{aligned} u'_\tau - y\, v' + p'_x &= S'_u \\ v'_\tau + y\, u' + p'_y &= S'_v \\ \theta'_\tau + w' &= S'_\theta \\ p'_z &= \theta' \\ u'_x + v'_y + w'_z &= 0 \\ \overline{S'_\theta} &= 0 \end{aligned}$$

Last condition is consistent with planetary scale mean heating being of lower order, $\overline{S}_\theta \sim \epsilon \sim 1.5\,\mathrm{K/day}$

Planetary scale quasi-linear dynamics: QLELWE

Setting zonal mean of the remaining terms to zero implies

$$
\begin{aligned}
\overline{U}_t - y\overline{V} + \overline{P}_X &= F^U \;\; - d_0\,\overline{U} \\
y\overline{U} + \overline{P}_y &= 0 \\
\overline{\Theta}_t + \overline{W} &= F^\theta \;\; - d_\theta\,\overline{\Theta} + \overline{S}_\theta \\
\overline{P}_z &= \overline{\Theta} \\
\overline{U}_X + \overline{V}_y + \overline{W}_z &= 0
\end{aligned}
$$

The fluxes from the synoptic scales are given by

$$
\begin{aligned}
F^U &= -\overline{(v'\,u')_y} - \overline{(w'\,u')_z} \\
F^\theta &= -\overline{(v'\,\theta')_y} - \overline{(w'\,\theta')_z}
\end{aligned}
$$

Each forcing effect, i.e. upscale vertical and meridional momentum and temperature transport and planetary scale mean heating can be considered separately and superposed

Solution of the synoptic scale dynamics

Model congestus, stratiform and deep convective latent heating

$$S'_\theta = G_x^1(X,x,y,t)\,\sin(z) + G_x^2(X,x,y,t)\,\sin(2z)$$

$$\Longrightarrow \qquad \overline{S'_\theta} = 0$$

Analytic Solution

$$\begin{aligned}
w' &= G_x^1\,\sin(z) + G_x^2\,\sin(2z) = S'_\theta \\
v' &= y\left[G_x^1\,\cos(z) + 2\,G_x^2\,\cos(2z)\right] \\
u' &= -\left[\left(2G^1 + yG_y^1\right)\,\cos(z) + 2\left(2G^2 + yG_y^2\right)\,\cos(2z)\right] \\
p' &= y^2\left[\,G^1\,\cos(z) + 2\,G^2\,\cos(2z)\right] \\
\theta' &= -y^2\left[\,G^1\,\sin(z) + 4\,G^2\,\sin(2z)\right].
\end{aligned}$$

Analytic Expression for the Fluxes

$$
\begin{aligned}
F^U = &\left[\frac{y^2}{2}\left(S^{11}+4S^{22}\right)\right]_y \\
&+\left[2y^2S^{12}_y+\tfrac{7}{2}yS^{12}-3\Upsilon^{12}-\tfrac{3}{2}yA^{12}\right]\cos(z) \\
&+\left[\tfrac{1}{2}y^2S^{11}_y+2yS^{11}\right]\cos(2z) \\
&+\left[2y^2S^{12}_y+\tfrac{17}{2}yS^{12}+3\Upsilon^{12}+\tfrac{3}{2}yA^{12}\right]\cos(3z) \\
&+4\left[\tfrac{1}{2}y^2S^{22}_y+2yS^{22}\right]\cos(4z)
\end{aligned}
$$

$$
\begin{aligned}
F^\theta = &\left[\tfrac{15}{2}y^2\Upsilon^{12}+3y^3\Upsilon^{12}_y\right]\sin(z) \\
&+\left[\tfrac{15}{2}y^2\Upsilon^{12}+y^3\Upsilon^{12}_y\right]\sin(3z)
\end{aligned}
$$

S^{11}, S^{22}, S^{12} are symmetric bilinear operators in G^1, G^2

$$S^{ij}=\frac{1}{2}\overline{G^i_x\,G^j_y+G^j_xG^i_y}$$

Υ^{12}, A^{12} are antisymmetric bilinear operators

$$\Upsilon^{12}=\overline{G^1_xG^2},\quad A^{12}=\frac{1}{2}\Upsilon^{12}_y$$

Carefully understanding these properties is essential to understanding the effects of synoptic scale heating on the upscale fluxes

Synoptic scale heating effects on Upscale Fluxes

- Zonal averaging $\Rightarrow$ only the same wavenumbers in x survive

Need consider only one wavelength in x

$$S'_\theta = F(X)\,[H_1(y)\cos(x+\phi_1(y))\,\sin(z) + H_2(y)\cos(x+\phi_2(y))\,\sin(2z)]$$

Therefore, the bilinear forms become

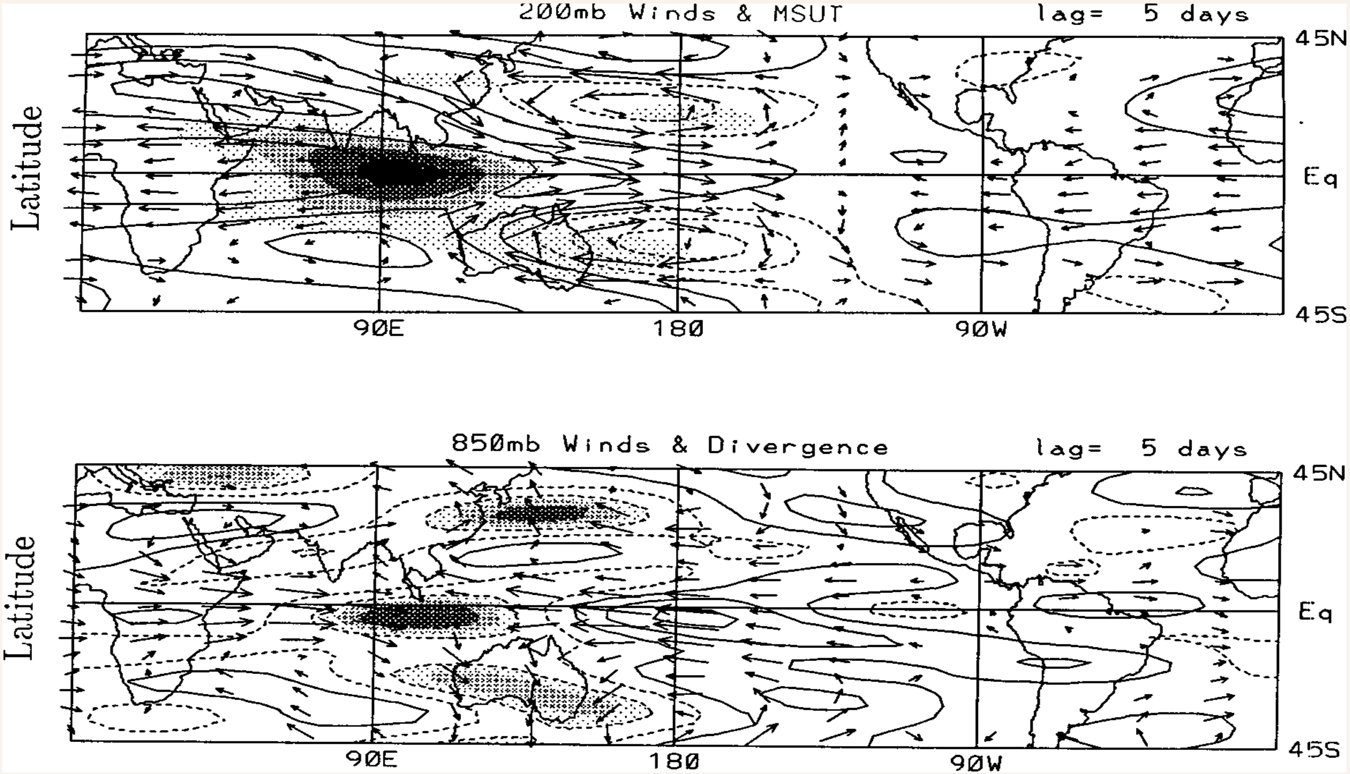
$$\begin{aligned}
\Upsilon^{12} &= \tfrac{1}{2}H_1\,H_2\,\sin(\phi_2-\phi_1)\\
A^{12} &= \tfrac{1}{4}\,[H_1\,H_2\,\sin(\phi_2-\phi_1)]_y \equiv \tfrac{1}{2}\Upsilon^{12}_y\\
S^{11} &= \tfrac{1}{2}(H_1)^2\,\phi_{1\,y}\\
S^{22} &= \tfrac{1}{2}(H_2)^2\,\phi_{2\,y}\\
S^{12} &= \tfrac{1}{4}\,[(H_1\,H_{2\,y}-H_2\,H_{1\,y})\,\sin(\phi_2-\phi_1)\\
&\quad +H_1\,H_2\,(\phi_{1\,y}+\phi_{2\,y})\,\cos(\phi_2-\phi_1)]
\end{aligned}$$

- Vertical tilt $\Longrightarrow \phi_1 \neq \phi_2$
- Meridional tilt $\Longrightarrow (\phi_i)_y \neq 0$

Synopsis

- Derivation of the IPESD (Biello & Majda, DAO)
- Analytic expressions for synoptic scale balanced dynamics from heating
- Analytic expressions for upscale flux convergence
  $\Longrightarrow$ we can trace specific properties of the upscale fluxes to specific aspects of the synoptic scale heating structure

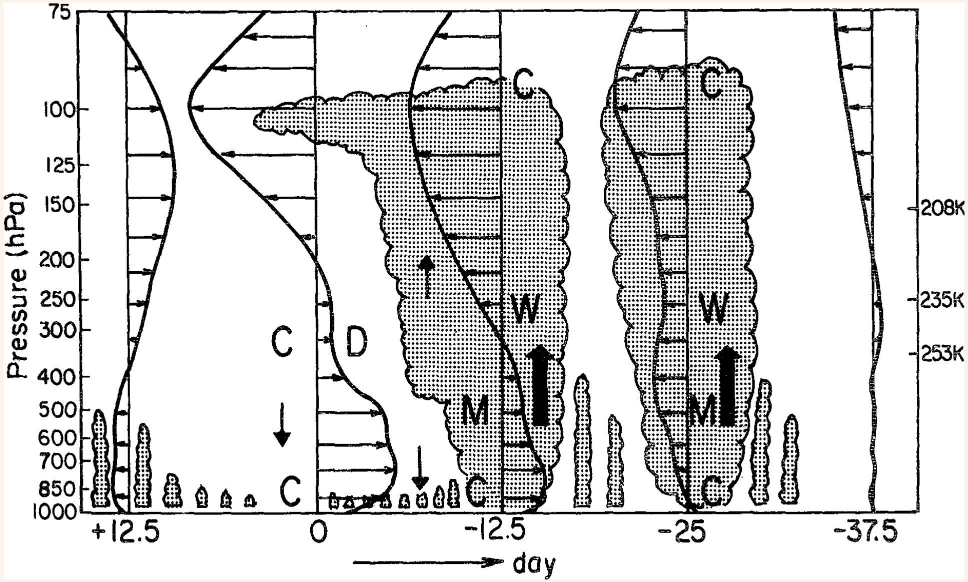
MJO: Large scale wind pattern



From Hendon & Salby *J. Atmos. Sci.*, **51**, p 2230, fig. 3.

- Top: Top of Troposphere, Winds and precipitation.
- Bottom: Bottom of Troposphere, winds and divergence. ***

MJO: Vertical Shear and Convection



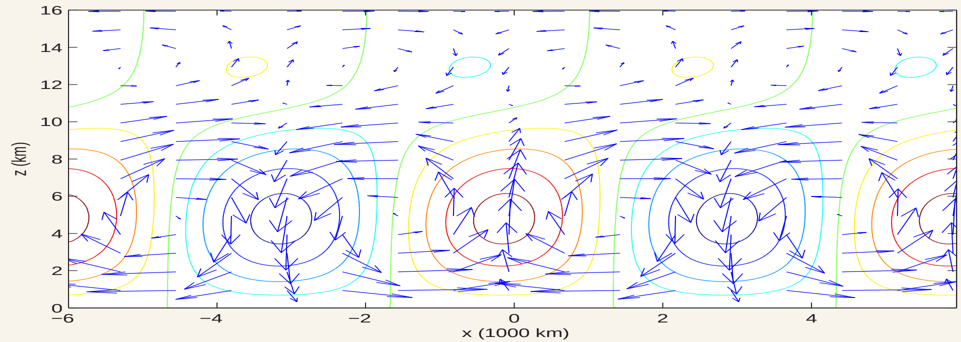
Lin & Johnson *J. Atmos. Sci.*, **53**, p 701, fig. 16.

- Schematic showing correlation of convection and vertical shear
- Time goes from right to left (reverse of previous slide) and can be interpreted as left = west, right = east.

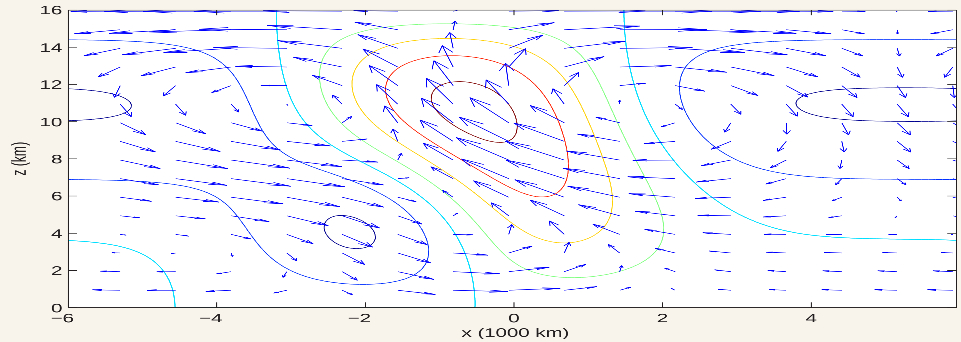
- Congestus clouds - weak winds/easterlies
- Westward tilted anvil - westerly onset
- Strong westerlies trail convection

MJO Model: Convection organized on small scales

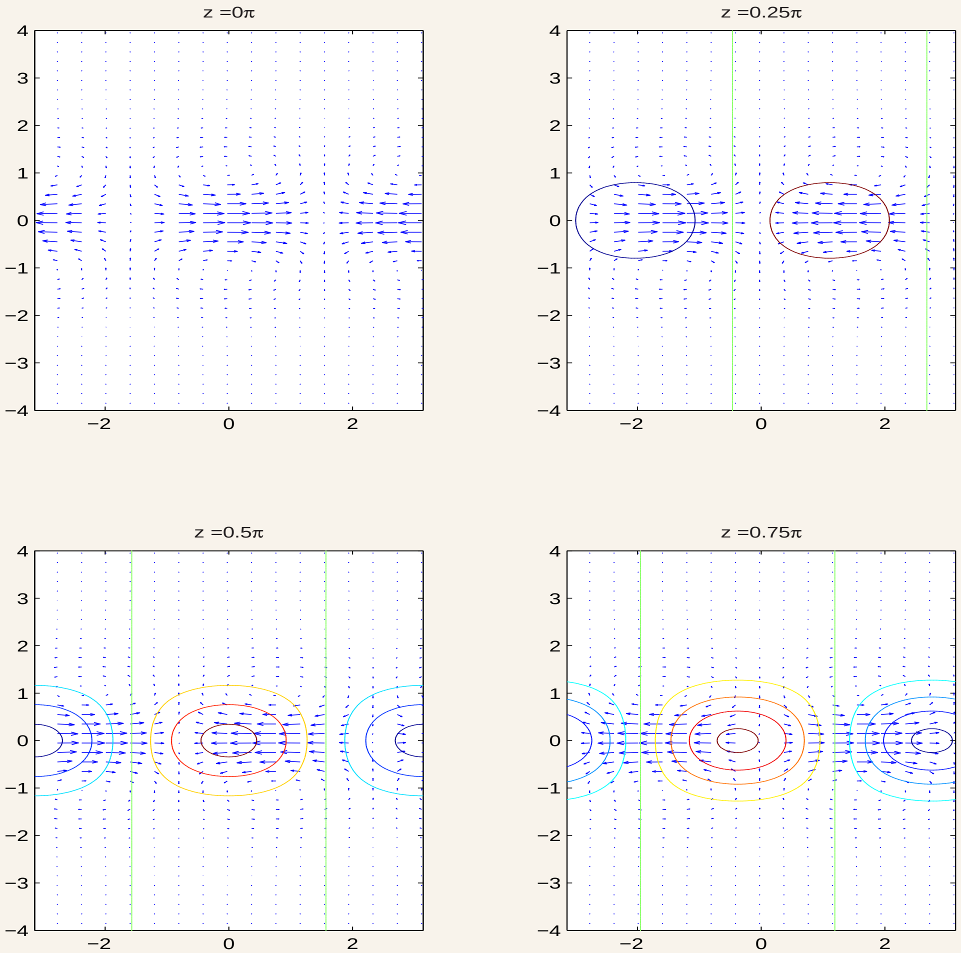
- Heating rate traces cloudiness (latent heat release).
- Fluctuations on 1500 km spatial scales
- Clouds/heating localized near equator above Western Pacific.



- East: Lower troposphere ***congestus clouds***
- West: High, westward tilted anvil ***superclusters***
- Flow vectors and heating contours
- Upscale flux, $\overline{\mathbf{N}\left(\psi', \psi'\right)} \neq 0$ $\Rightarrow$ Vertical/Longitudinal Tilt

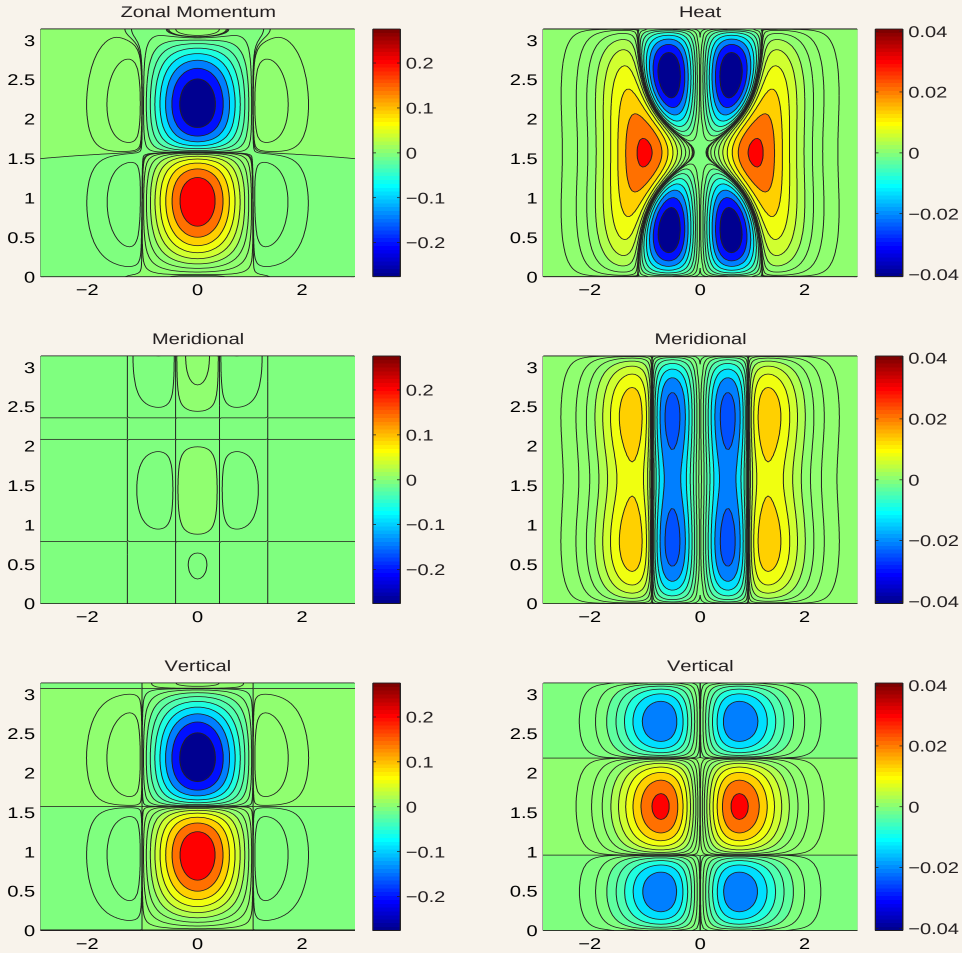


MJO Model: No Meridional Tilt



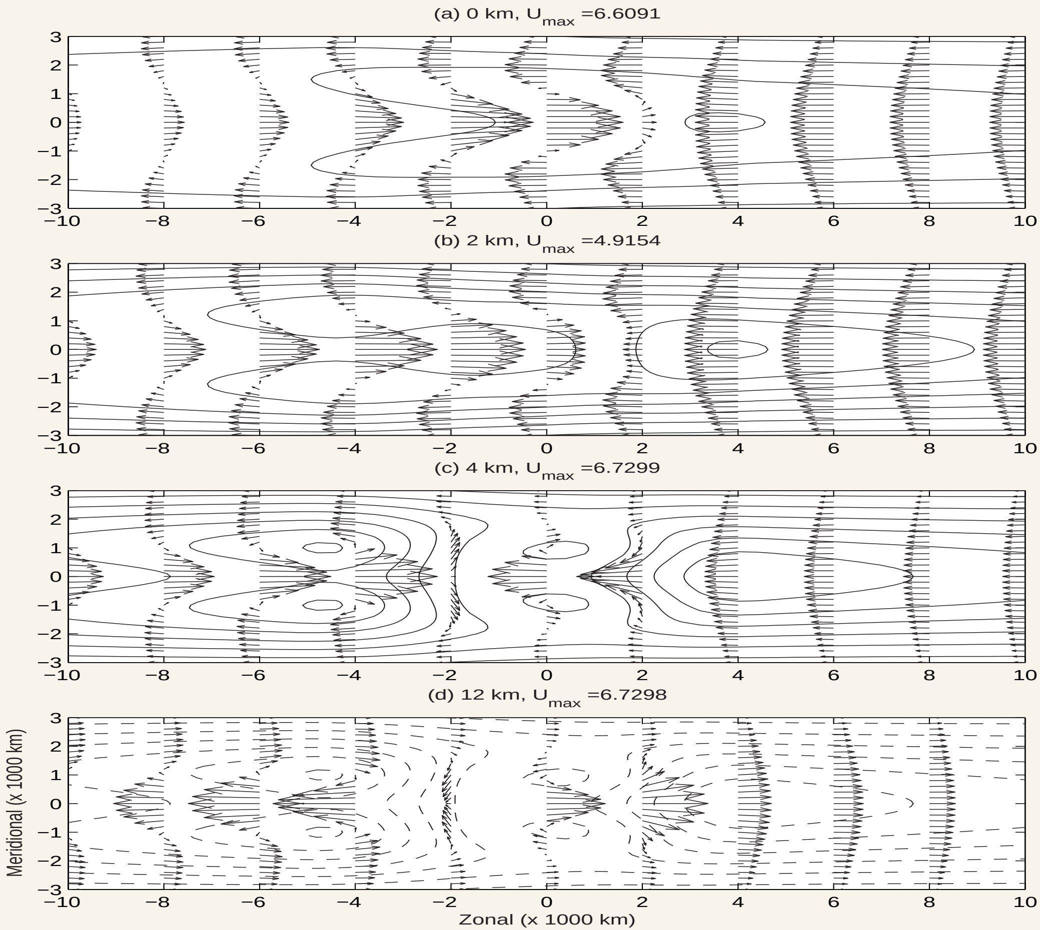
MJO Model: Upscale Fluxes

- Left
  Momentum Flux
- Right
  Temperature Flux
  much weaker



Equatorial MJO model: Flow in the Horizontal Plane

- Congestus heating in the east and westward tilted superclusters in the west of a moving warm pool.
- Planetary mean heating is weaker, but has same structure of synoptic scale fluctuations.
- Pressure and flow at $z = 0, 2, 4, 12$ km.



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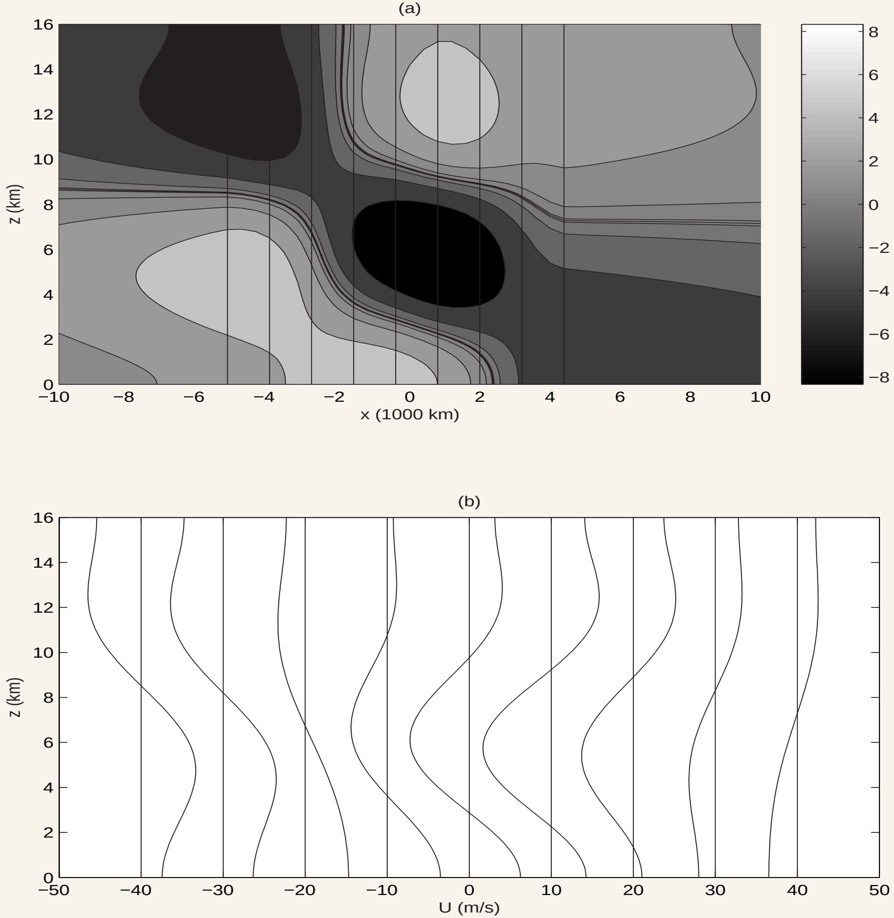
Equatorial MJO model: Winds above the equator

- Lower troposphere
  congestus
  heating in the east
- Westward tilted anvil
  superclusters in the west
- (a) Zonal velocity:
  westerly = light,
  easterly = dark
  versus height and longitude
  above equator
- (b) Height vs Velocity



Meridional Momentum Flux in a Multiscale Context

- Zonal mean of the meridional momentum flux vector

$$F_{mer} = \left\langle \vec{F}^U \right\rangle \cdot \hat{j} = \frac{1}{R} \int_0^R (uv)\ dx$$

- Vertical mean drives barotropic mean winds $\Rightarrow$ superrotation
- What does this mean in a multiscale context, i.e. when

$$u = u' + \overline{U}$$
$$v = v' + \epsilon \overline{V}$$

$$F_{mer} = \frac{2}{L\,R} \int_{-L}^{L} \int_0^R \left(u' + \overline{U}\right)\ \left(v' + \epsilon\,\overline{V}\right)\ dx\,dX$$

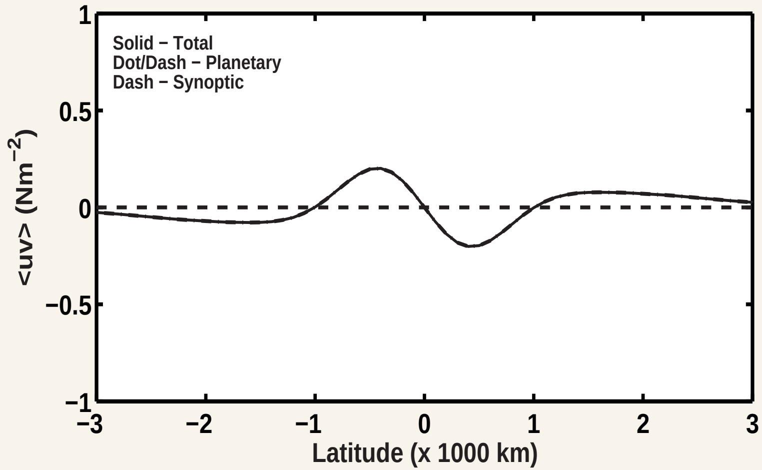
$$= \frac{1}{R} \int_0^R \left(\overline{u'\,v'} + \overline{u'\epsilon\overline{V}} + \overline{v'\overline{U}} + \epsilon\,\overline{U}\,\overline{V}\right)\ dX$$

$$= \frac{1}{R} \int_0^R \left(\overline{u'\,v'} + \epsilon\,\overline{U}\,\overline{V}\right)\ dX$$

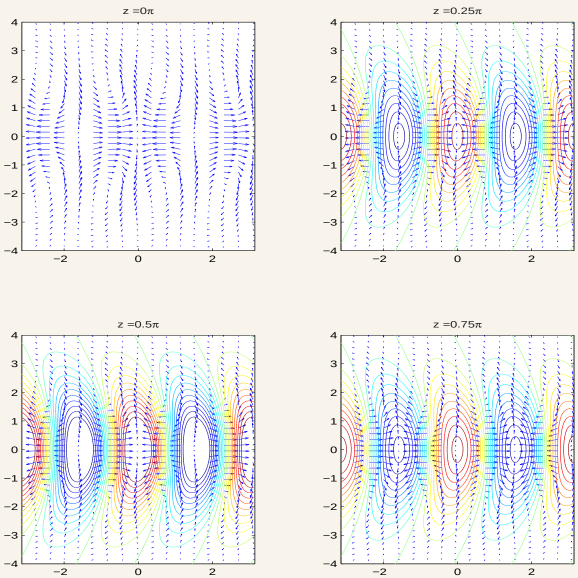
- Therefore synoptic and planetary scale contributions add linearly

Meridional Momentum Flux of the Canonical MJO Model

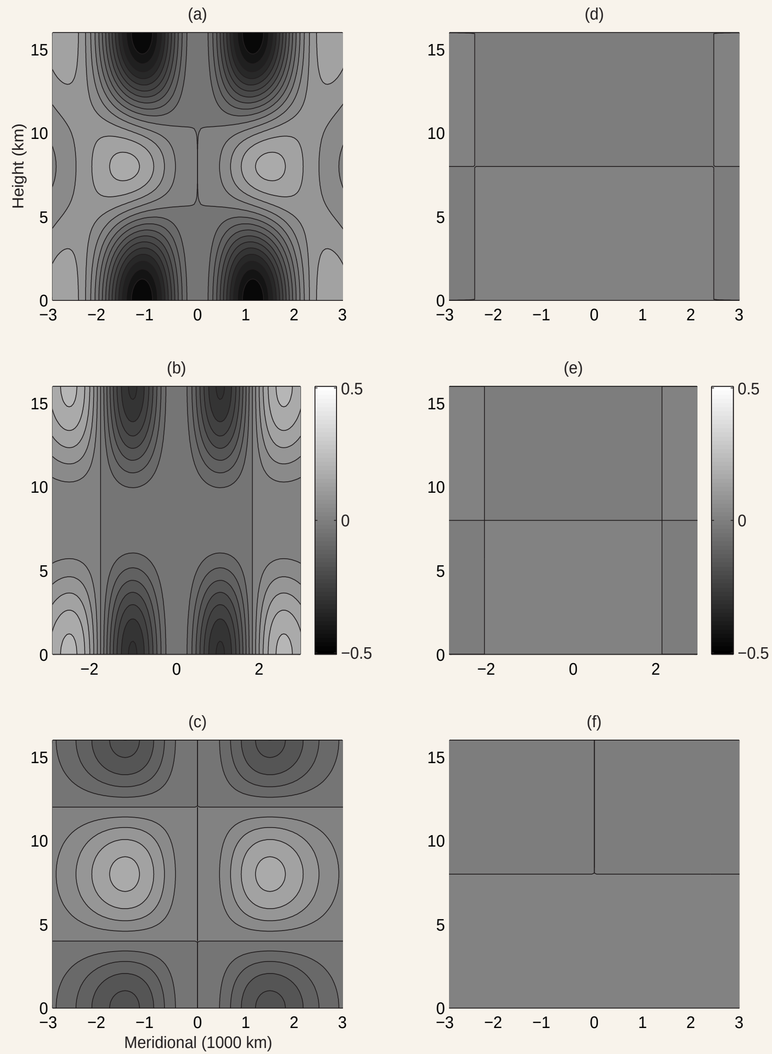
- Vertical mean F_{mer} versus latitude
- $\partial F_{mer}/\partial y < 0 \Longrightarrow$ Westerly Barotropic driving
- $\overline{u'v'} = 0$ by construction
- Only $\overline{U}\,\overline{V}$
- Planetary scale fluxes are equatorially confined
- Correlation of flow due to mean heating and synoptic scale flux



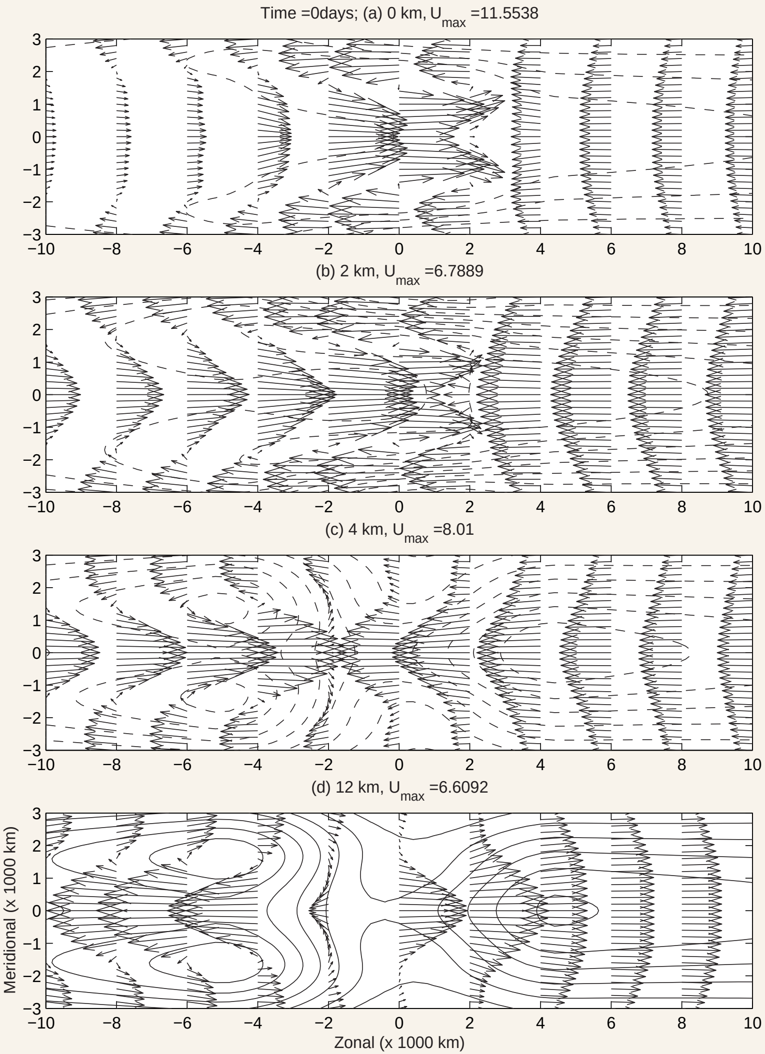
The effect Westward Meridional tilt: Fluxes

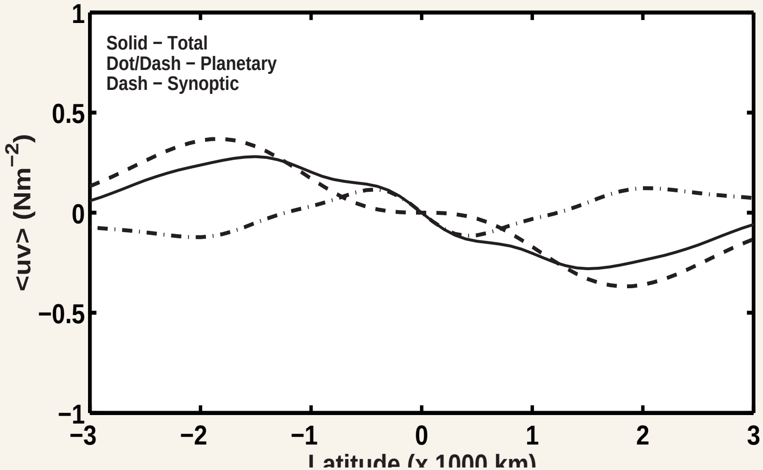


This is the additional flux due to the meridional tilt



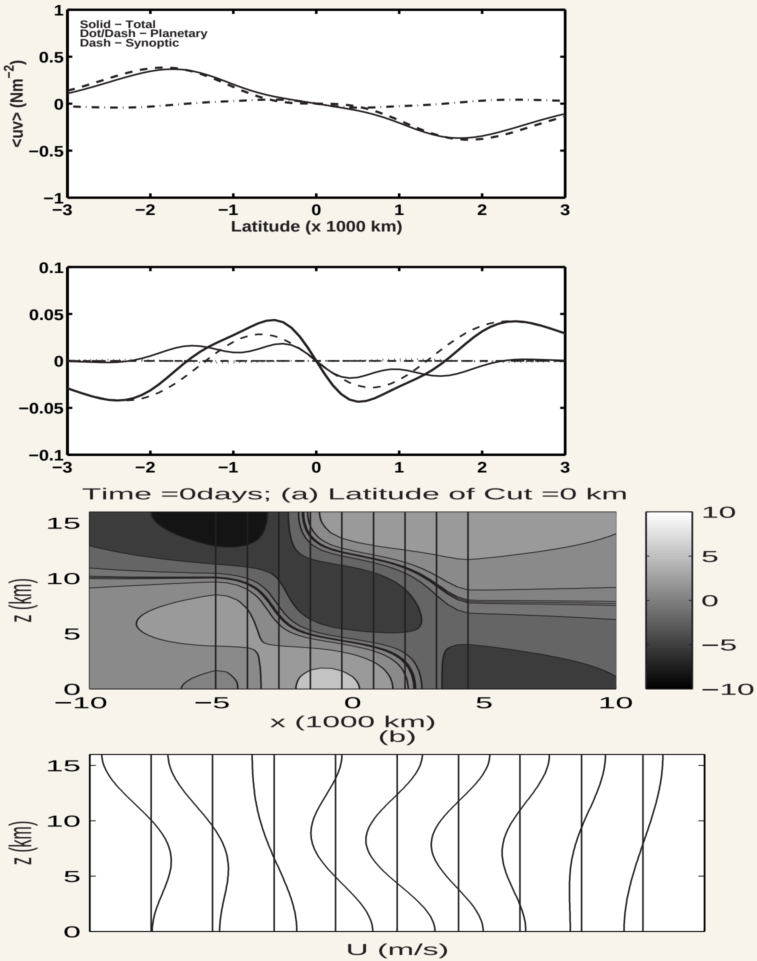
The effect Westward Meridional tilt: Flow

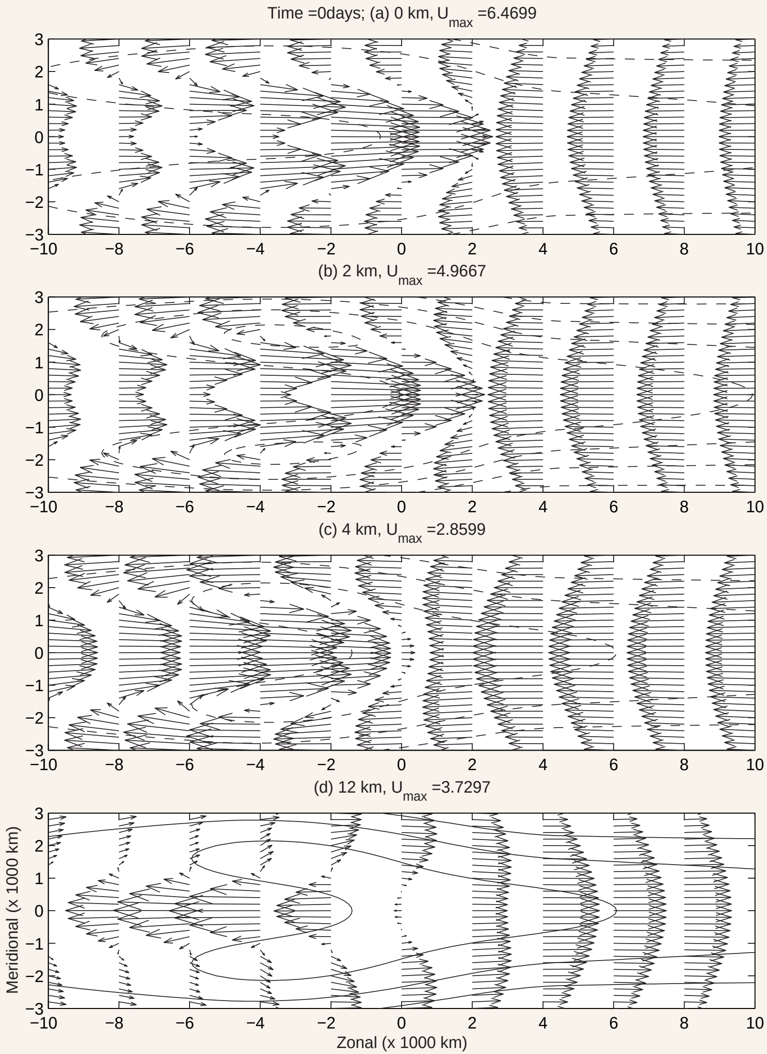




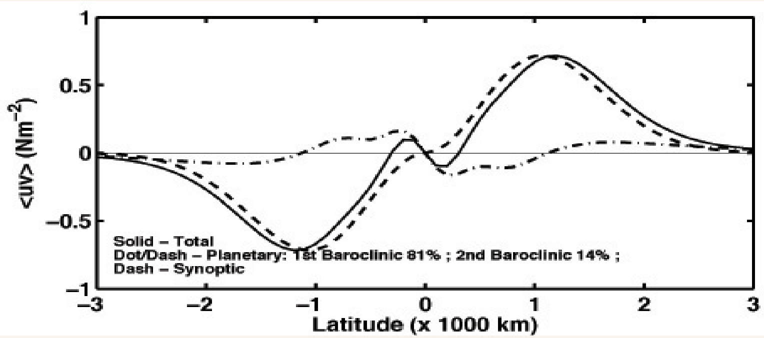
- Westward tilts cause jet to split
- Westerly fluxes throughout equatorial region

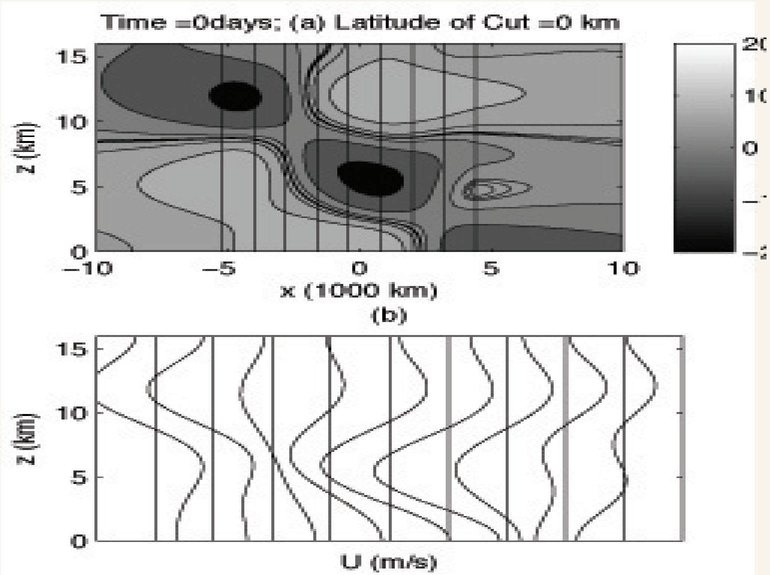
The effect Westward Meridional tilt alone: Flow

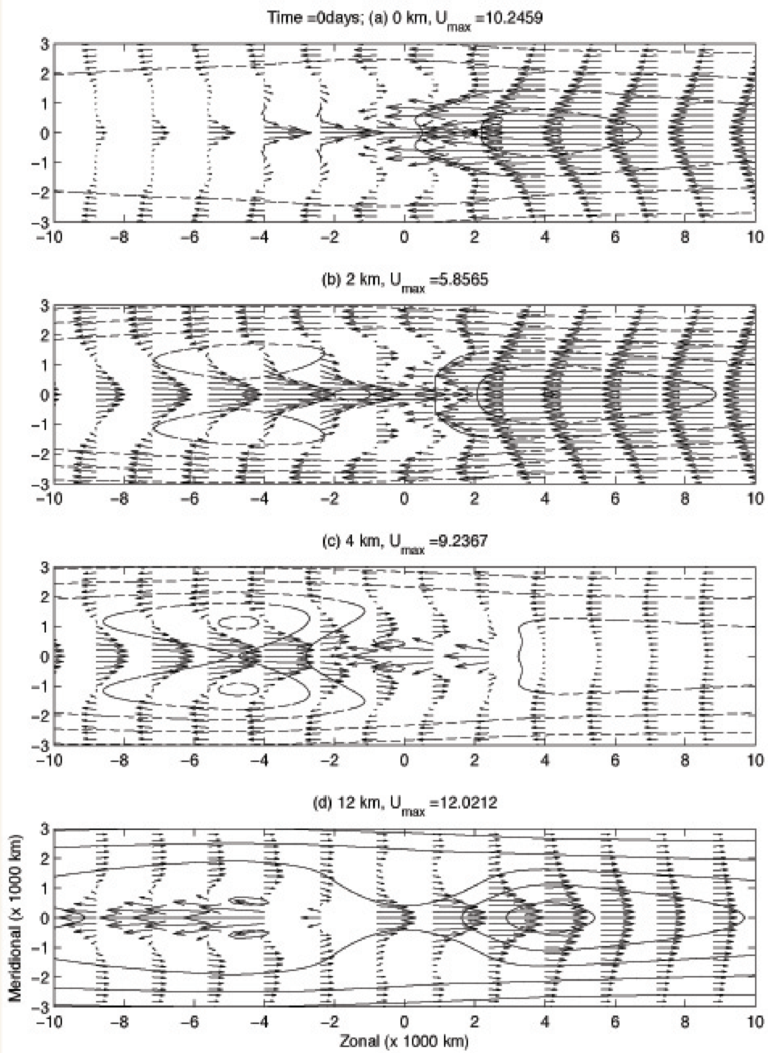




The effect Eastward Meridional tilt alone: Flow







How do vertical tilts drive equatorial superrotation?

Approximate using planetary balanced dynamics $\Longrightarrow$

$$U = \int_{x_0}^{x} \left[-2S^\theta - y\, S^\theta_y + F^U_{yy} \right] \, dx \qquad V = \; y\, S^\theta - F^U_y$$

Heating and vertical tilts yield planetary scale forcing:

$$S^\theta = S \, \cos\left(x/2\right) \, e^{-y^2/2} \qquad F^U = -F \, \sin\left(x\right) \, e^{-y^2/2}$$

The large scale meridional flux is simply

$$\overline{\langle U\, V \rangle} \propto -y\, S\, F\, e^{-y^2} + O(y^2)$$

- Convergence near equator for $SF > 0 \longrightarrow$ Westward anvils
- Convergence near equator for $SF < 0 \longrightarrow$ Eastward anvils

Synopsis: The effect of Meridional Tilt

- Analytic nature of the forcing allows us to systematically investigate the effect of different synoptic scale structures on the planetary scale flow
- Westward tilts to split the westerly wind burst
- Westward tilt alone does not result in the "upward/westward" tilt in observed MJOs
- Eastward tilts tend to confine the wind burst
- Equatorial superrotation arises due to a correlation of mean heating and the vertical upscale fluxes of zonal momentum

Active Moisture in IPESD: Incorporating Khouider/Majda Multicloud Models

$$
\begin{aligned}
\dot{\theta}_1 - \nabla \cdot u_1 &= P - Q_{R,1} - \frac{\theta_1}{\tau_R} \\
\dot{\theta}_2 - \frac{1}{4}\nabla \cdot u_2 &= -H_s + H_c - Q_{R,2} - \frac{\theta_2}{\tau_R} \\
\dot{\theta}_{eb} &= \frac{\theta_{eb}^* - \theta_{eb}}{\tau_e} - \frac{D}{h_b} \\
\dot{q} + Q\left(\nabla \cdot u_1 + \tilde{\lambda}\, \nabla \cdot u_2\right) &= -\frac{2\sqrt{2}}{\pi}\, P + \frac{D}{H_T}
\end{aligned}
$$

- advection terms are not written

Active Moisture in IPESD: Incorporating Khouider/Majda Multicloud Models

$$P = \frac{1-\Lambda}{1-\Lambda^*} \left[\bar{Q}_c + \frac{a_1\theta_{eb} + a_2 q - a_0(\theta_1 + \gamma_2\theta_2)}{\tau_{conv}}\right]^+$$

We will use unclipped P as the dynamical variable

$$D = \frac{m_0}{\bar{Q}_c} \left[\bar{Q}_c + \mu\left(H_s - H_c\right)\right]^+ \left(\theta_{eb} - \theta_{em}\right)$$

$$\Lambda = \begin{cases} 1 & \text{if} & \theta_{eb} - \theta_{em} \geq 20\,K \\ A\left(\theta_{eb} - \theta_{em}\right) + B & \text{if} & 10K < \theta_{eb} - \theta_{em} < 20K \\ \Lambda^* \;\left(= 0.2\right) & \text{if} & \theta_{eb} - \theta_{em} \leq 10K \end{cases}$$

$$\theta_{em} = q + \frac{2\sqrt{2}}{\pi}\left(\theta_1 + \alpha_2\theta_2\right)$$

Active Moisture in IPESD: Incorporating Khouider/Majda Multicloud Models

$$\dot{H}_c + N(\vec{u}_1, \nabla H_c) = \frac{1}{\tau_c}\left[\alpha_c \frac{\Lambda - \Lambda^*}{1-\Lambda^*}\,\kappa - H_c\right]$$
$$\dot{H}_s + N\vec{u}_1, \vec{u}_2, \nabla H_s) = \frac{1}{\tau_s}\left[\alpha_s\, P - H_s\right].$$

The congestus closure is denoted by κ and is either

$$\kappa = \begin{cases} \left[\bar{Q}_c + \frac{a_1\theta_{eb} - a_0(\theta_1 + \gamma_2\theta_2)}{\tau_{CAPE}}\right]^+ & \text{CAPE parametrization} \\ \frac{D}{H_T} & \text{downdraft parametrization} \end{cases}$$

Active Moisture: Ingredients needed to derive multi-scale asymptotics

1. non-dimensionalize variables consistent with IPESD scaling
2. determine radiative/convective equilibrium (RCE)
3. make multiscale ansatz as in IPESD
4. consistently determine multiscale behavior of P, H and θ_{eb}
5. understand how to multiscale a positive definite function, i.e. P
6. solvability condition

Active Moisture: $S_\theta = S_\theta' + \epsilon\, \overline{S}_\theta$

$$\mathbf{P}^+ = \left(\mathbf{P}^+\right)' + \epsilon\, \overline{\mathbf{P}^+} \qquad H = H' + \epsilon\, \overline{H}.$$

1. We expect dynamics whereby the filling fraction of $(\mathbf{P}^+)'$ is $O(\epsilon)$;
2. We expect there to be an $O(\epsilon)$ mean of $\mathbf{P}$ in regions where this mean is greater than zero;
3. We allow for dynamics where the mean of $\mathbf{P}$ is $O(1)$ in regions where this mean is less than zero;

$$\mathbf{P} = \mathbf{P}' + \begin{cases} \overline{\mathbf{P}} & \text{if} \quad \overline{\mathbf{P}} < 0 \\ \epsilon\overline{\mathbf{P}} & \text{if} \quad \overline{\mathbf{P}} > 0. \end{cases}$$

Active Moisture: Positive functions of two scales

$$\lambda = \lambda' + \epsilon\overline{\lambda}$$

We need the positive part

$$\begin{aligned}\lambda^+ &= (\lambda')^+ + \epsilon\, G(\overline{\lambda}, \lambda') + O(\epsilon^2) \\ &= \left[(\lambda')^+ - \overline{(\lambda')^+}\right] + \overline{(\lambda')^+} + \epsilon\, G(\overline{\lambda}, \lambda') + O(\epsilon^2)\end{aligned}$$

Can show that

$$\overline{(\lambda')^+} \leq c_1\, \epsilon$$

which will just renormalize the RCE values. But more importantly

$$\overline{G(\overline{\lambda}, \lambda')} \approx (\overline{\lambda})^+ + O(\epsilon)$$

Synoptic scale dynamics

$$\dot{\theta}'_1 - \nabla'_1 = \Upsilon(\mathbf{P}', \overline{\mathbf{P}}, \theta'_{eb}), \qquad \dot{\theta}'_2 - \tfrac{1}{4}\nabla'_2 = -H', \qquad \dot{\theta}'_{eb} = -\frac{\theta'_{eb}}{\tau_e} - \mathbf{D}$$

$$\begin{aligned}
\dot{\mathbf{P}}' + \nabla' &= -\alpha_1\,\Upsilon(\mathbf{P}', \overline{\mathbf{P}}, \theta'_{eb}) + \alpha_2 H' - \alpha_3\,\mathbf{D} - \frac{\theta'_{eb}}{\tau_e} \\
\dot{H}' &= \frac{1}{\tau_H}\left[\alpha_s \Upsilon(\mathbf{P}', \overline{\mathbf{P}}, \theta'_{eb}) - \alpha_c \left(\frac{\Lambda - \Lambda^*}{1 - \Lambda^*}\right)\mathbf{D} - H'\right]
\end{aligned}$$

where

$$\begin{aligned}
\Upsilon(\mathbf{P}', \overline{\mathbf{P}}, \theta'_{eb}) &= \frac{1 - \Lambda(\theta'_{eb})}{1 - \Lambda^*}\begin{cases} \frac{\left(\mathbf{P}' + \overline{\mathbf{P}}\right)^+}{\tau} & \overline{\mathbf{P}} < 0 \\ \frac{\mathbf{P}'^+}{\tau} & \overline{\mathbf{P}} > 0. \end{cases} \\
\mathbf{D} &= \frac{\Delta^*}{\tau_e}\left\{\left[1 + \frac{H'}{Q_0}\right]^+ \left[1 + \frac{\theta'_{eb}}{\Delta}\right] - 1\right\}
\end{aligned}$$

Planetary scale dynamics

$$\begin{aligned}
\partial_T\,\bar\theta_1 - \bar\nabla_1 &= -N(\theta',u') + \frac{1-\bar\Lambda}{1-\Lambda^*}\frac{\overline{\mathbf{P}}^+}{\tau} \\
\partial_T\,\bar\theta_2 - \frac{1}{4}\bar\nabla_2 &= -N(\theta',u') - \bar H \\
\bar\theta_{eb} &= -\tau_e\,N(\theta',u') - \Delta^*\,\Gamma(H',\bar H,\theta'_{eb},\bar\theta_{eb})
\end{aligned}$$

$$\bar H = \; -N(H',u') + \alpha_s\,\tfrac{1-\bar\Lambda}{1-\Lambda^*}\tfrac{\overline{\mathbf{P}}^+}{\tau} - \alpha_c\tilde h\,\left(\tfrac{\bar\Lambda-\Lambda^*}{1-\Lambda^*}\right)\,\tfrac{\Delta^*}{\tau_e}\,\Gamma(H',\bar H,\theta'_{eb},\bar\theta_{eb})$$

$$\partial_T\,\bar{\mathbf{P}} + \bar\nabla = -N(\mathbf{P}',u') + \alpha_2\,\bar H - \alpha_3\,\frac{\Delta^*}{\tau_e}\,\Gamma(H',\bar H,\theta'_{eb},\bar\theta_{eb}) - \frac{\bar\theta_{eb}}{\tau_e}$$

for $\bar{\mathbf{P}} < 0$ and

$$\alpha_1\,\frac{1-\bar\Lambda}{1-\Lambda^*}\frac{\overline{\mathbf{P}}^+}{\tau} = -N(\mathbf{P}',u') - \bar\nabla + \alpha_2\,\bar H - \alpha_3\,\frac{\Delta^*}{\tau_e}\,\Gamma(H',\bar H,\theta'_{eb},\bar\theta_{eb}) - \frac{\bar\theta_{eb}}{\tau_e}$$

for $\bar{\mathbf{P}} > 0$

Planetary scale dynamics

The function Γ can be estimated under certain assumptions

$$\begin{aligned}\Gamma(H', \bar{H}, \theta'_{eb}, \bar{\theta}_{eb}) &\equiv \frac{1}{\epsilon}\left\{\overline{\left(1+\frac{H'+\epsilon\bar{H}}{Q_0}\right)^{+}\left(1+\frac{\theta'_{eb}+\epsilon\bar{\theta}_{eb}}{\Delta}\right)-1}\right\} \\ &\approx -r+\frac{1}{\epsilon}\frac{\overline{H'\,\theta'_{eb}}}{\Delta\,Q_0}+\frac{\bar{H}}{Q_0}+\frac{\bar{\theta}_{eb}}{\Delta}+O(\epsilon).\end{aligned}$$

- r is a constant which renormalizes RCE
- under small correlation assumption, numerator of second term $O(\epsilon)$

Summary