On the Distribution of Subsidence in the Hurricane Eye

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IPAM: The Hurricane
12 February 2007

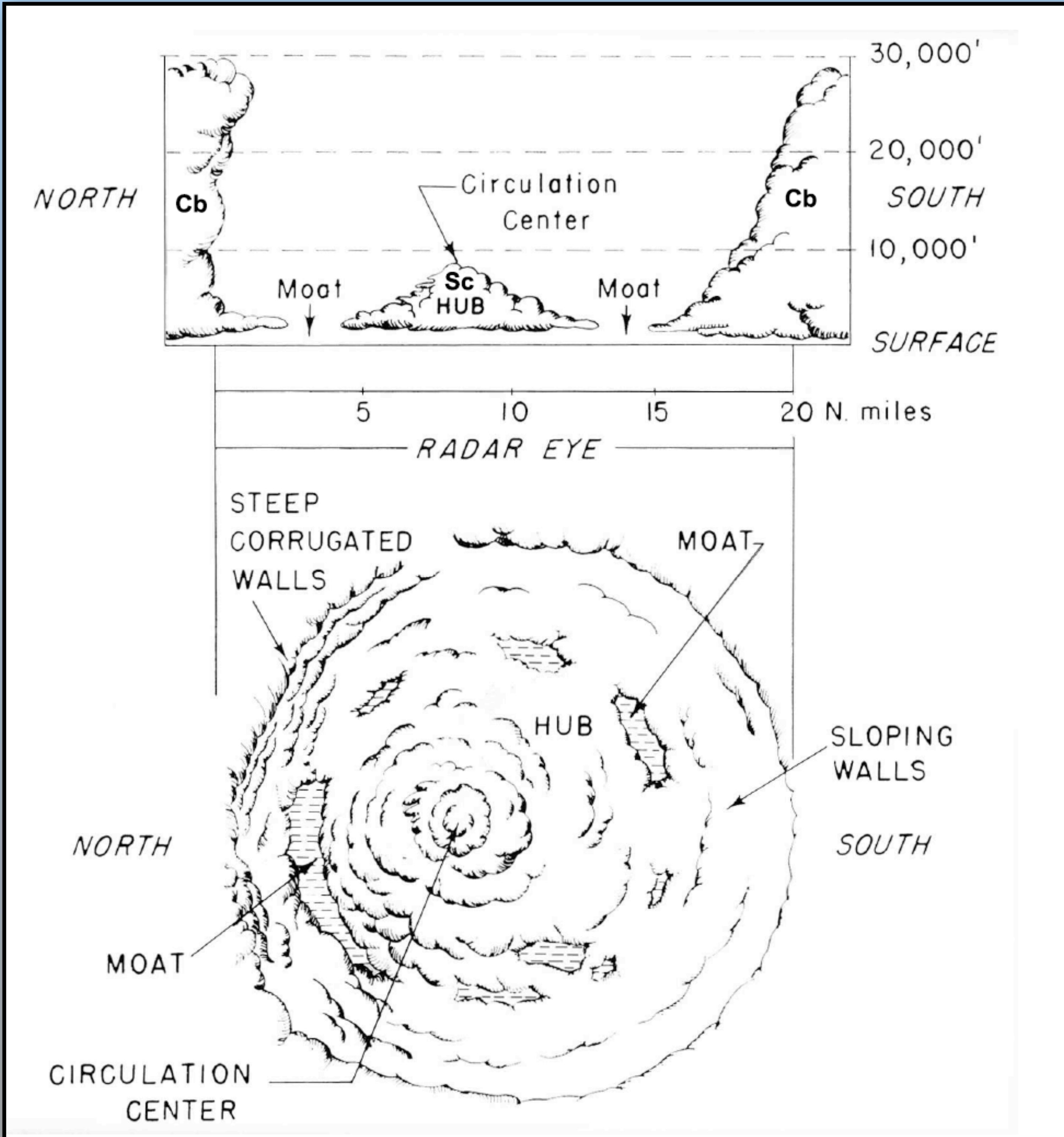
Hub Clouds, Moats, and Warm Rings

Simpson, R.H., and L.G. Starrett, 1955:
Bull. Amer. Meteor. Soc., 36, 459-468.

Fletcher, R.D., J.R. Smith, and R.C. Bundgaard, 1961:
Weatherwise, 14, 102-109.

LaSeur, N.E., and H.F. Hawkins, 1963:
Mon. Wea. Rev., 91, 694-709.

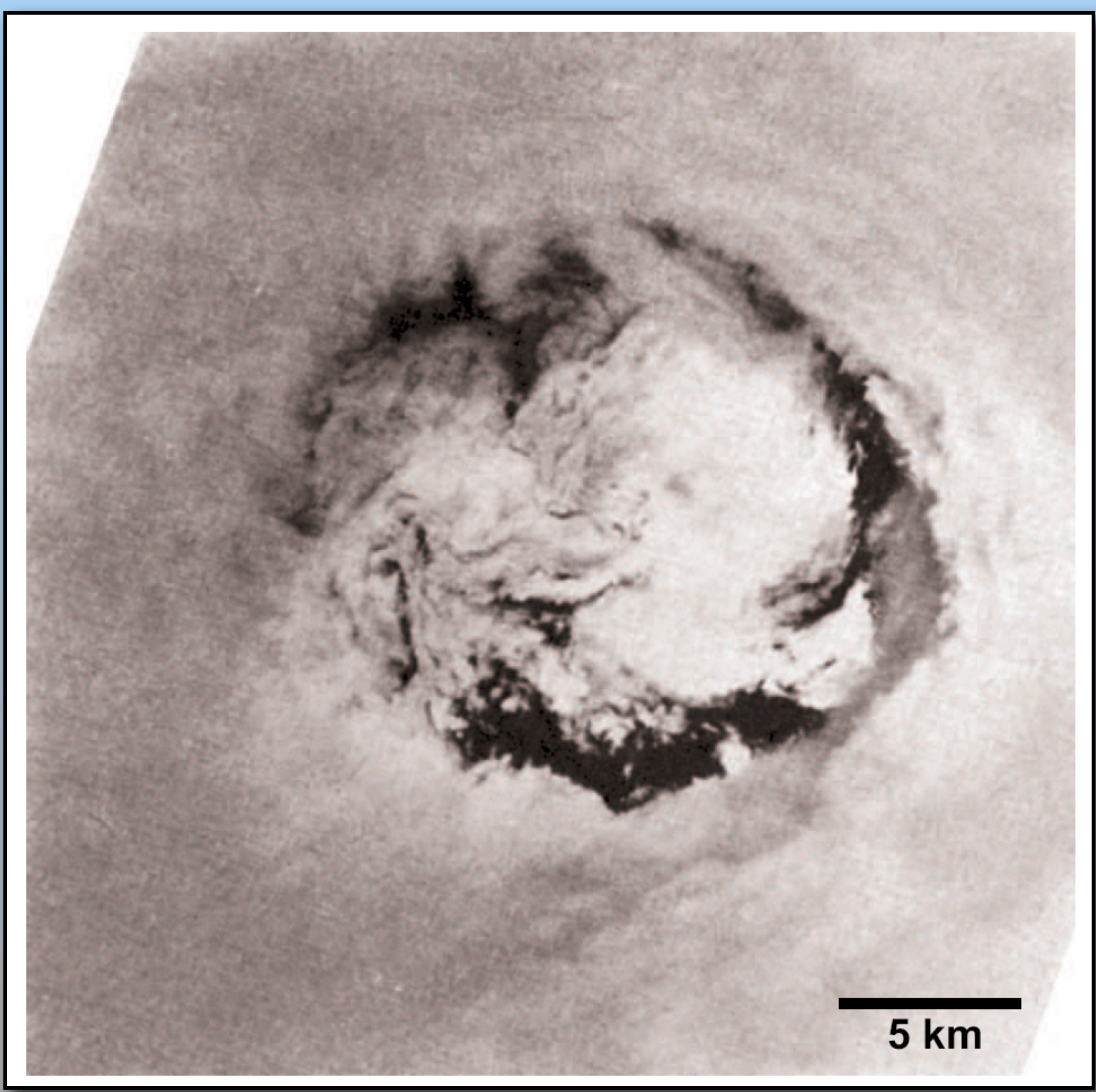
Simpson and Starrett (1955)



Schematic Diagram of the Eye of Hurricane Edna

9-10 Sept 1954

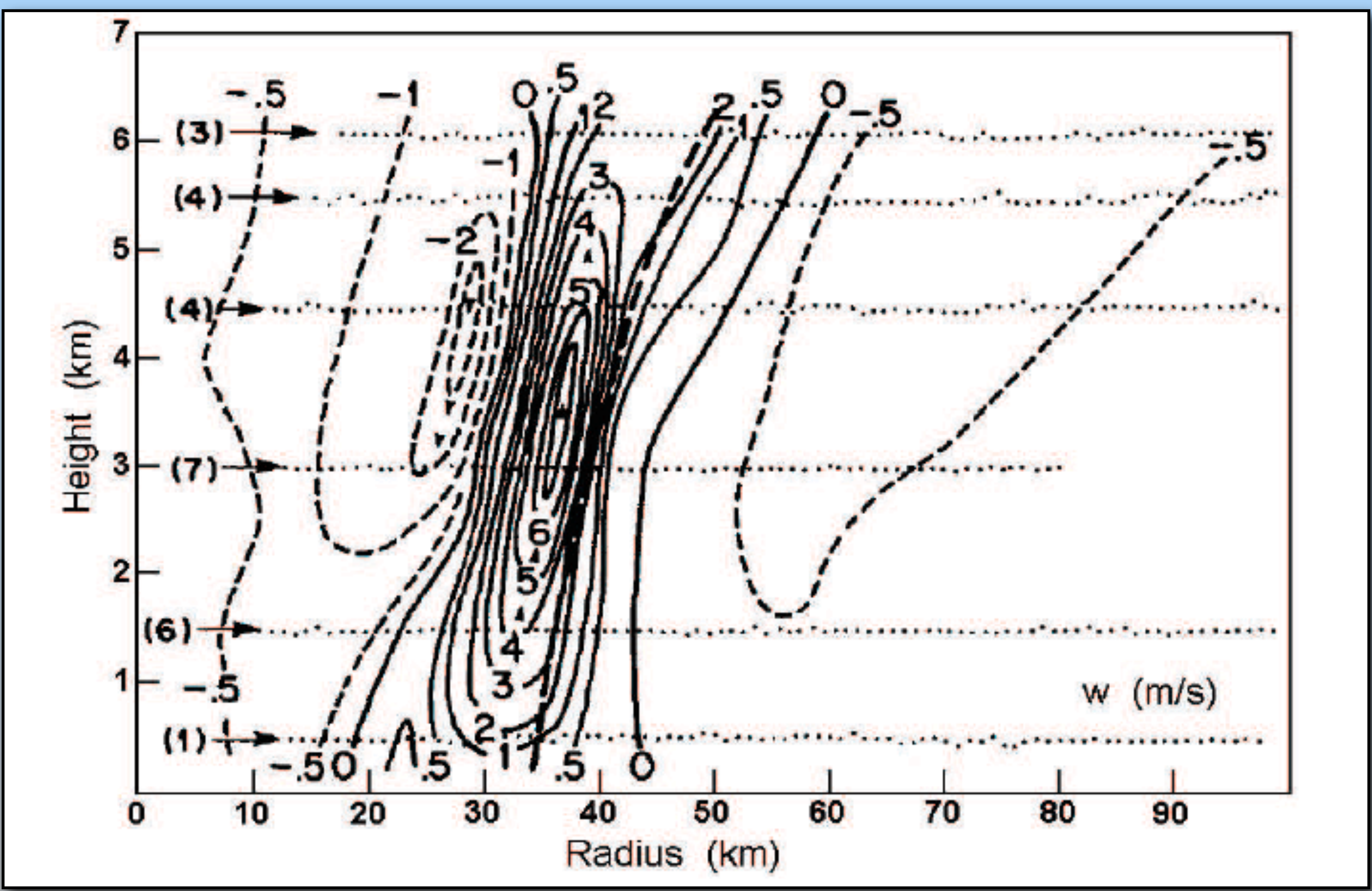
Fletcher et al. (1961)



U-2 Photograph
of
Typhoon Ida

25 Sept 1958

Hurricane Allen Vertical Velocities

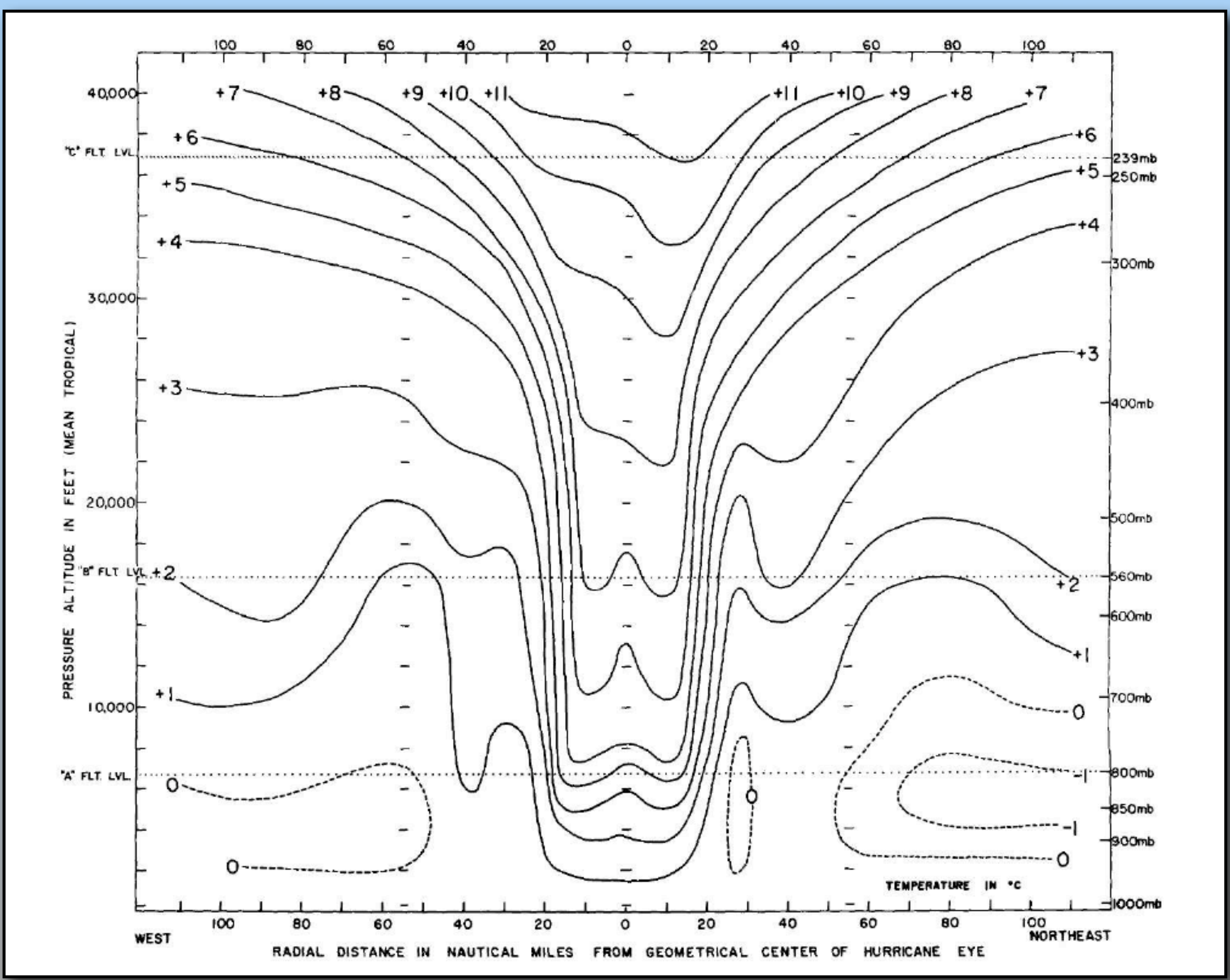


5 Aug 1980

Jorgensen (1984)

LaSeur and Hawkins (1963)

Hurricane Cleo (1958)



Temperature Anomaly

Balanced Vortex Model

- Assumptions:
  - Axisymmetric
  - Inviscid
  - Hydrostatic
  - Gradient Balanced

- Vertical Coordinate:

$$\boxed{z = H \ln\left(\frac{p_0}{p}\right)}$$

where

$$\boxed{H = \frac{RT_0}{g}}$$

constant scale height

Balanced Vortex Model

Governing Equations

$$\left(f + \frac{v}{r}\right) v = \frac{\partial \phi}{\partial r}$$

Gradient Balance

$$\frac{Dv}{Dt} + \left(f + \frac{v}{r}\right) u = 0$$

Momentum Eq.

$$\frac{\partial \phi}{\partial z} = \frac{g}{T_0} T$$

Hydrostatic Eq.

$$\frac{\partial (ru)}{r \partial r} + \frac{\partial w}{\partial z} - \frac{w}{H} = 0$$

Continuity Eq.

$$c_p \frac{DT}{Dt} + \frac{RT}{H} w = Q$$

Thermodynamic Eq.

Streamfunction for the Transverse Circulation

- The continuity equation

$$\boxed{\frac{\partial (ru)}{r\partial r} + \frac{\partial w}{\partial z} - \frac{w}{H} = 0}$$

is satisfied by

$$\boxed{e^{-z/H} u = -\frac{\partial \psi}{\partial z}} \quad \text{and} \quad \boxed{e^{-z/H} w = \frac{\partial (r\psi)}{r\partial r}}$$

Thermal Wind Equation

- Derived from the gradient balance and hydrostatic equations:

$$\boxed{\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial z} = \frac{g}{T_0}\frac{\partial T}{\partial r}}$$

- Taking the local time derivative:

$$\boxed{\frac{\partial}{\partial z}\left\{\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial t}\right\} = \frac{g}{T_0}\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial t}\right)}$$

Deriving the Transverse Circulation Equation

- Thermodynamic equation yields:

$$\frac{g}{T_0}\frac{\partial T}{\partial t} + A\frac{\partial(r\psi)}{r\partial r} + B\frac{\partial \psi}{\partial z} = \frac{g}{c_p T_0} Q$$

- Tangential wind equation yields:

$$-\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial t} + B\frac{\partial(r\psi)}{r\partial r} + C\frac{\partial \psi}{\partial z} = 0$$

- Variable coefficients:

$$A = e^{z/H}\frac{g}{T_0}\left(\frac{\partial T}{\partial z} + \frac{\kappa T}{H}\right)$$

Static Stability

$$B = -e^{z/H}\left(f + \frac{2v}{r}\right)\frac{\partial v}{\partial z} = -e^{z/H}\frac{g}{T_0}\frac{\partial T}{\partial r}$$

Baroclinity

$$C = e^{z/H}\left(f + \frac{2v}{r}\right)\left(f + \frac{\partial(rv)}{r\partial r}\right)$$

Inertial Stability

Sawyer-Eliassen Transverse Circulation Equation

$$\frac{\partial}{\partial r}\left(A\frac{\partial(r\psi)}{r\partial r}+B\frac{\partial\psi}{\partial z}\right)+\frac{\partial}{\partial z}\left(B\frac{\partial(r\psi)}{r\partial r}+C\frac{\partial\psi}{\partial z}\right)=\frac{g}{c_pT_0}\frac{\partial Q}{\partial r}$$

- Boundary Conditions:

$$\psi=0 \quad \text{if} \quad r=0, \quad \text{or} \quad z=0, z_T$$

$$r\psi\rightarrow 0 \quad \text{as} \quad r\rightarrow\infty$$

- Elliptic if: $AC-B^2>0$

Idealized Vortex

- Consider an idealized vortex such that:

$$\boxed{\begin{aligned} A &= e^{z/H} N^2 \\ B &= 0 \end{aligned}}$$

(Assume N is constant)

(Assume vortex is barotropic)

- The inertial stability can then be written:

$$\boxed{C = e^{z/H} \hat{f}^2}$$

where

$$\boxed{\hat{f}(r) = \left[\left(f + \frac{2v}{r} \right) \left(f + \frac{\partial (rv)}{r \partial r} \right) \right]^{1/2}}$$

effective Coriolis parameter

Simplified Transverse Circulation Equation

$$N^2 \frac{\partial}{\partial r}\left(\frac{\partial(r\psi)}{r\partial r}\right) + \hat{f}^2 e^{-z/H}\frac{\partial}{\partial z}\left(e^{z/H}\frac{\partial \psi}{\partial z}\right) = \frac{g e^{-z/H}}{c_p T_0}\frac{\partial Q}{\partial r}$$

- Now assume separable solutions:

$$Q(r,z) = \hat{Q}(r)\exp\left(\frac{z}{2H}\right)\sin\left(\frac{\pi z}{z_T}\right)$$

$$\psi(r,z) = \hat{\psi}(r)\exp\left(-\frac{z}{2H}\right)\sin\left(\frac{\pi z}{z_T}\right)$$

- The above PDE now reduces to an ODE.

Radial Structure Equation

$$r^2\frac{d^2\hat{\psi}}{dr^2} + r\frac{d\hat{\psi}}{dr} - \left(\mu^2 r^2 + 1\right)\hat{\psi} = \frac{gr^2}{c_p T_0 N^2}\frac{d\hat{Q}}{dr}$$

where

$$\mu = \frac{\hat{f}}{N}\left(\frac{\pi^2}{z_T^2} + \frac{1}{4H^2}\right)^{1/2}$$

inverse Rossby length

- We are particularly interested in the important role played by radial variations of the inverse Rossby length.

Idealized Barotropic Vortex

- Now consider the following barotropic vortex:

$$rv(r) = \frac{1}{2}\begin{cases} \left(\hat{f}_0 - f\right) r^2, & \text{if } 0 \le r \le r_1 \\ \left\{\hat{f}_0^2 r_1^4 + \hat{f}_1^2\left(r^4 - r_1^4\right)\right\}^{1/2} - f r^2, & \text{if } r_1 \le r \le r_2 \\ \left\{\hat{f}_0^2 r_1^4 + \hat{f}_1^2\left(r_2^4 - r_1^4\right) + \hat{f}_2^2\left(r^4 - r_2^4\right)\right\}^{1/2} - f r^2, & \text{if } r_2 \le r < \infty \end{cases}$$

- The resulting effective Coriolis parameter is piecewise constant:

$$\hat{f}(r) = \left\{\left(f + \frac{2v}{r}\right)\left(f + \frac{\partial(rv)}{r\partial r}\right)\right\}^{1/2} = \begin{cases} \hat{f}_0, & \text{if } 0 \le r < r_1 \text{ (eye)} \\ \hat{f}_1, & \text{if } r_1 < r < r_2 \text{ (eyewall)} \\ \hat{f}_2, & \text{if } r_2 < r < \infty \text{ (far-field)} \end{cases}$$

Diabatic Heating

- Now assume the heating is confined to the eyewall region:

$$\hat{Q}(r) = \begin{cases} 0 & \text{if } 0 \le r < r_1 \text{ (eye)} \\ Q_1 & \text{if } r_1 < r < r_2 \text{ (eyewall)} \\ 0 & \text{if } r_2 < r < \infty \text{ (far-field)} \end{cases}$$

- Parameter constraint (fixes the area-averaged rainfall):

$$(Q_1/c_p)(r_2^2 - r_1^2) = 125\,\mathrm{K\ day^{-1}}(50\ \mathrm{km})^2$$

Resulting Radial Structure Equations

- Governing equation for each region:

$$r^2\frac{d^2\hat{\psi}}{dr^2}+r\frac{d\hat{\psi}}{dr}-\left(\mu_j^2r^2+1\right)\hat{\psi}=0$$

j=0 eye
j=1 eyewall
j=2 far-field

- Jump conditions at the edges of the eyewall:

$$\left[\frac{d(r\hat{\psi})}{r\,dr}\right]_{r_1-}^{r_1+}=\frac{gQ_1}{c_pT_0N^2}$$

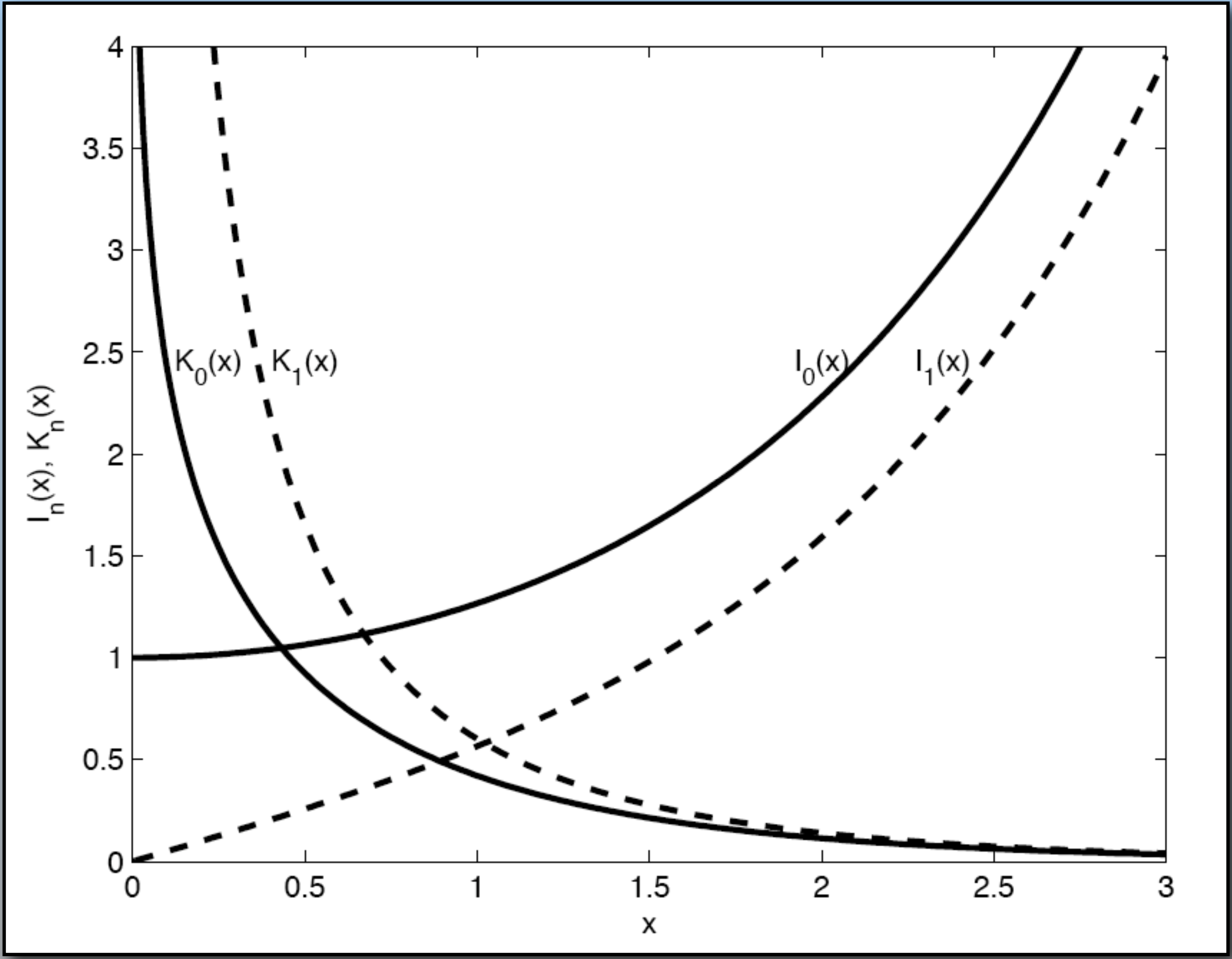
(inner edge of eyewall)

$$\left[\frac{d(r\hat{\psi})}{r\,dr}\right]_{r_2-}^{r_2+}=-\frac{gQ_1}{c_pT_0N^2}$$

(outer edge of eyewall)

- With our heating profile, solutions are simply the first-order modified Bessel functions.

Modified Bessel Functions



Streamfunction Solution

$$\hat{\psi}(r) = \begin{cases} \hat{\psi}_1 I_1(\mu_0 r)/I_1(\mu_0 r_1) & \text{if } 0 \leq r \leq r_1 \\ \{\hat{\psi}_1 F(r, r_2) + \hat{\psi}_2 F(r_1, r)\}/F(r_1, r_2) & \text{if } r_1 \leq r \leq r_2 \\ \hat{\psi}_2 K_1(\mu_2 r)/K_1(\mu_2 r_2) & \text{if } r_2 \leq r < \infty \end{cases}$$

where $F(x, y) = I_1(\mu_1 x) K_1(\mu_1 y) - K_1(\mu_1 x) I_1(\mu_1 y)$

- The parameters $\hat{\psi}_1$ and $\hat{\psi}_2$ are determined by the two jump conditions.
- Note that this solution is continuous at the edges of the eyewall.

Vertical Motion Solution

- Note the following derivative formulas for modified Bessel functions:

$$\frac{d[rI_1(\mu r)]}{rdr} = \mu I_0(\mu r)$$

$$\frac{d[rK_1(\mu r)]}{rdr} = -\mu K_0(\mu r)$$

- The vertical motion solution then becomes:

$$\frac{d(r\hat{\psi})}{r\,dr} = \begin{cases} \hat{\psi}_1\mu_0 I_0(\mu_0 r)/I_1(\mu_0 r_1) & \text{if } 0 \leq r < r_1 \\ \{\hat{\psi}_1\mu_1 G(r, r_2) - \hat{\psi}_2\mu_1 G(r, r_1)\}/F(r_1, r_2) & \text{if } r_1 < r < r_2 \\ -\hat{\psi}_2\mu_2 K_0(\mu_2 r)/K_1(\mu_2 r_2) & \text{if } r_2 < r < \infty \end{cases}$$

where

$$G(x, y) = I_0(\mu_1 x)K_1(\mu_1 y) + K_0(\mu_1 x)I_1(\mu_1 y)$$

Temperature Tendency

- The temperature tendency for a barotropic vortex is given by:

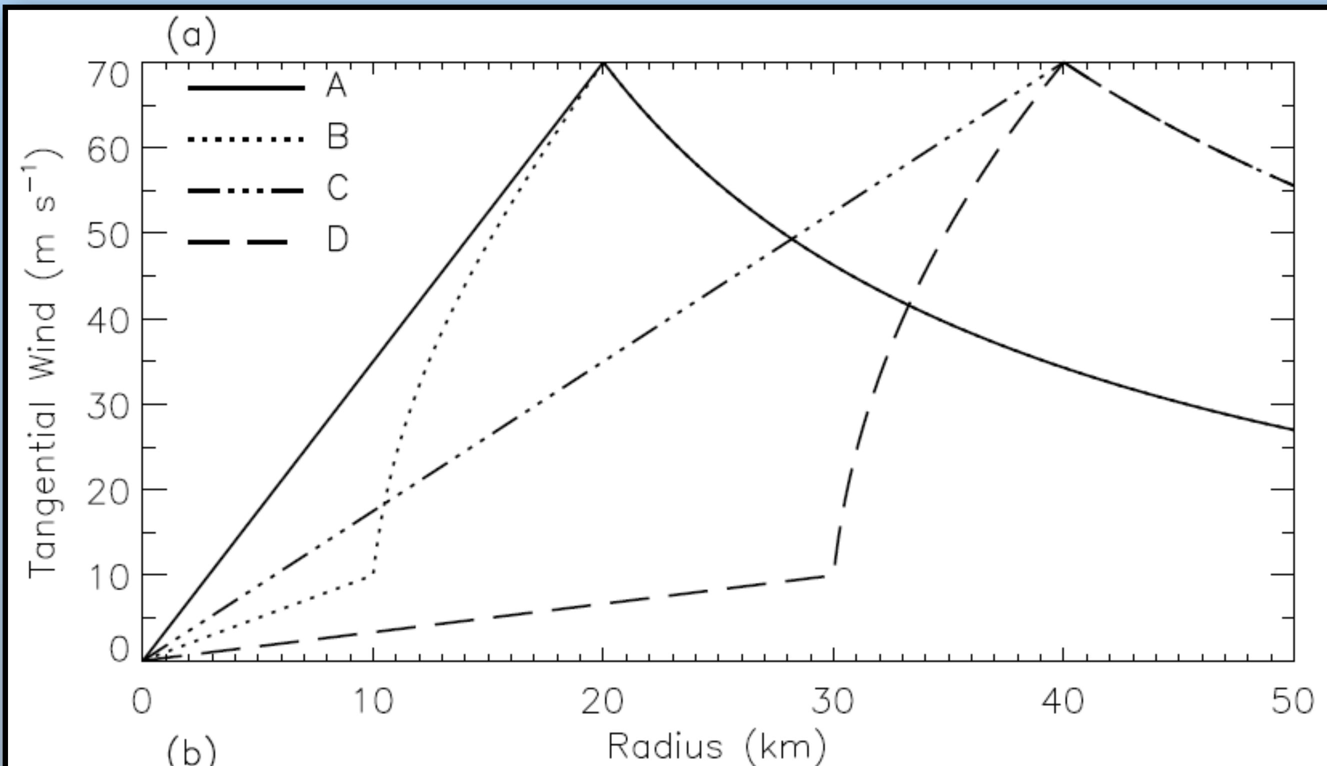
$$\boxed{\frac{\partial \hat{T}}{\partial t} = \frac{\hat{Q}}{c_p} - \frac{T_0 N^2}{g}\frac{d(r\hat{\psi})}{rdr}}$$

- With our heating profile, this becomes:

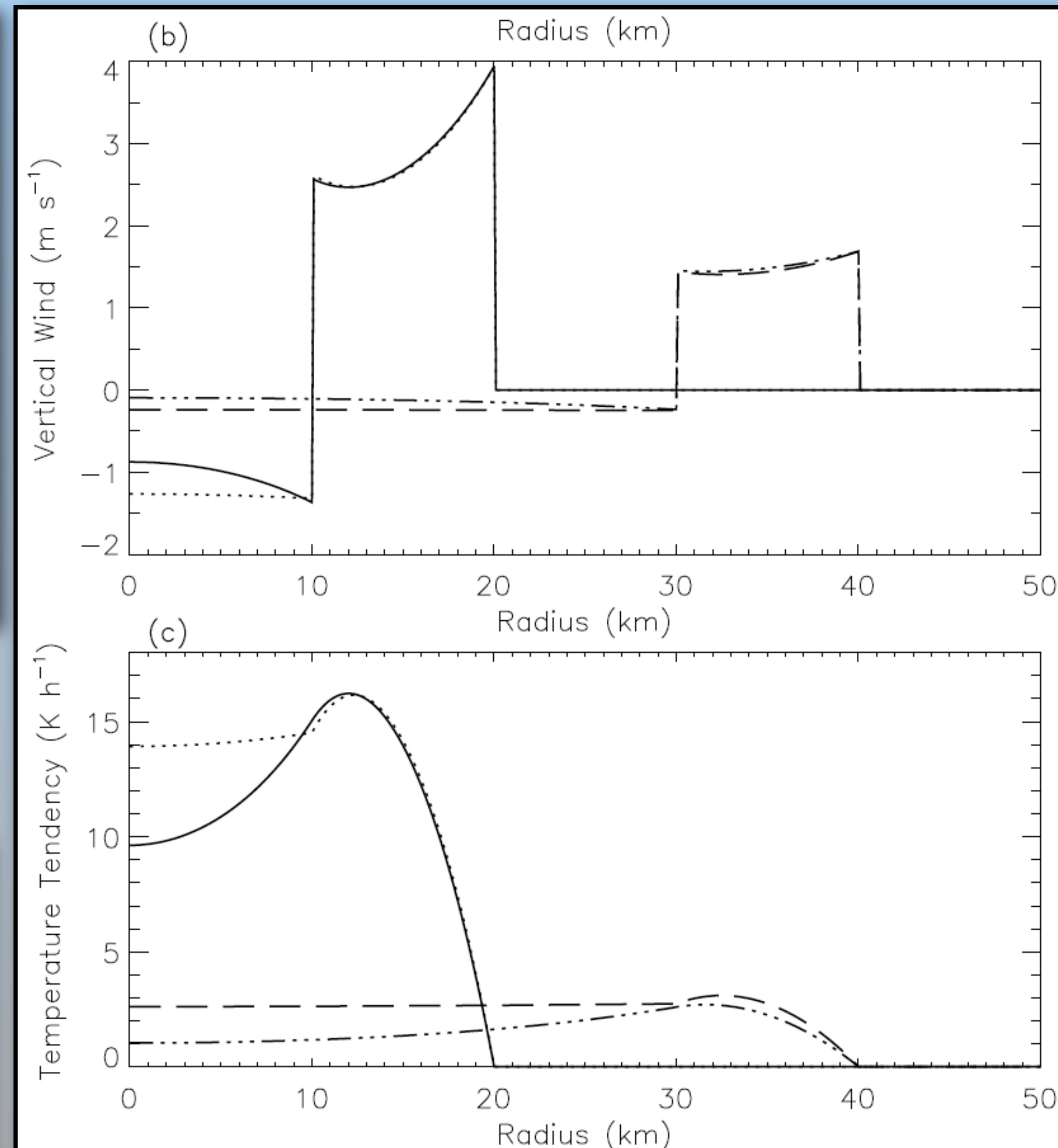
$$\boxed{\frac{\partial \hat{T}}{\partial t} = \frac{Q_1}{c_p}\begin{cases} \left\{1 - \left(\frac{1-\alpha}{1-\alpha\beta}\right)\mu_1 r_2 G(r_1, r_2) - \left(\frac{1-\beta}{1-\alpha\beta}\right)\right\}\frac{I_0(\mu_0 r)}{I_0(\mu_0 r_1)} & 0 \le r \le r_1 \\ 1 - \left(\frac{1-\alpha}{1-\alpha\beta}\right)\mu_1 r_2 G(r, r_2) - \left(\frac{1-\beta}{1-\alpha\beta}\right)\mu_1 r_1 G(r, r_1) & r_1 \le r \le r_2 \\ \left\{1 - \left(\frac{1-\alpha}{1-\alpha\beta}\right) - \left(\frac{1-\beta}{1-\alpha\beta}\right)\mu_1 r_1 G(r_2, r_1)\right\}\frac{K_0(\mu_2 r)}{K_0(\mu_2 r_2)} & r_2 \le r < \infty \end{cases}}$$

Plots for Different Vortices

Tangential Wind



Vertical Velocity



Temperature Tendency

Four Different Vortices

Case	r_1 (km)	r_2 (km)	$\hat{f}_0/f$	$\hat{f}_1/f$	$\hat{f}_2/f$
A	10	20	141.0	141.0	1.0
B	10	20	41.0	145.2	1.0
C	30	40	71.0	71.0	1.0
D	30	40	14.3	85.3	1.0

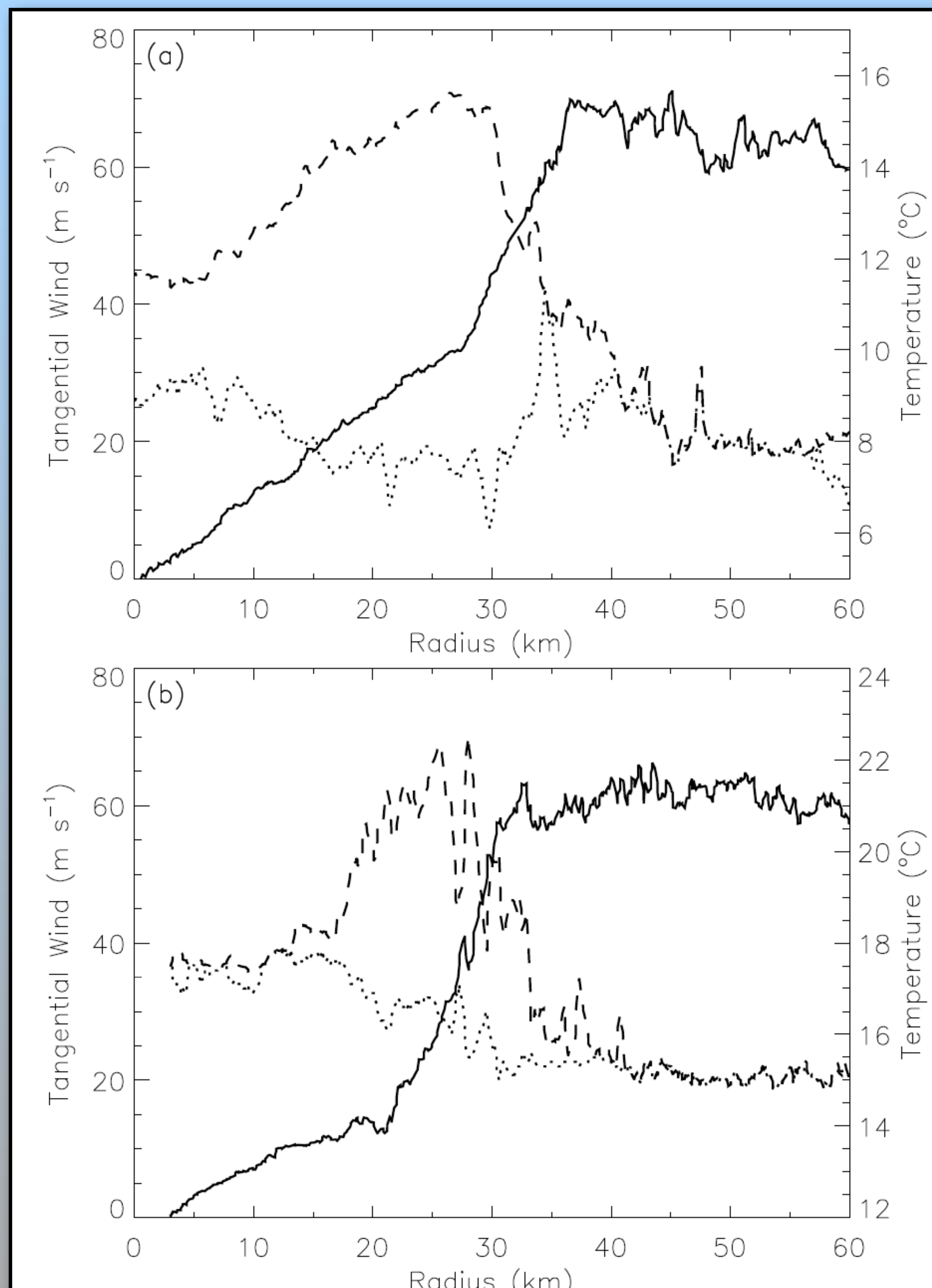
Hub Cloud of Hurricane Isabel

13 Sept 2003

Courtesy of Sim Aberson

Hurricane Isabel Flight-Level Data

Solid: Tangential Wind
Dashed: Temperature
Dotted: Dewpoint Temp.



3.7 km altitude

2.1 km altitude

13 Sept 2003

Conclusions

- There is less than 10% horizontal variation in the eye subsidence when the ratio of eye radius to Rossby length in the eye is less than 0.6
- The subsidence rate at the edge of the eye is more than twice as strong as that at the center of the eye when the ratio of eye radius to Rossby length in the eye is greater than 1.8 (i.e., large eyes and/or eyes with high inertial stability)

Conclusions (continued)

- The existence of a hub cloud at the center of the eye, cascading pileus in the upper troposphere on the edge of the eye, a clear inner moat in the lower troposphere on the edge of the eye, and a warm-ring structure are all associated with strong inertial stability in the eye and a relatively large eye radius.

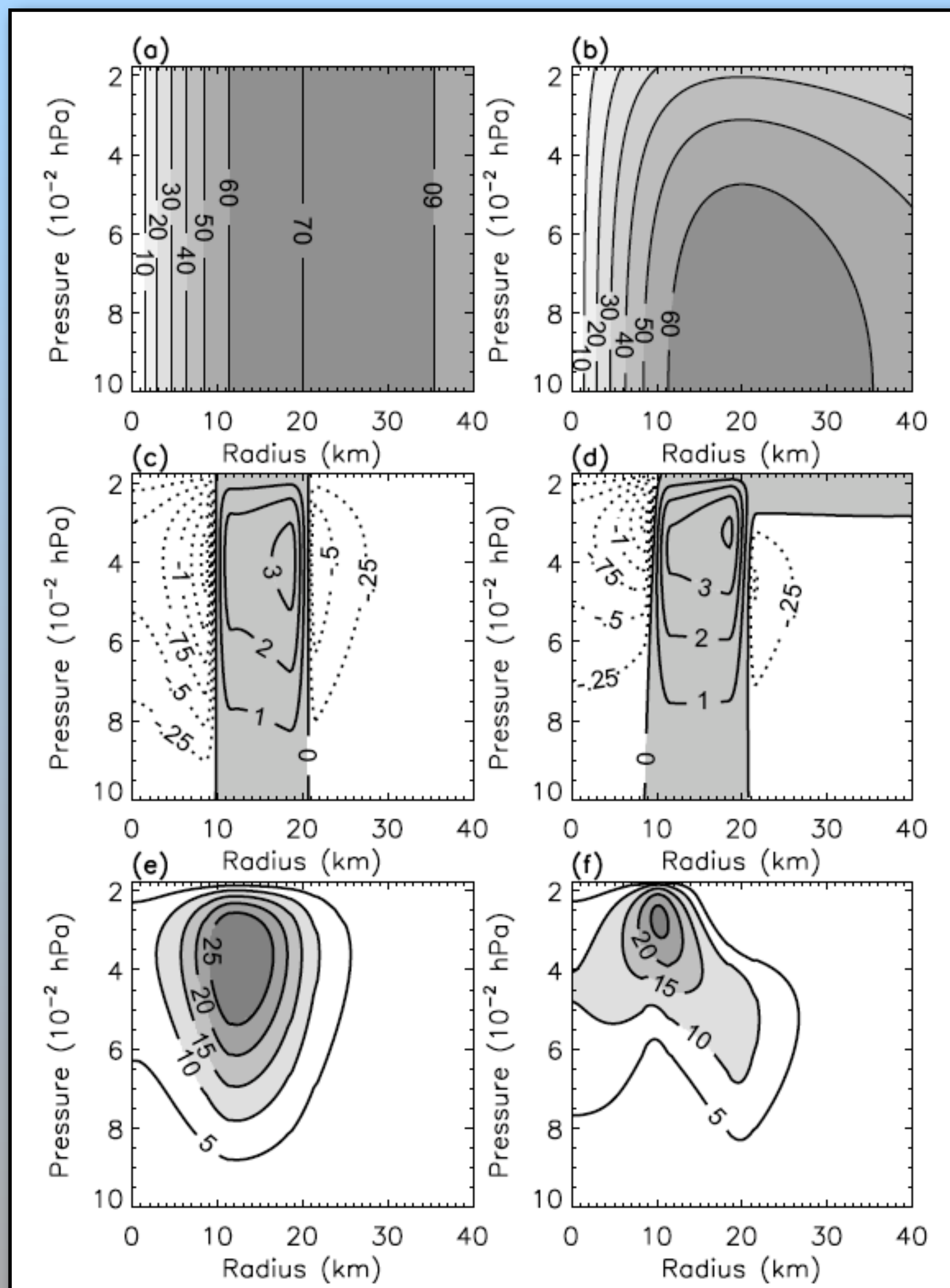
Modeling Results

Tangential Wind

Vertical Velocity

Temperature Tendency

Barotropic Vortex



Baroclinic Vortex