Multi-Scale Analyses for Intense Atmospheric Vortices

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Thanks to ...
Motivation

Modelling Approach

Scalings and Asymptotic Flow Regime

Structure and Motion of a Moist Vortex

Multi-Scale Interactions

A Deep Convective Column Model

Conclusions
Characteristics

$L \approx 250 \text{ km}$

$u_\theta \approx 40 \text{ m/s}$
Motivation
Goal:

Reduced equations for vortex core dynamics, vortex motion and the role of subscale moist processes

**Balanced motions on synoptic time scales**

**Multiple length & time scale interactions**
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Two universal dimensionless parameters* ...

$$\frac{c}{a\Omega} \sim 0.5 = O(1), \quad \frac{a\Omega^2}{g} \sim 6 \cdot 10^{-3} \ll 1$$

... and a more familiar one

$$\left( \frac{c}{a\Omega} \right)^2 \left( \frac{a\Omega^2}{g} \right) = \frac{h_{\text{sc}}}{a} \sim 1.5 \cdot 10^{-3} = \varepsilon^3$$

* Keller & Ting (1951), http://www.arxiv.org/abs/physics/0606114
Two universal dimensionless parameters ... 
\[
\frac{c}{a\Omega} \sim 0.5 = O(1), \quad \frac{a\Omega^2}{g} \sim 6 \cdot 10^{-3} \ll 1
\]

... and a more familiar one
\[
\left(\frac{c}{a\Omega}\right)^2 \left(\frac{a\Omega^2}{g}\right) = \frac{h_{sc}}{a} \sim 1.5 \cdot 10^{-3} = \varepsilon^3
\]

Then, e.g., for the middle latitudes
\[
L_{meso} = \frac{h_{sc}}{\varepsilon}, \quad L_{syn} = \frac{h_{sc}}{\varepsilon^2}, \quad a = \frac{h_{sc}}{\varepsilon^3},
\]
**General multiple scales expansions**

\[
U(t, x, z; \varepsilon) = \sum_i \varepsilon^i U^{(i)}(t, \varepsilon^t, \varepsilon^2 t, ..., \frac{x}{\varepsilon}, \varepsilon^x, \varepsilon^2 x, ..., \frac{z}{\varepsilon}, z)
\]

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<td>(U^{(i)}(t, \varepsilon x, z))</td>
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<td>(U^{(i)}(t, \frac{x}{\varepsilon}, \frac{z}{\varepsilon}))</td>
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<td>(U^{(i)}(\varepsilon^2 t, \varepsilon^2 x, \varepsilon^2 y, z))</td>
<td>Equatorial Kelvin, Yanai &amp; Rossby Waves</td>
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</tbody>
</table>

... etc.

**Modelling Approach**
Motivation

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Conclusions
“Lothar”-Storms and H1-Hurricanes

\[ u_{\text{ref}} \sim 10 \text{ m/s} \]

\[ u_{\text{ref}} / \varepsilon^{1/2} \]

\[ h_{\text{sc}} \sim 10 \text{ km} \]

\[ h_{\text{sc}} / \varepsilon^{3/2} \]

\[ h_{\text{sc}} / \varepsilon^{2} \]
Outer Expansion: \textbf{QG–scaling}

\[ \sum_{i} \varepsilon^{i} U^{(i)}(\varepsilon^{2} t, \varepsilon^{2} x, z) \]

\[ u_{\text{ref}} / \varepsilon^{1/2} \]

\[ h_{\text{sc}} \sim 10 \text{ km} \]

\[ h_{\text{sc}} / \varepsilon^{2} \]

\[ h_{\text{sc}} / \varepsilon^{3/2} \]
Inner Expansion: **Gradient Wind**–scaling

\[ u_{\text{ref}} \sim 10 \text{ m/s} \]

\[ u_{\text{ref}} / \varepsilon^2 \]

\[ h_{\text{sc}} / \varepsilon^2 \]

\[ h_{\text{sc}} / \varepsilon \]

\[ h_{\text{sc}} \sim 10 \text{ km} \]

\[ u_{\text{ref}} \sim 10 \text{ m/s} \]

\[ u_{\text{ref}} / \varepsilon^2 \]

\[ h_{\text{sc}} \sim 10 \text{ km} \]

\[ h_{\text{sc}} / \varepsilon \]

\[ h_{\text{sc}} / \varepsilon^2 \]

\[ \sum_j \varepsilon^{-\frac{j}{2}} U^{(\frac{j}{2})}(\varepsilon^2 t, \varepsilon^{\frac{3}{2}} x', z) \]

**Scalings and Asymptotic Flow Regime**
Inner Expansion: **Gradient Wind**–scaling

\[ x' = x - \frac{1}{\varepsilon^2} X_C(\varepsilon^2 t, z) \]

\[ u_{ref} \sim 10 \text{ m/s} \]

\[ h_{sc} \sim 10 \text{ km} \]

**Scalings and Asymptotic Flow Regime**
Steps of the analysis:

leading axisymmetric balances  ⇒  Eliassen-type
“balanced vortex models”

1st order first Fourier modes
matching
solvability conditions
⇒  centerline motion and tilt

2nd order axisymmetric balances
solvability conditions
⇒  core structure evolution
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Leading order results

\[
\frac{1}{r} \frac{\partial (ru_r, 0)}{\partial r} + \frac{1}{\bar{\rho}} \frac{\partial (\bar{\rho}w_0)}{\partial z} = 0
\]

\[
\frac{\partial \pi}{\partial z} = \Theta
\]

- homentropic background:
  \[
  \bar{\rho}(z), \bar{\rho}(z) \quad \text{with} \quad \bar{\Theta} \equiv 1
  \]
- anelastic
- hydrostatic
Leading order results cont’d

\[\frac{\partial \pi}{\partial \theta} = \frac{\partial u_\theta}{\partial \theta} = 0\]

\[\frac{\partial \pi}{\partial r} - \frac{u_\theta^2}{r} - \Omega u_\theta = 0\]

\[\left( u_r, 0 \frac{\partial}{\partial r} + w_0 \frac{\partial}{\partial z} \right) \left( ru_\theta + r^2 \Omega_0 \right) = 0\]

• axisymmetric leading order core structure
• gradient wind balance
• angular momentum conservation along stream surfaces
Leading order results cont’d

\[
\begin{align*}
    w_0 \left( \frac{d\Theta_2}{dz} + \frac{L^* dq_{vs}^{(0)}}{\bar{\rho}} \frac{dq_{vs}}{dz} \right) & = 0 \\
    u_{r,0} \frac{\partial \Theta^{(3)}}{\partial r} + w_0 \left( \frac{\partial \Theta^{(3)}}{\partial z} + \mathcal{J}(z) \right) & = \hat{S}
\end{align*}
\]

- \( w_0 \equiv 0 \) or moist adiabatic background stratification
- \( \mathcal{J}(z) = \frac{\Gamma^* L^* dq_{vs}^{(1)}}{\bar{\rho}} \frac{dq_{vs}}{dz} \) known moist thermodynamic function

*Structure and Motion of a Moist Vortex*
Leading order results cont’d

\[ w_0 \left( \frac{d\Theta_2}{dz} + \frac{L^* dq_{VS}^{(0)}}{\overline{p}} \frac{dz}{dz} \right) = 0 \]

\[ u_{r,0} \frac{\partial \Theta^{(3)}}{\partial r} + w_0 \left( \frac{\partial \Theta^{(3)}}{\partial z} + J(z) \right) = \tilde{S}_0 \]

- \[ w_0 \frac{L^* dq_{VS}^{(0)}}{\overline{p}} \frac{dz}{dz} \] vortex-scale latent heating at \( O(\varepsilon^{7/2}) \)

- \[ \tilde{S}_0 \] unresolved-scale source term at \( O(\varepsilon^{9/2}) \)

\textit{Structure and Motion of a Moist Vortex}
Quasi-steady Eliassen-type balanced moist vortex model

\[ \frac{\partial \pi}{\partial z} = \Theta \]

\[ \frac{1}{r} \frac{\partial (ru_r,0)}{\partial r} + \frac{1}{\bar{\rho}} \frac{\partial (\bar{\rho}w_0)}{\partial z} = L^* \]

\[ r^3 \frac{\partial \pi}{\partial r} - M^2 + \frac{\Omega_0^2 r^4}{4} = 0 \]

\[ \left( u_{r,0} \frac{\partial}{\partial r} + w_0 \frac{\partial}{\partial z} \right) M = K^* \]

\[ \left( u_{r,0} \frac{\partial}{\partial r} + w_0 \frac{\partial}{\partial z} \right) \Theta^{(3)} + w_0 \mathcal{J}(z) = \tilde{S}^{(9/2)}_0 \]

* \( K, L \) are non-zero for stronger tilt and asymmetry
Forbidden Streamlines of the secondary circulation

\[ 2ru_\theta + r^2\Omega_0 = M(\psi) \]
Streamlines of the secondary circulation in the farfield

\[
\frac{d\Gamma}{dz} w_0 - 2\pi \Omega_0 u_{r,0} \to 0 \quad \text{as} \quad (r \to \infty)
\]
How does the air get back down?
Vortex core structure

- quasi-steady Elliassen-type balanced vortex model
- on-scale latent heat release* merely “unfreezes” vertical motion
- higher-order diabatics determine structure
- * ... unavailable in the downward branch (?)
Structure and Motion of a Moist Vortex
Matching

Self-induced farfield core velocity

\[ |V| (\cos(\theta) e_r - \sin(\theta) e_\theta) \]

Self-induced core velocity
Matching

Self-induced farfield core velocity pushes vortex in addition to background flow advection

Seek far-field behavior of Waveno. 1 Fourier modes of core flow
First order, first Fourier mode asymmetries \((k \in \{1, 2\})\)

\[
\mathcal{L}[U_k] = \underbrace{S_{\theta_k}^{(\frac{3}{2})} R_s}_{\text{subscale heating}} + \underbrace{\frac{\partial}{\partial z} X C_k^{(\frac{3}{2})} R_x}_{\text{weak vortex tilt}} + \underbrace{V C_k^{(0)} R_v}_{\text{vortex motion}}
\]

\(\mathcal{L}[\cdot]\) : known linear operator

\[
U = \left( u^{(\frac{1}{2})}, u^{(\frac{1}{2})}, w^{(\frac{1}{2})}, \pi^{(\frac{1}{2})}, \theta^{(\frac{1}{2})} \right)
\]

\(R_s, R_x, R_v\) : known constants in \(\mathbb{R}^5\)

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*Structure and Motion of a Moist Vortex*
Matching

\[ V_{C_k}^{(0)} = V_{B_k}^{(0)}(z) + S_{\theta_k}^{(9/2)} I_s(z) + \frac{\partial}{\partial z} \left( \frac{\partial}{\partial z} X_C^{(9/2)} I_x(z) \right) + V_{C_k}^{(0)} I_v(z) \]

- subscale heating
- weak vortex tilt
- vortex motion

- tilt \( \partial_z X_C \) adjusts to eliminate \( z \)-dependence

- vortex motion as fct. of
  - background flow
  - asymmetric subscale heating
  - weak vortex tilt
  - self-induced Coriolis effects
  - axisymmetric vortex core structure

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Conclusions
How did the air get back down?

$w_0 \left( \frac{d\Theta_2}{dz} + \frac{L^*}{\bar{p}} \frac{dq^{(0)}_{vs}}{dz} \right) = 0$

$w_0 \left( \frac{\partial \Theta}{\partial r} + w_0 \left( \frac{\partial \Theta}{\partial z} + J(z) \right) \right) = \tilde{S}_0$

$w_0 \frac{L^*}{\bar{p}} \frac{dq^{(0)}_{vs}}{dz}$ vortex-scale latent heating $O(\varepsilon^{7/2})$

$\tilde{S}_0$ unresolved-scale source term $O(\varepsilon^{9/2})$
Hypothesis: Downdrafts via organized convection

WTG-adjustment in stable environment

\[ w \frac{d\Theta}{dz} = S \]

Saturated air

\[ S = -w \frac{\Gamma L dq_{vs}}{p_0} \frac{dS}{dz} \geq 0 \]

Undersaturated air

\[ S = -C_{ev} (q_{vs} - q_v) q_r \leq 0 \]

Water flux balance

\[ (\dot{m} q_{vs})_{\uparrow} = (\dot{m} q_{vs})_{\downarrow}, \quad \dot{m}_{\downarrow} - \dot{m}_{\uparrow} = \dot{m}_{\downarrow} \left(1 - \frac{q_{vs,\downarrow}}{q_{vs,\uparrow}}\right) \]
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Conclusions
A Deep Convective Column Model

\[ w = \mathcal{O}(\varepsilon) \]

\[ w = \mathcal{O}(1) \]

\[ \theta^{(4)} \text{-variations} \]

\[ \theta^{(3)} \text{-variations} \]

\( \mathcal{O}(h_{sc}) \)

\( \mathcal{O}(h_{sc}/\varepsilon) \)

\( \mathcal{O}(\varepsilon h_{sc}) \)

\( \mathcal{O}(h_{sc}) \)

\( \mathcal{O}(h_{sc}/\varepsilon) \)
\[ U(x, z, t; \varepsilon) = \sum_{i} \varepsilon^{i} U^{(i)}(t, \frac{x}{\varepsilon}, z, \varepsilon t, \varepsilon x), \]

\[ x = \frac{x'}{h_{sc}} , \quad z = \frac{z'}{h_{sc}}, \quad t = \frac{t'}{h_{sc}/u_{ref}} \]

\( z \rightarrow \) pressure scale height

\( t \rightarrow \) deep convective time scale

\( x/\varepsilon \rightarrow \) narrow deep convective turrets

\( \varepsilon t, \varepsilon x \rightarrow \) meso-scale vortex formation

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_A Deep Convective Column Model_
“pressure-less” column model (barotropic background)

\[
\begin{align*}
\frac{dw}{dt} + \mathbf{v} \cdot \nabla w + w w_z &= \frac{\theta}{w} + D_w \\
\frac{d\theta}{dt} + \mathbf{v} \cdot \nabla \theta + w \theta_z &= \frac{d\Theta}{dz} w + D_\theta + S_\theta \\
\mathbf{v} \cdot \nabla \mathbf{v} + w \mathbf{v}_z + \frac{1}{\rho} \nabla p &= w \Omega \times \mathbf{k} + D_v \\
\bar{\rho} \nabla \cdot \mathbf{v} + (\bar{\rho} w)_z &= 0
\end{align*}
\]
“pressure-less” column model (barotropic background)

\[ w_t + \mathbf{v} \cdot \nabla \mathbf{w} + \mathbf{w} \mathbf{w}_z = \frac{\theta}{\rho} + D_w \]

\[ \theta_t + \mathbf{v} \cdot \nabla \theta + \mathbf{w} \theta_z = \frac{d\Theta}{dz} \mathbf{w} + D_\theta + S_\theta \]

\[ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{w} \mathbf{v}_z + \frac{1}{\rho} \nabla p = \mathbf{w} \Omega \times \mathbf{k} + D_v \]

\[ \overline{\rho} \nabla \cdot \mathbf{v} + (\overline{\rho} \mathbf{w})_z = 0 \]

Accumulated column fluxes drive meso-scale flow \((R_0 = O(1))\)
Meso-scale WTG-flow

\[ \nu_t + \nu \cdot \nabla \nu + w\nu_z + \Omega_0 k \times \nu + \frac{1}{\rho} \nabla P = 0 \]

\[ \omega \frac{d\Theta_2}{dz} = \overline{S_\theta} \]

\[ \overline{\rho} \nabla \cdot \nu + (\overline{\rho}w)_z = 0 \]
Meso-scale WTG-flow

\[ v_t + v \cdot \nabla v + w v_z + \Omega_0 k \times v + \frac{1}{\rho} \nabla P = 0 \]

\[ w \frac{d\Theta_2}{dz} = \overline{S_\theta} \]

\[ \overline{\rho} \nabla \cdot v + (\overline{\rho} w)_z = 0 \]

Slight “cheat”, since column requires barotropic background
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Reduced equations for vortex core dynamics, vortex motion and the role of subscale moist processes

shown:

• (generalized) Eliassen-type balanced core structure
• vortex motion and tilt
• role of higher order diabatic effects
• multi-scale column model for incipient stage (related models may explain large-scale descend)

• Open issue: closedness of the secondary circulation
Reduced equations for vortex core dynamics, vortex motion and the role of subscale moist processes

**not shown:**

- core structure evolution equations
- buoyancy-controlled, WTG-type regimes (*dry or farther from moist adiabatic*)
- Eliassen-type model for stronger tilt
- regimes with intense near-surface boundary layer
- precession of a dry vortex