Multi-Scale Analyses for Intense Atmospheric Vortices

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Motivation

Modelling Approach

Scalings and Asymptotic Flow Regime

Structure and Motion of a Moist Vortex

Multi-Scale Interactions

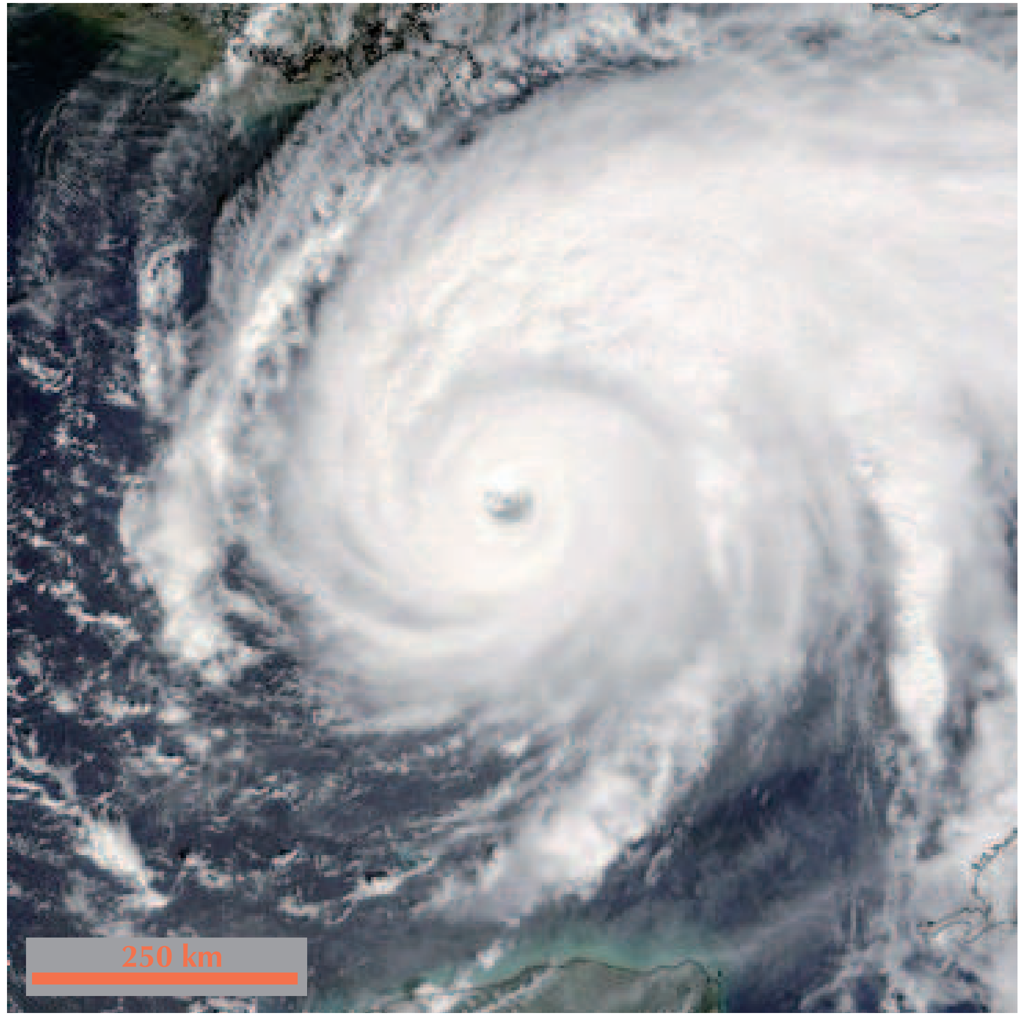
A Deep Convective Column Model

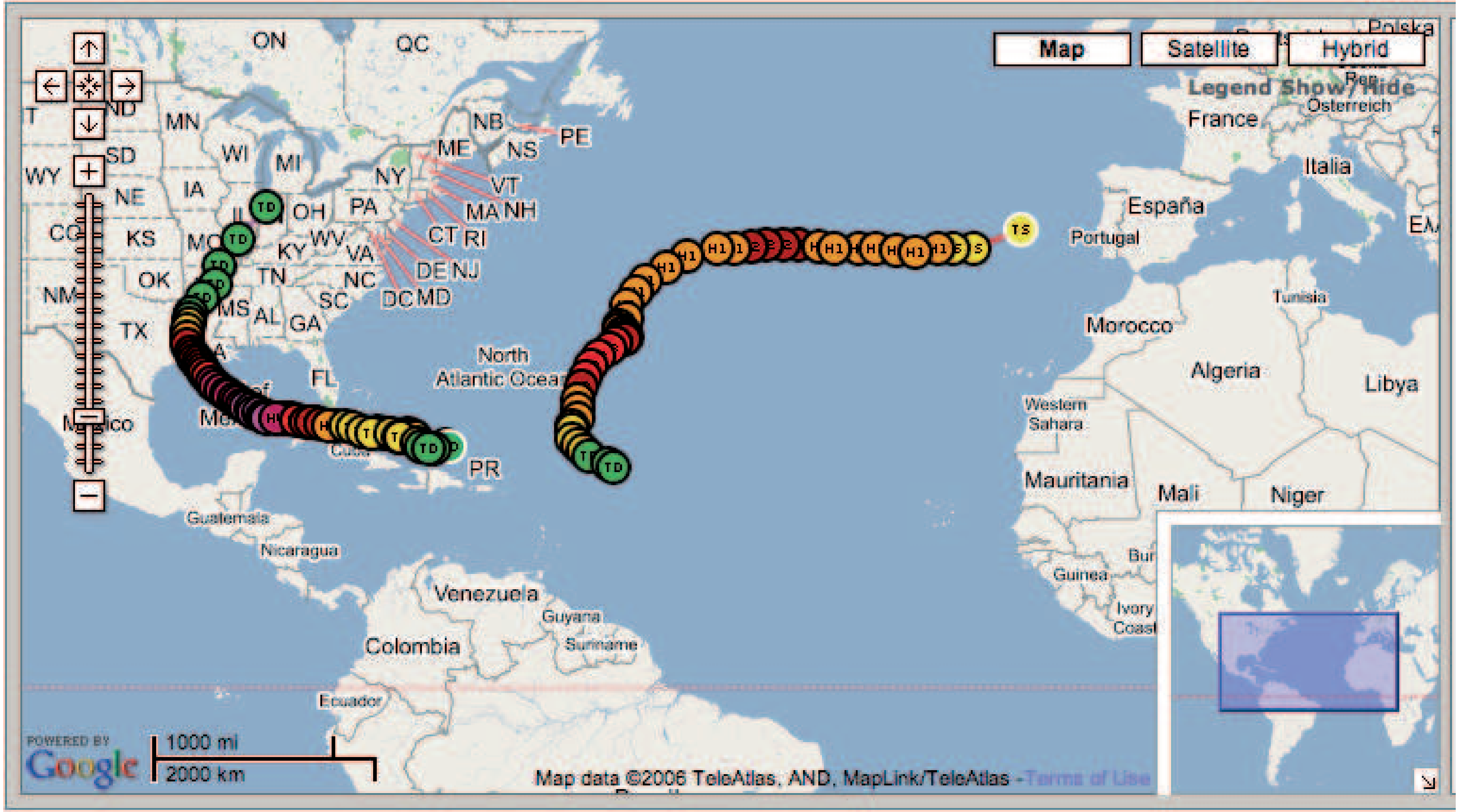
Conclusions

Characteristics

$L \approx 250$ km

$u_\theta \approx 40$ m/s





Hurricane Tracking & Google Maps

Hurricane Rita (2005), Hurricane Gordon (2006)

http://stormadvisory.org/map/atlantic/

Goal:

Reduced equations for

vortex core dynamics, vortex motion

and the role of subscale moist processes

Balanced motions on synoptic time scales

Multiple length & time scale interactions

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Two universal dimensionless parameters* ...

$$\frac{c}{a\Omega} \sim 0.5 = O(1)\,, \qquad \frac{a\Omega^2}{g} \sim 6\cdot 10^{-3} \ll 1$$

... and a more familiar one

$$\left(\frac{c}{a\Omega}\right)^2 \left(\frac{a\Omega^2}{g}\right) = \frac{h_{\rm sc}}{a} \sim 1.5\cdot 10^{-3} \overset{!}{=} \varepsilon^3$$

* Keller & Ting (1951), `http://www.arxiv.org/abs/physics/0606114`

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Then, e.g., for the middle latitudes

$$L_{\mathrm{meso}} = \frac{h_{\mathrm{sc}}}{\varepsilon} \quad L_{\mathrm{syn}} = \frac{h_{\mathrm{sc}}}{\varepsilon^2}\,, \quad a = \frac{h_{\mathrm{sc}}}{\varepsilon^3}\,,$$

General multiple scales expansions

$$\boldsymbol{U}(t,\boldsymbol{x},z;\boldsymbol{\varepsilon}) = \sum_i \boldsymbol{\varepsilon}^i \boldsymbol{U}^{(i)}\left(\frac{t}{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon} t, \boldsymbol{\varepsilon}^2 t, \ldots, \frac{\boldsymbol{x}}{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}\boldsymbol{x}, \boldsymbol{\varepsilon}^2\boldsymbol{x}, \ldots, \frac{z}{\boldsymbol{\varepsilon}}, z\right)$$

Coordinate scalings	Simplified model obtained
$\boldsymbol{U}^{(i)}(t,\boldsymbol{x},z)$	Anelastic & pseudo-incompressible models
$\boldsymbol{U}^{(i)}(t,\boldsymbol{\varepsilon}\boldsymbol{x},z)$	Linear large scale internal gravity waves
$\boldsymbol{U}^{(i)}(\frac{t}{\boldsymbol{\varepsilon}},\boldsymbol{x},\frac{z}{\boldsymbol{\varepsilon}})$	Linear small scale internal gravity waves
$\boldsymbol{U}^{(i)}(\boldsymbol{\varepsilon}^2 t,\boldsymbol{\varepsilon}^2\boldsymbol{x},z)$	Mid-latitude Quasi-Geostrophic model
$\boldsymbol{U}^{(i)}(\boldsymbol{\varepsilon}^2 t,\boldsymbol{\varepsilon}^2\boldsymbol{x},z)$	Equatorial Weak Temperature Gradient models
$\boldsymbol{U}^{(i)}(\boldsymbol{\varepsilon}^2 t,\boldsymbol{\varepsilon}^{-1}\xi(\boldsymbol{\varepsilon}^2\boldsymbol{x}),z)$	Semi-geostrophic model
$\boldsymbol{U}^{(i)}(\boldsymbol{\varepsilon}^{\frac{5}{2}} t,\boldsymbol{\varepsilon}^{\frac{7}{2}} x,\boldsymbol{\varepsilon}^{\frac{5}{2}} y,z)$	Equatorial Kelvin, Yanai & Rossby Waves
$\ldots$	etc.

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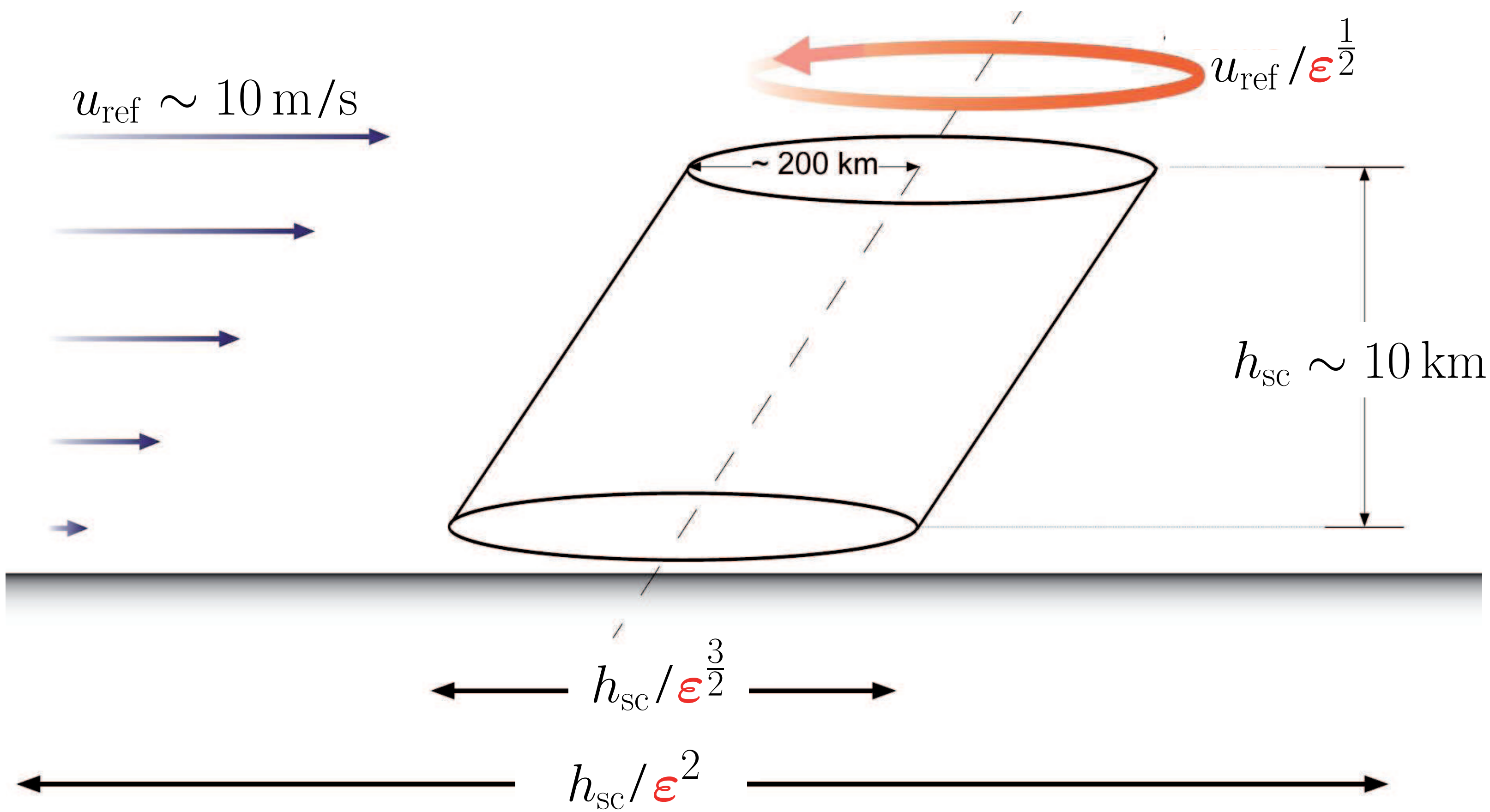
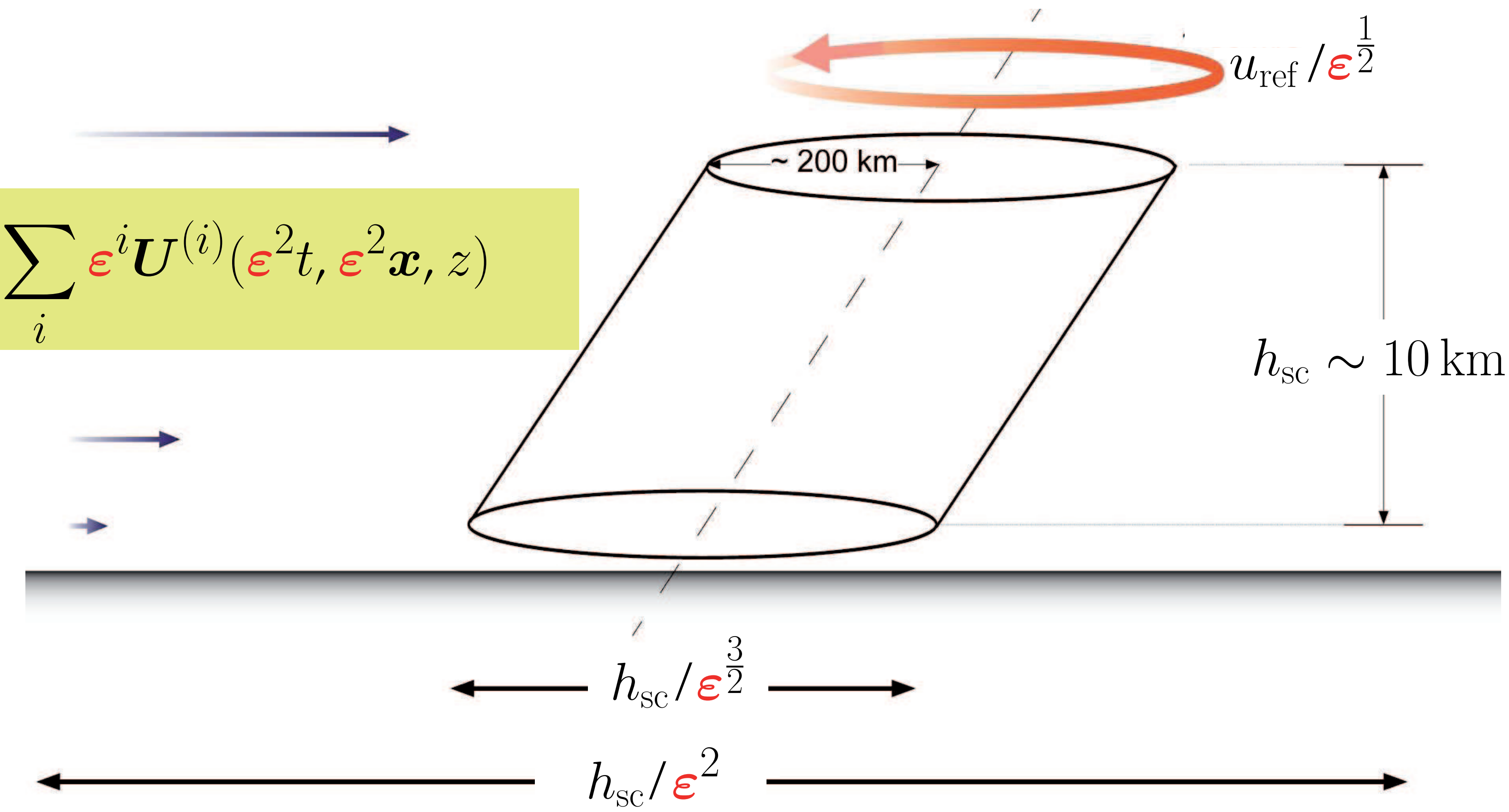
“Lothar”-Storms and H1-Hurricanes

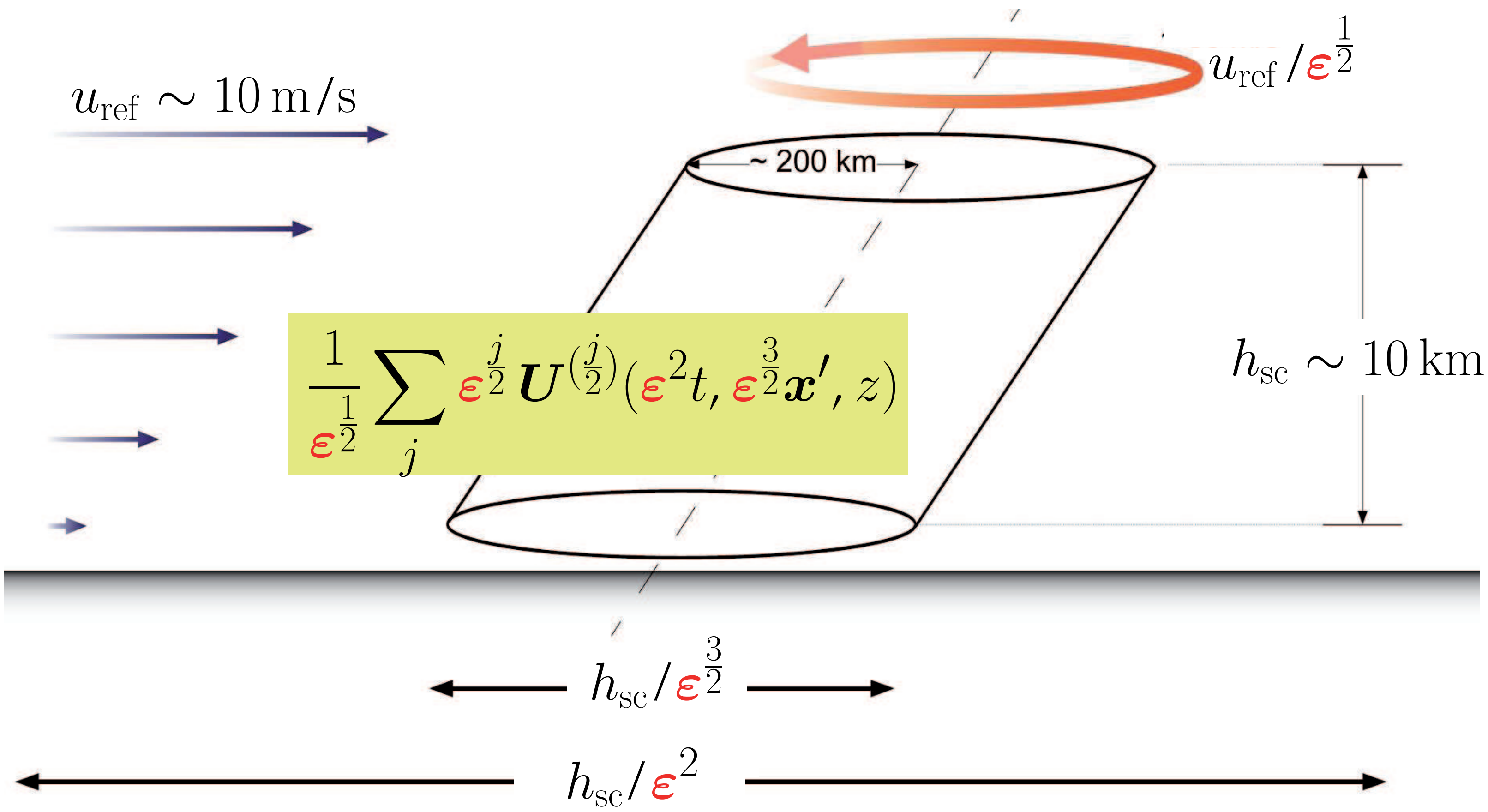


Outer Expansion: QG–scaling

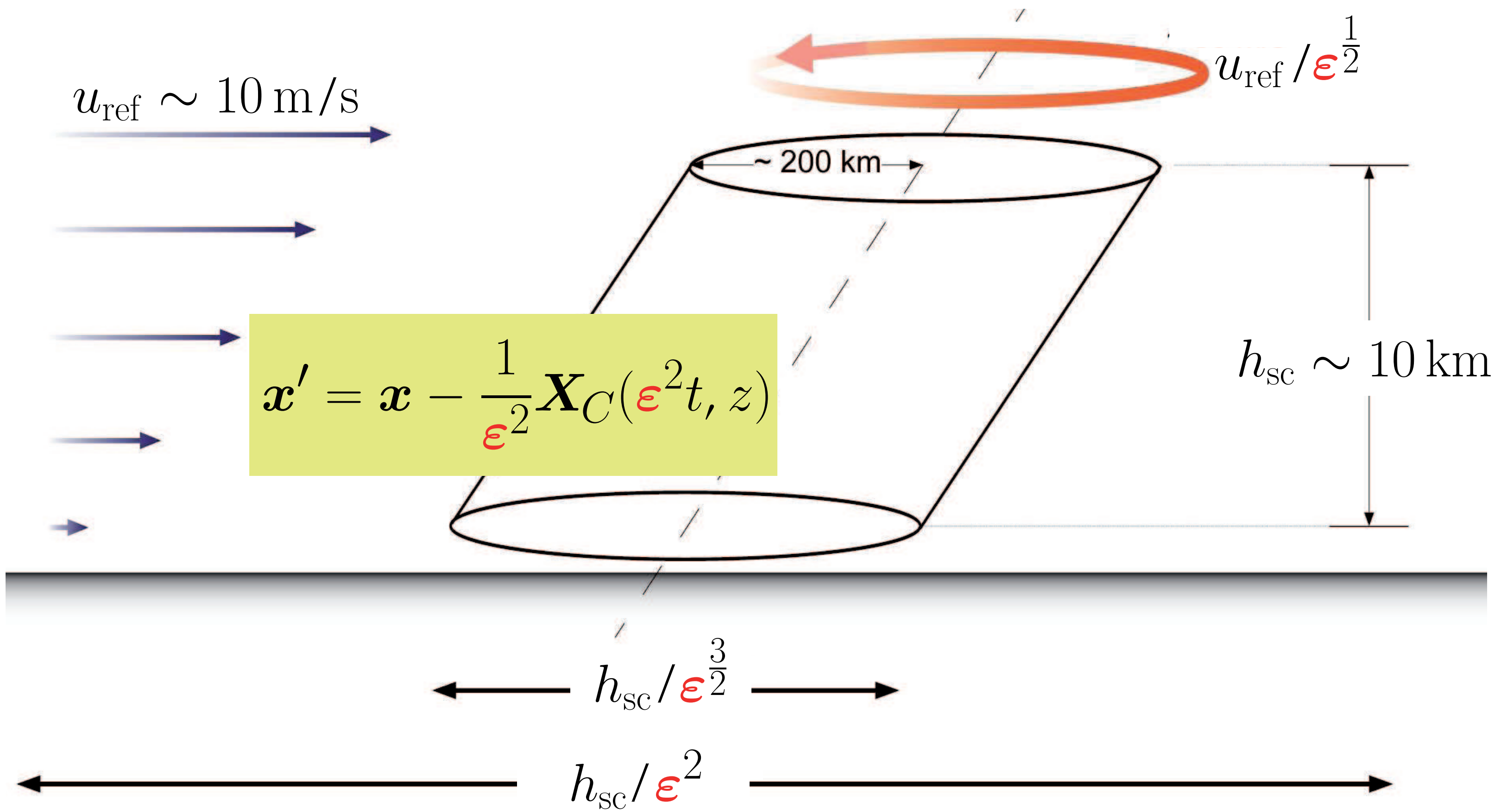
$$\sum_i \varepsilon^i \boldsymbol{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$$



Inner Expansion: Gradient Wind–scaling



Inner Expansion: Gradient Wind–scaling



Steps of the analysis:

leading axisymmetric balances	$\Rightarrow$	Eliassen-type "balanced vortex models"
1st order first Fourier modes matching solvability conditions	$\Rightarrow$	centerline motion and tilt
2nd order axisymmetric balances solvability conditions	$\Rightarrow$	core structure evolution

Leading order results

$$\frac{1}{r}\frac{\partial(r u_{r,0})}{\partial r} + \frac{1}{\overline{\rho}}\frac{\partial(\overline{\rho} w_0)}{\partial z} = 0$$

$$\frac{\partial \pi}{\partial z} = \Theta$$

- homentropic background:

$$\overline{\rho}(z), \overline{p}(z) \quad \text{with} \quad \overline{\Theta} \equiv 1$$

- anelastic
- hydrostatic

Leading order results *cont'd*

$$
\begin{aligned}
\frac{\partial \pi}{\partial \theta} = \frac{\partial u_\theta}{\partial \theta} \; &= \; 0 \\
\frac{\partial \pi}{\partial r} - \frac{u_\theta^2}{r} - \Omega u_\theta \; &= \; 0 \\
\left(u_{r,0} \frac{\partial}{\partial r} + w_0 \frac{\partial}{\partial z} \right) \left(r u_\theta + r^2 \Omega_0 \right) \; &= \; 0
\end{aligned}
$$

- axisymmetric leading order core structure
- gradient wind balance
- angular momentum conservation along stream surfaces

Leading order results *cont'd*

$$w_0 \left(\underline{\frac{d\Theta_2}{dz} + \frac{L^*}{\overline{p}} \frac{dq_{\text{vs}}^{(0)}}{dz}} \right) = 0$$

$$u_{r,0} \frac{\partial \Theta^{(3)}}{\partial r} + w_0 \left(\frac{\partial \Theta^{(3)}}{\partial z} + \mathcal{J}(z) \right) = \tilde{\boldsymbol{S}}$$

- $w_0 \equiv 0$ or moist adiabatic background stratification
- $\mathcal{J}(z) = \frac{\Gamma^* L^*}{\overline{p}} \frac{dq_{\text{vs}}^{(1)}}{dz}$ known moist thermodynamic function

Leading order results *cont'd*

$$\boldsymbol{w_0} \left(\frac{d\Theta_2}{dz} + \frac{\boldsymbol{L^*}}{\boldsymbol{\overline{p}}} \frac{\boldsymbol{dq_{\mathrm{vs}}^{(0)}}}{\boldsymbol{dz}} \right) = 0$$

$$u_{r,0} \frac{\partial \Theta^{(3)}}{\partial r} + w_0 \left(\frac{\partial \Theta^{(3)}}{\partial z} + \mathcal{J}(z) \right) = \tilde{\boldsymbol{S}}_0$$

- $w_0 \dfrac{L^*}{\overline{p}} \dfrac{dq_{\mathrm{vs}}^{(0)}}{dz}$ vortex-scale latent heating at $O(\varepsilon^{7/2})$
- $\tilde{\boldsymbol{S}}_0$ **unresolved-scale source term** at $O(\varepsilon^{9/2})$

Quasi-steady Eliassen-type balanced moist vortex model

$$\frac{\partial \pi}{\partial z} = \Theta$$

$$\frac{1}{r}\frac{\partial (r u_{r,0})}{\partial r} + \frac{1}{\overline{\rho}}\frac{\partial (\overline{\rho} w_0)}{\partial z} = L^{*}$$

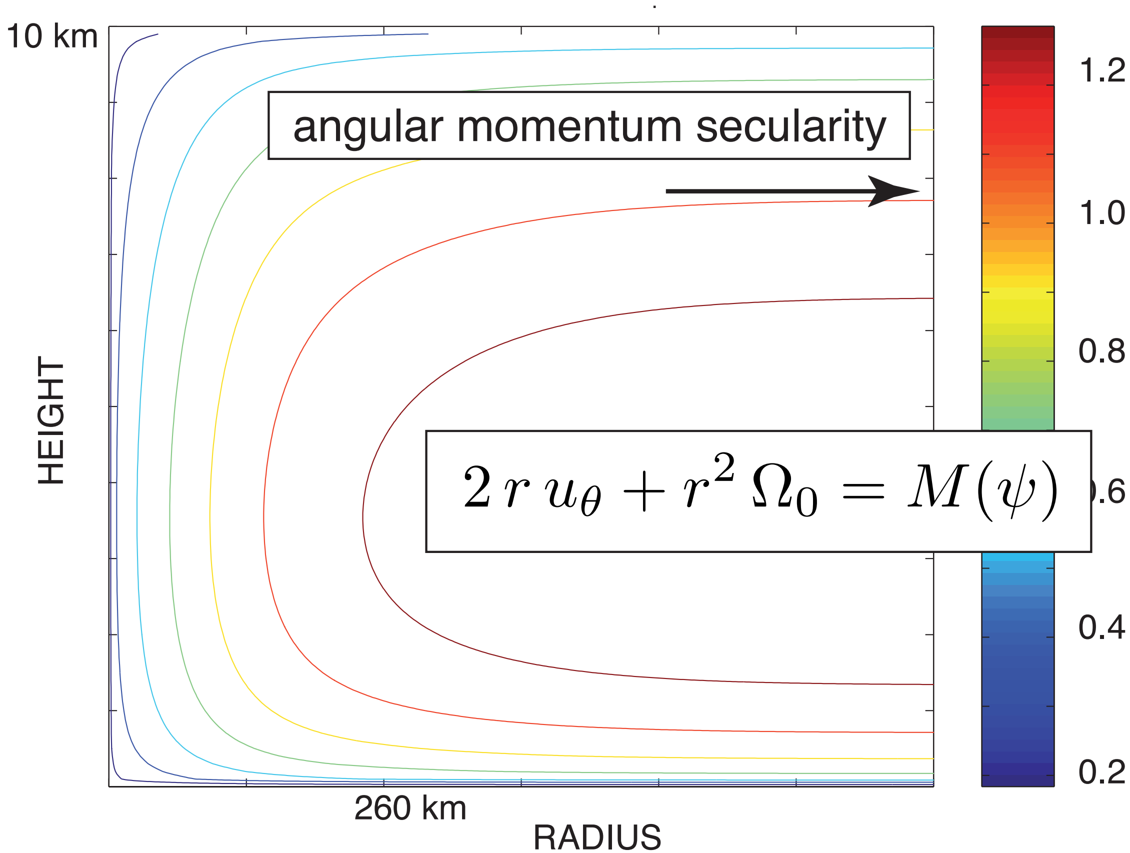
$$r^3 \frac{\partial \pi}{\partial r} - M^2 + \frac{\Omega_0^2 r^4}{4} = 0 \qquad \left(M = r u_\theta + \frac{\Omega_0 r^2}{2} \right)$$

$$\left(u_{r,0} \frac{\partial}{\partial r} + w_0 \frac{\partial}{\partial z} \right) M = K^{*}$$

$$\left(u_{r,0} \frac{\partial}{\partial r} + w_0 \frac{\partial}{\partial z} \right) \Theta^{(3)} + w_0 \mathcal{J}(z) = \tilde{\boldsymbol{S}}_0^{(\frac{9}{2})}$$

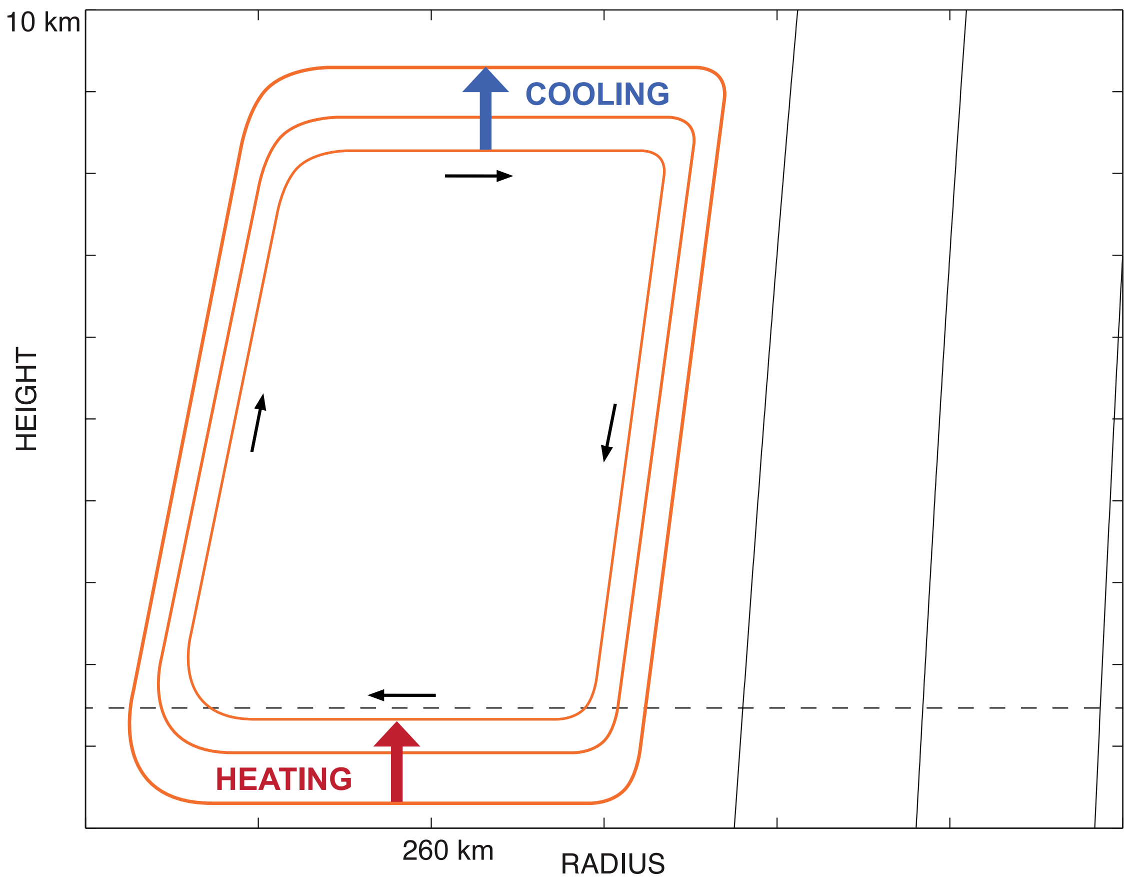
* K, L are non-zero for stronger tilt and asymmetry

forbidden Streamlines of the secondary circulation



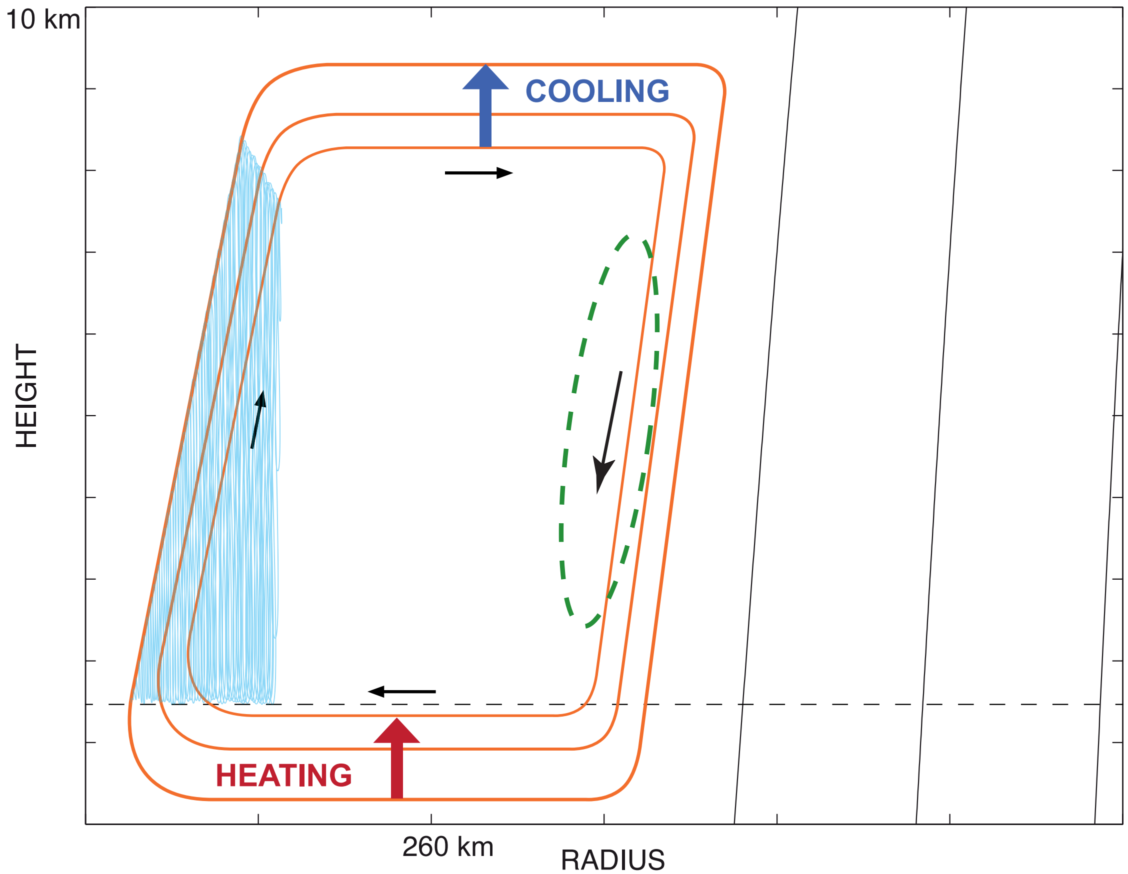
$$2\, r\, u_\theta + r^2\, \Omega_0 = M(\psi)$$

Streamlines of the secondary circulation in the farfield



$$\frac{d\Gamma}{dz} w_0 - 2\pi\Omega_0 \, u_{r,0} \longrightarrow 0 \qquad \text{as} \qquad (r \longrightarrow \infty)$$

Streamlines of the secondary circulation



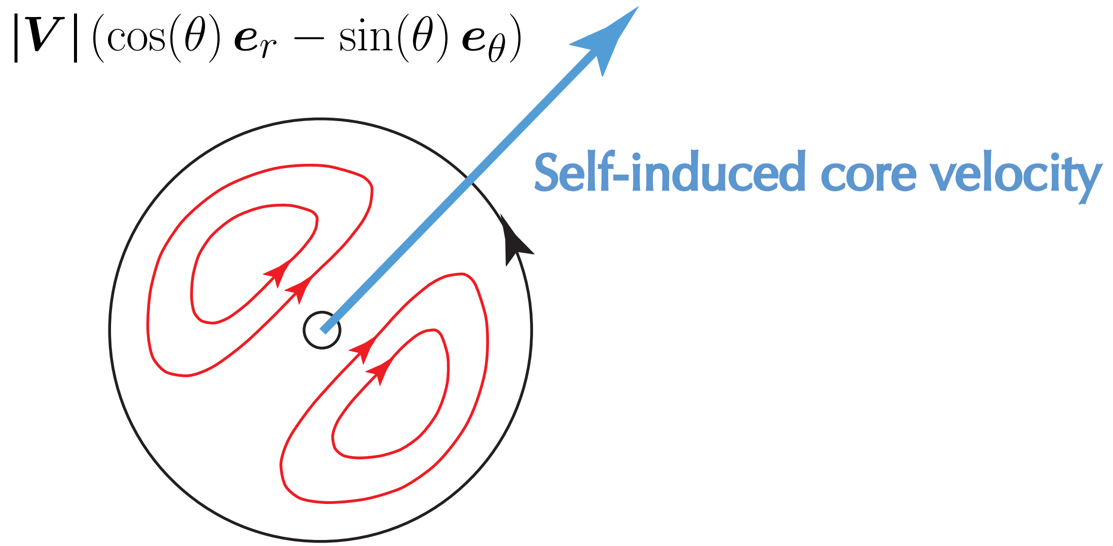
How does the air get back down?

Vortex core structure

- quasi-steady Elliassen-type balanced vortex model
- on-scale latent heat release* merely “unfreezes” vertical motion
- higher-order diabatics determine structure
- * ... unavailable in the downward branch (?)

Matching

Self-induced farfield core velocity

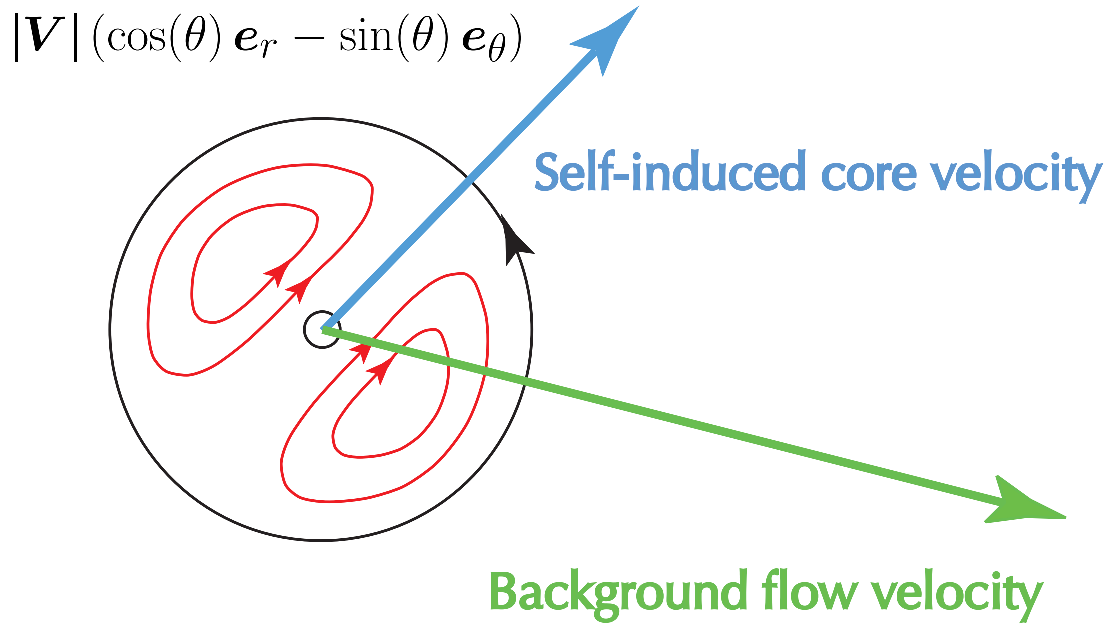


Matching

Self-induced farfield core velocity pushes vortex in addition to background flow advection

$$\Downarrow$$

Seek far-field behavior of Waveno. 1 Fourier modes of core flow



First order, first Fourier mode asymmetries $\quad (k \in \{1, 2\})$

$$\mathcal{L}[U_k] = \underbrace{\boldsymbol{S_\theta}{}_k^{(\frac{9}{2})} R_s}_{\text{subscale heating}} + \underbrace{\frac{\partial}{\partial z}\boldsymbol{X_C}{}_k^{(\frac{2}{2})} R_x}_{\text{weak vortex tilt}} + \underbrace{\boldsymbol{V_C}{}_k^{(0)} R_v}_{\text{vortex motion}}$$

$\mathcal{L}[\cdot]$: known linear operator

$$U = \left(u_r^{(\frac{1}{2})}, u_\theta^{(\frac{1}{2})}, w^{(\frac{4}{2})}, \pi^{(\frac{7}{2})}, \theta^{(\frac{7}{2})}\right)$$

R_s, R_x, R_v : known constants in $\mathbb{R}^5$

Matching

$$\boldsymbol{V_C}_k^{(0)} = \boldsymbol{V_B}_k^{(0)}(z) + \underbrace{\boldsymbol{S_\theta}_k^{(\frac{9}{2})} I_s(z)}_{\text{subscale heating}} + \underbrace{\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\boldsymbol{X_C}_k^{(\frac{2}{2})} I_x(z)\right)}_{\text{weak vortex tilt}} + \underbrace{\boldsymbol{V_C}_k^{(0)} I_v(z)}_{\text{vortex motion}}$$

- tilt $\partial_z \boldsymbol{X_C}$ adjusts to eliminate z-dependence
- vortex motion as fct. of
  - background flow $\boldsymbol{V_B}$
  - asymmetric subscale heating $\boldsymbol{S_\theta}_k$
  - weak vortex tilt $\partial_z \boldsymbol{X_C}$
  - self-induced Coriolis effects $\Omega_0\, \boldsymbol{V_C}$
  - axisymmetric vortex core structure $I_{s,x,v}(z)$

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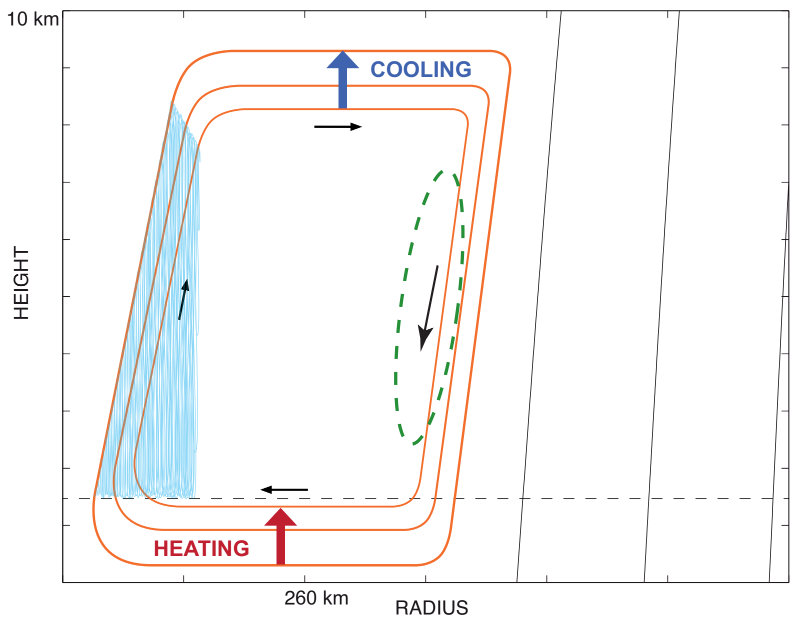
Required multi-scale ingredients



How did the air get back down?

$$w_0 \left(\frac{d\Theta_2}{dz} + \frac{L^*}{\overline{p}} \frac{dq_{\mathrm{vs}}^{(0)}}{dz} \right) = 0$$

$$u_{r,0} \frac{\partial \Theta}{\partial r} + w_0 \left(\frac{\partial \Theta}{\partial z} + \mathcal{J}(z) \right) = \tilde{S}_0$$

$w_0 \dfrac{L^*}{\overline{p}} \dfrac{dq_{\mathrm{vs}}^{(0)}}{dz}$ vortex-scale latent heating $O(\varepsilon^{7/2})$

$\tilde{S}_0$ **unresolved-scale source term** $O(\varepsilon^{9/2})$

Hypothesis: Downdrafts via organized convection

WTG-ajdustment in stable environment

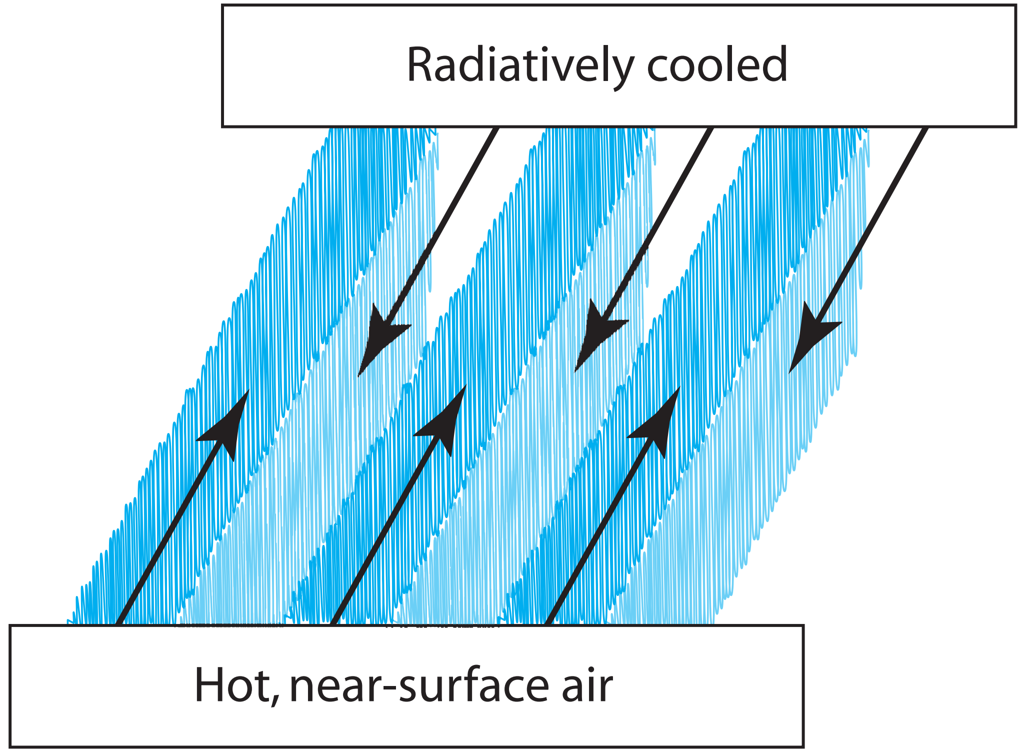
$$w \frac{d\Theta}{dz} = S$$

Saturated air

$$S = -w \frac{\Gamma L}{p_0} \frac{dq_{vs}}{dz} \geq 0$$

Undersaturated air

$$S = -C_{\rm ev} \left(q_{\rm vs} - q_v \right) q_{\rm r} \leq 0$$



Water flux balance

$$(\dot{m}\, q_{vs})_{\uparrow} = (\dot{m}\, q_{vs})_{\downarrow} \,, \qquad \dot{m}_{\downarrow} - \dot{m}_{\uparrow} = \dot{m}_{\downarrow} \left(1 - \frac{q_{vs,\downarrow}}{q_{vs,\uparrow}} \right)$$

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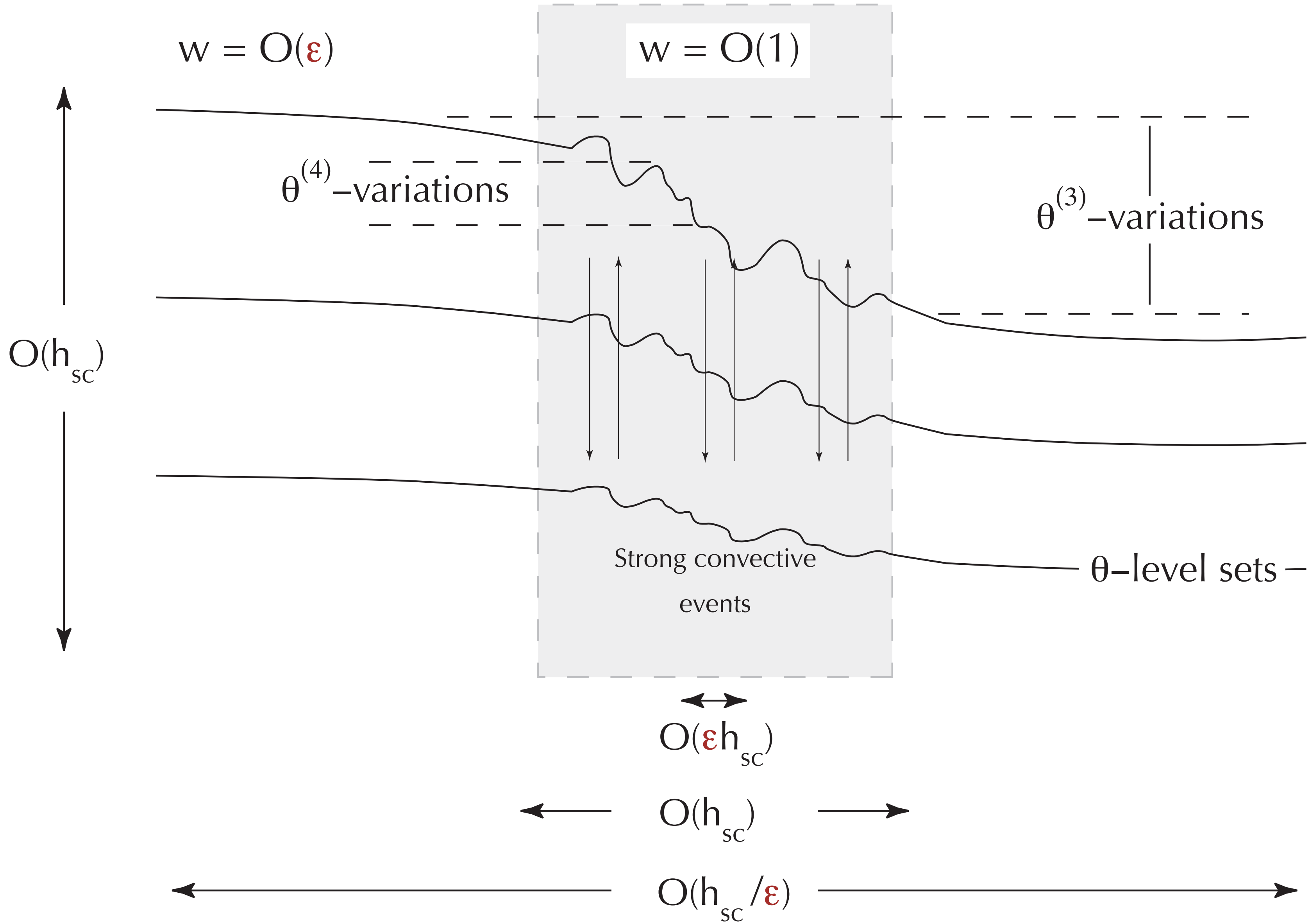

A Deep Convective Column Model

$$\mathbf{U}(\boldsymbol{x}, z, t; \varepsilon) = \sum_i \varepsilon^i\, \mathbf{U}^{(i)}(t, \frac{\boldsymbol{x}}{\varepsilon}, z, \varepsilon t, \varepsilon \boldsymbol{x}),$$

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{h_{\mathrm{sc}}}, \qquad z = \frac{z'}{h_{\mathrm{sc}}}, \qquad t = \frac{t'}{h_{\mathrm{sc}}/u_{\mathrm{ref}}}$$

z $\longrightarrow$ pressure scale height

t $\longrightarrow$ deep convective time scale

$\boldsymbol{x}/\varepsilon$ $\longrightarrow$ narrow deep convective turrets

$\varepsilon t, \varepsilon \boldsymbol{x}$ $\longrightarrow$ meso-scale vortex formation

“pressure-less” column model (barotropic background)

$$w_t + \boldsymbol{v} \cdot \nabla w + w w_z = \underline{\theta} \qquad + D_w$$

$$\theta_t + \boldsymbol{v} \cdot \nabla \theta + w \theta_z = \underline{\frac{d\Theta}{dz} w} \qquad + D_\theta + S_\theta$$

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + w \boldsymbol{v}_z + \frac{1}{\overline{\rho}} \nabla p = \underline{w\, \boldsymbol{\Omega} \times \boldsymbol{k}} + D_{\boldsymbol{v}}$$

$$\overline{\rho} \nabla \cdot \boldsymbol{v} + (\overline{\rho} w)_z = 0$$

"pressure-less" column model (barotropic background)

$$
\begin{aligned}
w_t + \boldsymbol{v}\cdot\nabla w + w w_z &= \underline{\theta} \qquad\qquad + D_w \\
\theta_t + \boldsymbol{v}\cdot\nabla\theta + w\theta_z &= \underline{\frac{d\Theta}{dz}\, w} \qquad + D_\theta + S_\theta \\
\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + w\boldsymbol{v}_z + \frac{1}{\overline{\rho}}\nabla p &= \underline{w\,\boldsymbol{\Omega}\times\boldsymbol{k}} + D_{\boldsymbol{v}} \\
\overline{\rho}\nabla\cdot\boldsymbol{v} + (\overline{\rho} w)_z &= 0
\end{aligned}
$$

Accumulated column fluxes drive meso-scale flow ($\mathrm{Ro} = O(1)$)

Meso-scale WTG-flow

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + w \boldsymbol{v}_z + \Omega_0 \boldsymbol{k} \times \boldsymbol{v} + \frac{1}{\overline{\rho}} \nabla P = 0$$

$$w \frac{d\Theta_2}{dz} = \underline{\overline{S_\theta}}$$

$$\overline{\rho} \nabla \cdot \boldsymbol{v} + (\overline{\rho} w)_z = 0$$

Meso-scale WTG-flow

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + w \boldsymbol{v}_z + \Omega_0 \boldsymbol{k} \times \boldsymbol{v} + \frac{1}{\overline{\rho}} \nabla P = 0$$

$$w \, \frac{d\Theta_2}{dz} = \underline{\overline{S_\theta}}$$

$$\overline{\rho} \nabla \cdot \boldsymbol{v} + (\overline{\rho} w)_z = 0$$

Slight "cheat", since column requires barotropic background

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Reduced equations for

vortex core dynamics, vortex motion

and the role of subscale moist processes

shown:

- (generalized) Eliassen-type balanced core structure
- vortex motion and tilt
- role of higher order diabatic effects

- multi-scale column model for incipient stage
  (related models may explain large-scale descend)

- Open issue: closedness of the secondary circulation

Reduced equations for

vortex core dynamics, vortex motion

and the role of subscale moist processes

not shown:

- core structure evolution equations
- buyoancy-controlled, WTG-type regimes
  (dry or farther from moist adiabatic)
- Eliassen-type model for stronger tilt
- regimes with intense near-surface boundary layer

- precession of a dry vortex