

Singular Perturbations in Stochastic Control and Hamilton-Jacobi-Bellman Equation

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Problem

$$dX_t = \mathbf{f}(X_t, Y_t, \mathbf{u}_t) dt + \sqrt{2}\sigma^\varepsilon(X_t, Y_t, \mathbf{u}_t) dW_t, \quad X_0 = x \in \mathbb{R}^n$$
$$dY_t = \frac{1}{\varepsilon} \mathbf{b}(X_t, Y_t) dt + \sqrt{\frac{2}{\varepsilon}} \mathbf{g}(X_t, Y_t) dW_t, \quad Y_0 = y \in \mathbb{R}^m$$

Goal: study the limit as $\varepsilon \rightarrow 0$

Issues:

High dimension	}	\implies	Can we do something?
Controlled dynamics			
Unbounded domain			
Unbounded data			
Possible degeneracy			

Yes, but...

References & Motivation

Literature since 60s (engineering, PDEs, probability, large deviations, ...)

Kokotovic, Khalil, O'Reilly (1986): book (theory for control app.)

Lions P-L, Papanicolaou, Varadhan (1986): pioneering paper

Jensen R, Lions P-L (1984), Bensoussan (1988): HJB

Dmitiriev MG, Kurina GA (2006), Naidu DS (2002): survey papers

Kushner (1990), Kabanov, Pergamenschikov (2003): books – probabilistic methods

Pardoux E, Veretennikov (2001-2003), Borkar, Gaitsgory (2007): papers – probabilistic methods

Alvarez O, Bardi M (2002,2003,2010), A., B., Marchi C (2007,2008)

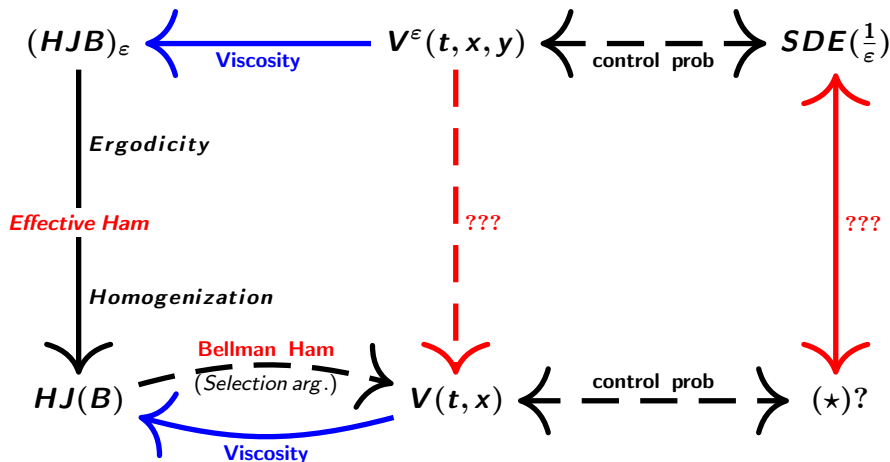
Bardi M, Cesaroni A (2011), B., C., Ghilli D. (2014)

...and references therein!

Application (physics, finance, economy, machine learning, ...)

- pricing/trading derivative securities in financial markets w/ stoch. volatility
- optimal economic growth (HARA functions), portfolio optimization, risky assets
- many examples: https://en.wikipedia.org/wiki/Multiscale_modeling

An overview



System of Singularly Perturbed SDEs

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a complete filtered probability space,
 $(W_t)_t$ an \mathcal{F}_t -adapted standard r -dimensional Brownian motion,

$$\begin{aligned}dX_t &= \mathbf{f}(X_t, Y_t, \mathbf{u}_t) dt + \sqrt{2}\boldsymbol{\sigma}^\varepsilon(X_t, Y_t, \mathbf{u}_t) dW_t, & X_0 &= x \in \mathbb{R}^n \\dY_t &= \frac{1}{\varepsilon}\mathbf{b}(X_t, Y_t) dt + \sqrt{\frac{2}{\varepsilon}}\boldsymbol{\varrho}(X_t, Y_t) dW_t, & Y_0 &= y \in \mathbb{R}^m\end{aligned}\quad (1)$$

\mathbf{U} compact set, $\varepsilon > 0$, $\mathbf{f}, \boldsymbol{\sigma}^\varepsilon, \mathbf{b}, \boldsymbol{\varrho}$ Lipschitz continuous unif. in \mathbf{u} s.t.

$$|\mathbf{f}(x, y, \mathbf{u})|, \|\boldsymbol{\sigma}^\varepsilon(x, y, \mathbf{u})\| \leq C(1 + |x| + |y|), \quad \forall x, y, \forall \varepsilon > 0$$

$$\|\mathbf{b}(x, y)\|, \|\boldsymbol{\varrho}(x, y)\| \leq C'(1 + |x| + |y|), \quad \forall x, y,$$

$\lim_{\varepsilon \rightarrow 0} \boldsymbol{\sigma}^\varepsilon(x, y, \mathbf{u}) = \boldsymbol{\sigma}(x, y, \mathbf{u})$ loc. unif., and *possibly degenerate*,

$\exists A, R > 0$ s.t. $\langle y, \mathbf{b}(x, y) \rangle < -A\|y\|$, $\forall \|y\| \geq R, \forall x \in \mathbb{R}^n$

$\boldsymbol{\varrho}(x, y)\boldsymbol{\varrho}^\top(x, y)$ unif. bounded, *non-degenerate*¹

$$\mathbf{b}(\cdot, \cdot), \boldsymbol{\varrho}(\cdot, \cdot) \in C^{1,\alpha}(\mathbb{R}^n \times \mathbb{R}^m)$$

¹possibly degenerate provided it satisfies Hörmander's condition.

Stochastic Control Problem

Pay-off function $\mathbf{J} : [0, T] \ni (t, x, y, u) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{U} \rightarrow \mathbb{R}$, $\lambda > 0$

$$\mathbf{J}(t, x, y, u) := \mathbb{E}_{x,y} \left[e^{\lambda(t-T)} \mathbf{g}(X_T) + \int_t^T \ell(X_s, Y_s, u_s) e^{\lambda(s-T)} ds \right],$$

Value function

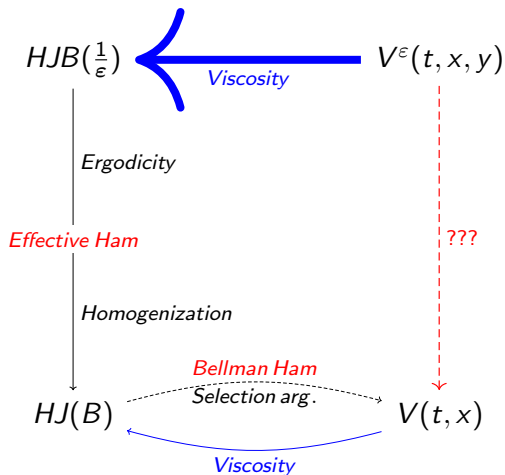
$$\mathbf{V}^\varepsilon(t, x, y) := \sup_{u \in \mathcal{U}} \{ \mathbf{J}(t, x, y, u), \text{ s.t. } (X^\varepsilon, Y^\varepsilon) \text{ in (1)} \} \quad (2)$$

\mathcal{U} the set of \mathcal{F}_t -progressively measurable processes taking values in U .

The utility function $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}$ and the running cost

$\ell : \mathbb{R}^n \times \mathbb{R}^m \times U \rightarrow \mathbb{R}$ are continuous functions and satisfy

$$\exists K > 0 \text{ such that } |\mathbf{g}(x)|, |\ell(x, y, u)| \leq K(1 + |x|^2 + |y|^2), \forall x, y$$



HJB equation

A fully nonlinear degenerate parabolic equation in $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$

$$\begin{cases} -V_t^\varepsilon + F^\varepsilon \left(x, y, V^\varepsilon, D_x V^\varepsilon, \frac{D_y V^\varepsilon}{\varepsilon}, D_{xx}^2 V^\varepsilon, \frac{D_{yy}^2 V^\varepsilon}{\varepsilon}, \frac{D_{xy}^2 V^\varepsilon}{\sqrt{\varepsilon}} \right) = 0, \\ V^\varepsilon(T, x, y) = g(x), \quad \text{in } \mathbb{R}^n, \end{cases}$$

The Hamiltonian $F^\varepsilon : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^n \times \mathbb{S}^m \times \mathbb{M}^{n,m} \rightarrow \mathbb{R}$ is

$$F^\varepsilon(x, y, s, p, q, M, N, Z) := H^\varepsilon(x, y, p, M, Z) - \mathcal{L}(x, y, q, N) + \lambda s,$$

where

$$H^\varepsilon(x, y, p, M, Z) := \min_{u \in U} \left\{ -\text{tr}(\sigma^\varepsilon \sigma^{\varepsilon T} M) - f \cdot p - 2\text{tr}(\sigma^\varepsilon \varrho^T Z^T) - \ell \right\}$$

$$\mathcal{L}(x, y, q, N) := b \cdot q + \text{tr}(\varrho \varrho^T N)$$

σ^ε , f , b and ℓ are computed at (x, y, u) and $\varrho = \varrho(x, y)$

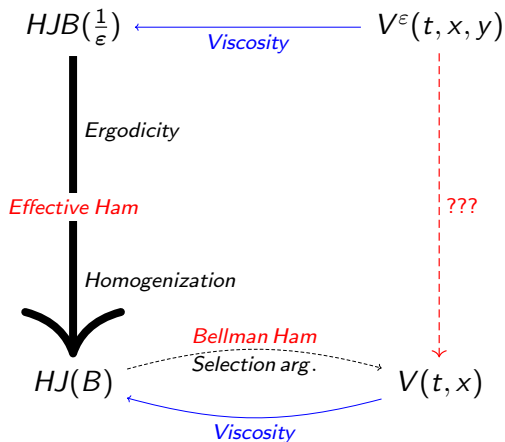
Proposition

For any $\varepsilon > 0$, the function V^ε is the unique continuous viscosity solution to the Cauchy problem with at most quadratic growth in x and y , i.e.

$\exists K > 0$ such that

$$|V^\varepsilon(t, x, y)| \leq K(1 + |x|^2 + |y|^2), \quad \forall t \in [0, T], x \in \mathbb{R}^n, y \in \mathbb{R}^m$$

Notice that V^ε is **not bounded** in y , but has a quadratic growth, and also the coefficients of the PDE are not bounded...



$$dY_y(\cdot) = \mathbf{b}(\mathbf{x}, Y_y(\cdot)) dt + \sqrt{2}\mathbf{g}(\mathbf{x}, Y_y(\cdot)) dW_t, \quad Y_y(0) = y \in \mathbb{R}^m, \quad \mathbf{x} \text{ fixed}$$

It is well known² that an **invariant measure** $\mu_{\mathbf{x}}$ of $Y_y(\cdot)$

- ▶ exists, is unique, has finite moments and Lip. Cont. density³ w.r.t. \mathbf{x}
- ▶ satisfies $\| \mathbb{P}_{Y_y(t)}(\cdot) - \mu_{\mathbf{x}}(\cdot) \|_{TV} \leq C(1 + |y|^d)(1 + t)^{-(1+k)}$

Moreover, we prove for $\tau_n := \inf \{t \geq 0 \text{ s.t. } \|Y_y(t)\| \geq n\}$ s.t.

Lemma

$$\exists \eta > 0, \forall \beta > 0, \mathbb{E} \left[\exp \left(-\frac{\tau_n}{n^\beta} \right) \right] \leq C n^\beta e^{-n\eta} \xrightarrow{n \rightarrow +\infty} 0, \quad (\text{loc. unif. } y)$$

²Veretennikov, "On polynomial mixing and convergence rate for stochastic difference and differential equations." Theory of Probability & Its Applications 44.2 (2000)

³Pardoux & Veretennikov, "On Poisson equation and diffusion approximation 2." The Annals of Probability (2003)

Ergodicity (II)

Recall: $dY_y(\cdot) = \mathbf{b}(x, Y_y(\cdot)) dt + \sqrt{2}\varrho(x, Y_y(\cdot)) dW_t$, $Y_y(0) = y_0 \in \mathbb{R}^m$
with the assumption

$$\exists A, R > 0 \text{ s.t. } \langle y, \mathbf{b}(x, y) \rangle < -A\|y\|, \quad \forall \|y\| \geq R, \forall x \in \mathbb{R}^n$$

Idea of proof⁴:

(i) Construct (Z_t, L_t) reflected process (by Skorokhod's lemma), s.t.

$$dZ_t := -\eta \|\nabla h(Y_t)^\top \varrho(Y_t)\|^2 dt + \sqrt{2} \nabla h(Y_t)^\top \varrho(Y_t) dW_t + dL_t$$

$h : \mathbb{R}^m \rightarrow \mathbb{R}$, C^2 s.t. $h(y) = \|y\|$ if $\|y\| \geq R$, and is $< R$ otherwise,

(ii) Find $\eta > 0$ s.t. $\|Y_t\| \leq Z_t$ a.s., when $Z_0 = R \vee \|y_0\|$

(iii) Upperbound $\mathbb{E}_Z [e^{-\delta\tau_n}]$ and conclude.

⁴In the line of proof [Prop.1.4] in: Herrmann, Imkeller, Peithmann, "Transition times and stochastic resonance for multidimensional diffusions with time periodic drift: A large deviations approach", Ann. Appl. Probab. (2006)

Effective Hamiltonian (I)

Cell problem: fix $(\bar{x}, \bar{p}, \bar{M})$

$$-\underbrace{\mathcal{L}(\bar{x}, y, D\chi, D^2\chi)}_{\text{infinitesimal generator}} + \underbrace{H(\bar{x}, y, \bar{p}, \bar{M}, 0)}_{\text{Hamiltonian}} = \mathbf{c}, \quad y \in \mathbb{R}^m, \quad (CP)$$

Effective Hamiltonian = $\mathbf{c} \in \mathbb{R}$ (unique) s.t. $\exists \chi$ viscosity sol. of (CP).

\rightsquigarrow It is sufficient to consider an approximation of (CP).

δ -Cell problem: fix $(\bar{x}, \bar{p}, \bar{M})$

$$\delta\omega_\delta - \mathcal{L}(\bar{x}, y, D\omega_\delta, D^2\omega_\delta) + H(\bar{x}, y, \bar{p}, \bar{M}, 0) = 0, \quad y \in \mathbb{R}^m \quad (CP_\delta)$$

Effective Hamiltonian = $-\lim_{\delta \rightarrow 0} \delta\omega_\delta(\cdot)$. (ω_δ is an *approximate-corrector*)

Effective Hamiltonian (II) – construction

↪ An **approximation** of the δ -Cell problem:

Let $\{D_n\}_n \subset \mathbb{R}^m$, ∂D_n smooth, $D_n \xrightarrow[n \rightarrow \infty]{} \mathbb{R}^m$ (e.g. D_n ball of radius n).

Consider the Dirichlet-Poisson problem, for $\mathbf{h}(\mathbf{y}) := \mathbf{H}(\bar{x}, \mathbf{y}, \bar{p}, \bar{M}, 0)$

$$\begin{cases} \delta \omega(\mathbf{y}) - \mathcal{L}\omega(\mathbf{y}) = -h(\mathbf{y}), & \text{in } D_n \\ \omega(\mathbf{y}) = 0, & \text{on } \partial D_n \end{cases}$$

It has a unique solution $\omega^{\delta, n}(\mathbf{y}) = \mathbb{E} \left[- \int_0^{\tau_n} h(Y_{\mathbf{y}}(t)) e^{-\delta t} dt \right]$
where τ_n is the first exit time of $Y_{\mathbf{y}}(\cdot)$ from D_n .

Proposition

Let $\delta(n) = O(n^{-(4+\alpha)})$, for some $\alpha > 0$, the one has

$$\lim_{n \rightarrow \infty} \left| \delta(n) \omega^{\delta(n), n}(\mathbf{y}) - \left(- \int_{\mathbb{R}^m} h(\mathbf{y}) d\boldsymbol{\mu}(\mathbf{y}) \right) \right| = 0, \quad \text{loc. unif. in } \mathbf{y},$$

where $\boldsymbol{\mu}$ is the unique invariant probability measure for the process $Y_{\mathbf{y}}(\cdot)$.

Convergence of the value function (I)

Recall the $(HJB)_\varepsilon$ equation

$$\begin{cases} -V_t^\varepsilon + F^\varepsilon \left(x, y, V^\varepsilon, D_x V^\varepsilon, \frac{D_y V^\varepsilon}{\varepsilon}, D_{xx}^2 V^\varepsilon, \frac{D_{yy}^2 V^\varepsilon}{\varepsilon}, \frac{D_{xy}^2 V^\varepsilon}{\sqrt{\varepsilon}} \right) = 0, \\ V^\varepsilon(T, x, y) = g(x), \quad \text{in } \mathbb{R}^n, \end{cases}$$

The Hamiltonian F^ε is

$$F^\varepsilon(x, y, s, p, q, M, N, Z) := H^\varepsilon(x, y, p, M, Z) - \mathcal{L}(x, y, q, N) + \lambda s,$$

Now, the *effective HJB* equation

$$\begin{cases} -V_t + \bar{H}(x, D_x V, D_{xx}^2 V) + \lambda V(x) = 0, & (t, x) \in (0, T) \times \mathbb{R}^n \\ V(T, x) = g(x), & \text{in } \mathbb{R}^n, \end{cases}$$

The *effective* Hamiltonian is

$$\bar{H}(\bar{x}, \bar{p}, \bar{M}) := \int_{\mathbb{R}^m} H(\bar{x}, y, \bar{p}, \bar{M}, 0) d\mu(y)$$

Convergence of the value function (II)

Theorem

The solution V^ε to $(HJB)_\varepsilon$ converges uniformly on compact subsets of $[0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ to the unique continuous viscosity solution to the limit problem \widetilde{HJB} satisfying a quadratic growth condition in x , i.e.

$$\exists K > 0 \text{ such that } |V(t, x)| \leq K(1 + |x|^2), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n$$

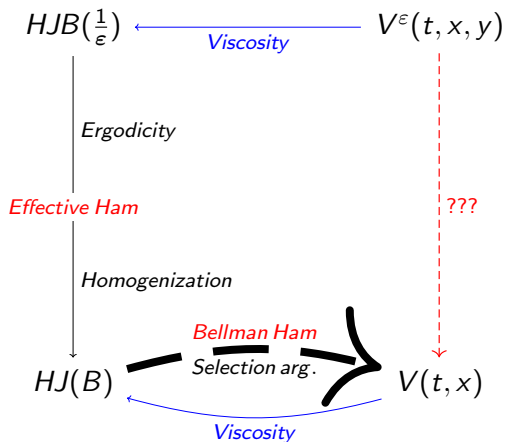
Proof.

Viscosity methods in the same spirit as in the proof for V^ε with in addition

- ▶ Liouville-type result^a to control \mathcal{L} in $F^\varepsilon(\bullet) = H^\varepsilon(\bullet) - \mathcal{L}(\bullet) + \lambda\bullet$
- ▶ adaptation of the *perturbed test function*^b to produce maximas. □

^aMannucci, P., Marchi, C., & Tchou, N. "The ergodic problem for some subelliptic operators with unbounded coefficients." NoDEA(2016)

^bintroduced in: Evans LC, "The perturbed test function method for viscosity solutions of nonlinear PDE." (1989)



Proposition

Under the standing assumptions, the effective Hamiltonian writes

$$\bar{H}(x, p, M) = \min_{\nu \in \mathcal{U}^{\text{ex}}(x)} \int_{\mathbb{R}^m} \left[-\text{trace}(\sigma \sigma^\top M) - f \cdot p - \ell \right] d\mu_x(y)$$

where σ, f and ℓ are computed at (x, y, u) , and $\mathcal{U}^{\text{ex}}(x)$ is the set of progressively measurable processes taking values in the extended control set $U^{\text{ex}}(x) := L^2((\mathbb{R}^m, \mu_x), U)$.

The extended controls are

$$\begin{aligned} \nu(\cdot) : t \mapsto \nu_t(\cdot) &\in L^2((\mathbb{R}^m, \mu_{\hat{x}_t}), U) \\ &= \left\{ \phi(\cdot) : y \mapsto \phi(y) \in U \mid \int_{\mathbb{R}^m} |\phi(y)|^2 d\mu_{\hat{x}_t}(y) < \infty \right\} \end{aligned}$$

\rightsquigarrow This is an **exchange** operation " $\min \int = \int \min$ "

Limit Control Problem (I)

A guess for the limit dynamics:

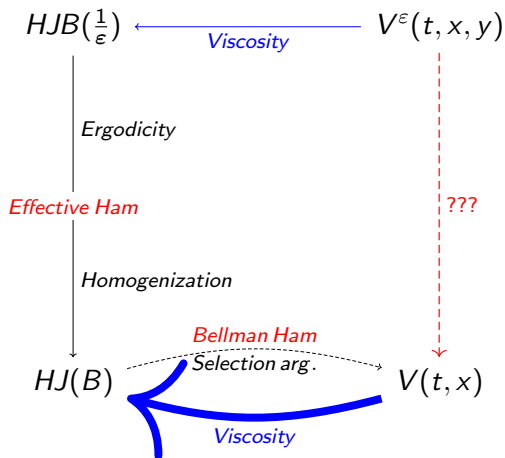
$$\left\{ \begin{aligned} d\hat{X}_t &= \int_{\mathbb{R}^m} f(\hat{X}_t, y, \mathbf{v}_t(y)) d\mu_{\hat{X}_t}(y) dt \\ &\quad + \sqrt{2} \sqrt{\int_{\mathbb{R}^m} \sigma \sigma^\top(\hat{X}_t, y, \mathbf{v}_t(y)) d\mu_{\hat{X}_t}(y)} dW_t, \quad (3) \\ \mathbf{v}_t(\cdot) &\in \mathcal{U}^{\text{ex}}(\hat{X}_t), \quad \text{and} \quad \hat{X}_0 = x \in \mathbb{R}^n. \end{aligned} \right.$$

The *effective* optimal control problem

$$V(t, x) = \sup \{ \hat{J}(t, x, \mathbf{v}(\cdot)), \quad \text{subject to (3)} \} \quad (4)$$

where the *effective* pay off $\hat{J}(t, x, \mathbf{v}(\cdot))$ is

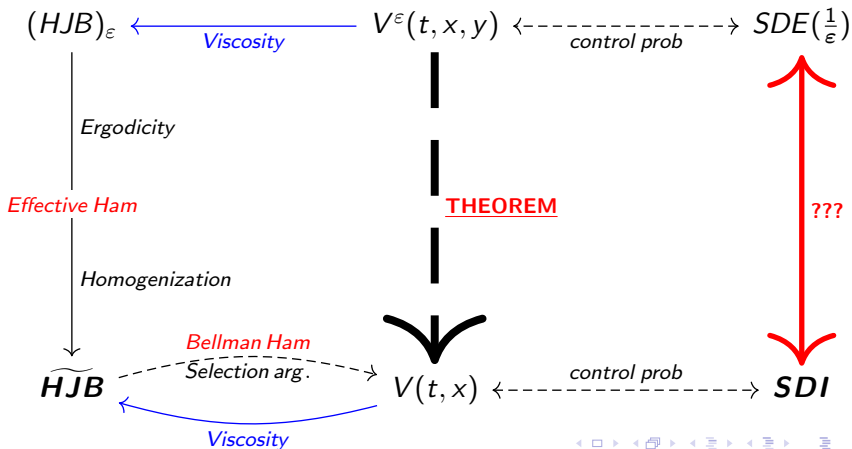
$$\mathbb{E}_x \left[e^{\lambda(t-T)} g(\hat{X}_T) + \int_t^T \int_{\mathbb{R}^m} \ell(\hat{X}_s, y, \mathbf{v}_s(y)) d\mu_{\hat{X}_s}(y) e^{\lambda(s-T)} ds \right]$$



Limit Control Problem (II)

Theorem

The value function (4) is the unique viscosity solution to the Cauchy problem \widetilde{HJB} . It is in particular, the limit of V^ε defined in (2) for $(HJB)_\varepsilon$.



Convergence of Trajectories

Key observation:

The *convergence Theorem* for the value function holds independently of the choice of the cost functional.

\rightsquigarrow As $\varepsilon \rightarrow 0$,

$SDE(\frac{1}{\varepsilon})$ and SDI always produce the same value for every choice of an optimal control problem.

So we can *hope* for *at least* a convergence of the type

$$\lim_{\varepsilon \rightarrow 0} \max_{t \in [0, T]} \|\phi(X_t^\varepsilon) - \phi(\hat{X}_t)\| = 0$$

where ϕ is a real valued continuous function .

Work in Progress

Control of { Smoluchowski equation // Stochastic Gradient Descent⁵ }

Let \mathbf{V} be a confining potential. Let $\beta, \gamma > 0$ and $\sigma \geq 0$.

$$dX_t = -u_t \underbrace{\gamma^{-1}(X_t - Y_t)}_{\nabla_x F(x,y)} dt + u_t \sqrt{2\sigma} dW_t, \quad X_0 = x \in \mathbb{R}^n \quad (5)$$

$$dY_t = -\frac{1}{\varepsilon} \nabla_y F(X_t, Y_t) dt + \sqrt{\frac{2}{\varepsilon}} \beta^{-1/2} dW_t, \quad Y_0 = y \in \mathbb{R}^n$$

where $F(x, y) := \mathbf{V}(y) + \frac{1}{2\gamma} |x - y|^2$. Therefore as $\varepsilon \rightarrow 0$ one expects

$$d\hat{X}_t = -v_t \nabla \mathbf{V}_\gamma(\hat{X}_t) dt + v_t \sqrt{2\sigma} dW_t \quad (6)$$

where \mathbf{V}_γ is the local entropy, and β is the inverse temperature

$$\mathbf{V}_\gamma := -\frac{1}{\beta} \log (G_{\beta^{-1}\gamma} * \exp(-\beta \mathbf{V})).$$

⁵Highly inspired & motivated by: Chaudhari, Oberman, Osher, Soatto & Carlier (2018). *Deep relaxation: partial differential equations for optimizing deep neural networks*

Thank You

- For further info/comments/remarks/...
check the poster and several others here:

hdhj2020-posters.slack.com