Stochastic heat equation and polymers on graphs

Ofer Zeitouni

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Background: stochastic heat equation, parabolic Anderson model

$$\partial_t u = \frac{1}{2}\Delta u + \lambda V(t,x)u, \quad x \in \mathbb{R}^d, d \geq 3.$$

Here, $V(t,x)$ is random field, molification of space-time white noise:

$$V(t,x) = \int_{\mathbb{R}^{d+1}} \phi(t-s)\psi(x-y)dW(s,y), \quad \phi, \psi \text{ compactly supported, } \psi \text{ isotropic.}$$

Hopf-Cole Logarithmic transformation: $w(t,x) = \log u(t,x)$ satisfies the KPZ equation

$$\partial_t w = \frac{1}{2}\Delta w + \frac{1}{2}|\nabla w|^2 + \lambda V(t,x)$$

Rescale: $u_\varepsilon(t,x) := u(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ satisfies

$$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \frac{\lambda}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})u_\varepsilon. \quad u_\varepsilon(0,x) = u_0(x) \in \mathcal{C}_b(\mathbb{R}^d).$$

The noise $\varepsilon^{-2}V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ does not converge to white noise $\dot{W}$ - rather to $\varepsilon^{d/2-1}\dot{W}$.

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The Feynmann-Kac representation

$$\partial_t u = \frac{1}{2}\Delta u + \lambda V(t,x)u$$

$$u(t,x) = E_B^x\left(u_0(X_t)\exp(\lambda\int_0^t V(t-\tau, B_\tau)d\tau)\right)$$

In particular, if V is white in time, can be made into a martingale (in t) using time reversal and substraction of the (deterministic) quadratic variation.

$$d_t u = \frac{1}{2}\Delta u dt + \lambda d_t V(t,x)u$$

$$u(t,x) = E_B^x\left(u_0(X_t)\exp(\int_0^t V(t-\tau, B_\tau)d\tau - \frac{\lambda^2 t}{2}R_V(0))\right)$$

In non-white in time case, the correction term is itself not deterministic.

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Special case: V - white in time, $u_0 = 1$.

Theorem (Mukherjee, Shamov, Z. '16 ($d \geq 3$))

There exits $\lambda_* \in (0, \infty)$ so that:

- *(Weak disorder) For $\lambda < \lambda^*$, solutions converge weakly in distribution to a deterministic limit, and $u_\varepsilon(x)$ converges to a random variable $Z_\infty > 0$.*
- *(Strong disorder) For $\lambda > \lambda^*$, $u_\epsilon(0) \to 0$ in probability.*

$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \frac{\lambda}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) u_\varepsilon.$

Theorem (Ryzhik, Gu, Z. '17)

($d \geq 3$, $\lambda < \lambda_0 < \lambda_*$ "deep in L^2 region") There exist c_1, c_2 depending on λ such that for any $t > 0$ and $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x) \exp\Big\{ -\frac{c_1 t}{\varepsilon^2} - c_2 \Big\} g(x) dx \to_{\varepsilon \to 0} \int_{\mathbb{R}^d} \bar{u}(t,x) g(x) dx,$$

$$\frac{1}{\varepsilon^{d/2-1}} \int_{\mathbb{R}^d} (u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)]) \exp\Big\{ -\frac{c_1 t}{\varepsilon^2} - c_2 \Big\} g(x) dx \Rightarrow_{\varepsilon \to 0} \int_{\mathbb{R}^d} \mathbf{U}(t,x) g(x) dx.$$

$\bar{u}$ - solution of effective heat equation

$$\partial_t \bar{u} = \frac{1}{2} \nabla \cdot \boldsymbol{a}_{\text{eff}} \nabla \bar{u}, \quad \bar{u}(0,x) = u_0(x), \boldsymbol{a}_{\text{eff}} \in \mathbb{R}^{d\times d}_{\text{sym}} \text{ effective diffusion},$$

U *solves the additive stochastic heat equation*

$$\partial_t \mathbf{U} = \frac{1}{2} \nabla \cdot \boldsymbol{a}_{\text{eff}} \nabla \mathbf{U} + \lambda \nu_{\text{eff}} \bar{u} \dot{W}, \quad \mathbf{U}(0,x) = 0, \nu_{\text{eff}}^2 > 0 \text{ effective variance}$$

$\partial_t u_\varepsilon = \frac{1}{2}\Delta u_\varepsilon + \frac{\lambda}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) u_\varepsilon.$

$$\text{Heat equation: } \partial_t \bar{u} = \frac{1}{2}\nabla \cdot \boldsymbol{a}_{\text{eff}} \nabla \bar{u}, \quad \bar{u}(0,x) = u_0(x)$$

$$\text{Edwards-Wilkinson (additive noise): } \partial_t \mathbf{U} = \frac{1}{2}\nabla \cdot \boldsymbol{a}_{\text{eff}} \nabla \mathbf{U} + \lambda \nu_{\text{eff}} \bar{u} \dot{W}, \quad \mathbf{U}(0,x) = 0$$

Related results:

Magnen, Unterberger '17, applies also to Hopf-Cole transform (KPZ) and gives same EW limit. Different methods.

Mukherjee '17 averaged CLT;

Comets, Cosco, Mukherjee '18 rates of convergence to limit Z_∞, fluctuations from limit, link with GFF. Stochastic analysis methods.

Dunlap, Gu, Ryzhik, Z. '18–'19: Further link with homogenization, application to KPZ, representation in terms of correctors, representation of $\boldsymbol{a}_{\text{eff}}, \nu_{\text{eff}}$

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$d = 2$ - results for appropriate tuning $\lambda = \lambda_\epsilon$ - Caravenna, Sun, Zygouras '17+, ...

$d = 1$ - continuous polymers Bertini-Giacomin '97, Alberts, Khanin, Quastel '14, regularity structures Hairer '13, ...

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The polymer connection: Feynmann Kac

We proceed in white-in-time setup. Recall that

$$d_t u = \frac{1}{2}\Delta u dt + \lambda u d_t V(t,x)$$

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$$u(t) = E_B\left(\exp(\beta\int_0^t V(\tau, B_\tau)d\tau - \frac{\beta^2 t}{2}R_V(0))\right)$$

Switch to discrete model (directed polymer):

$$Z_n(x) = E_S^x\left(\exp(\beta\sum_{i=1}^n V(i,S_i) - C_\beta n)\right)$$

where $\{S_n\}$ is random walk on $\mathbb{Z}^d$, and $\{V(i,x)\}_{i\in\mathbb{Z}_+, x\in\mathbb{Z}^d}$ are i.i.d. C_β chosen such that expectation remains 1. Martingale! (with respect to randomness of V).

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Z_n, being positive martingale, converges a.s.: $Z_n \to Z_\infty$. Also have the polymer (random) measure

$$dQ_n(\mathbf{S}) = \frac{e^{\beta \sum_{i=1}^{n} V(i,S_i)}}{Z_n} dP(\mathbf{S}).$$

Definition

Weak disorder if $Z_\infty > 0$, Strong disorder if $Z_\infty = 0$.

By Kolmogorov 0-1, these are 0-1 properties, ie $\mathbb{P}(\text{weak disorder}) \in \{0, 1\}$.
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Setup: infinite connected graph $G = (V, E)$, $\{S_i\}$ - nearest neighbor random walk on G, started in $x \in V$. Extends to weighted graphs

Some results translate immediately. For example:

1. The limit $Z_n(x) \to Z_\infty(x)$ exists a.s. Further, $\mathbb{P}(Z_\infty(x) > 0) \in \{0, 1\}$ and is independent of $x \in V$.
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Since $\mathbb{E}Z_\infty(x)$ is harmonic, the last property implies that $\{\text{UI} \iff \text{weak disorder}\}$ if G has the Liouville property. Liouville: $\mathbb{Z}^d$, infinite percolation cluster of $\mathbb{Z}^d$; not trees!
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Some results translate immediately. For example:

1. The limit $Z_n(x) \to Z_\infty(x)$ exists a.s. Further, $\mathbb{P}(Z_\infty(x) > 0) \in \{0, 1\}$ and is independent of $x \in V$.
2. There exists $\beta_c \in [0, \infty]$ so that weak disorder for $\beta \in [0, \beta_c)$ and strong disorder for $\beta > \beta_c$.
3. Relation of strong disorder to localization of endpoint.
4. $\{\{Z_n(x)\}_n \text{ is UI for some } x\} \iff \{\mathbb{E}Z_\infty(x) = 1 \text{ for some } x\} \iff \{\inf_{x\in V}\mathbb{E}Z_\infty(x) > 0\} \iff \{\mathbb{E}Z_\infty(x) = 1, \{Z_n(x)\} \text{ UI } \forall x \in V\}$.

Also, some regularity helps: $\{\text{UI} \iff \text{weak disorder}\}$ if there is a finite set V_0 s.t. $\forall x \in V$, the rooted graph (x, G) is isomorphic to some (v, G), $v \in V_0$.

Polymers - generalities

Theorem (Cosco, Seroussi, Z.- Sufficient conditions for $\beta_c = 0$)

1. *If G is positive recurrent then $\beta_c = 0$.*
2. *If heat kernel admits Gaussian upper bound with spectral dimension $d < 2$ (with respect to reversing measure π), volume growth with exponent d and isoperimetric bound $d-1$ for π, then $\beta_c = 0$.*

Proof by fractional moments method - change law of $V(i,x)$ in some sets that are visited much.

Recall $\beta_2 = \max\{\beta > 0 : \mathbb{E}Z_n^2 \text{ is uniformly bounded}\}$. Independent of starting point x.For $\mathbb{Z}^d$ and $d \geq 5$, $\beta_2 < \beta_c$.

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More generally,

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To see this, consider event $\mathcal{A} = \{S \text{ never enters pipes}\}$. $P(\mathcal{A} > 0)$.Define $W_n(x) = E^x(e^{\beta \sum_{i=1}^n V(i,S_i) - c_\beta n} \mathbf{1}_{\mathcal{A}})$. For β small, $EW_n(x)^2 < \infty$. But $W_n(x) < Z_n(x)$ is a martingale, hence impossible that $Z_n(x) \to 0$.

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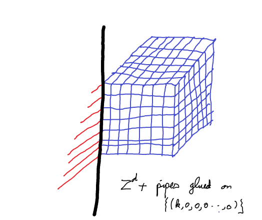

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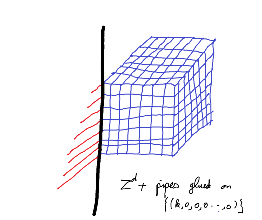

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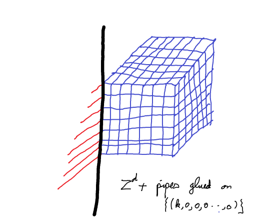

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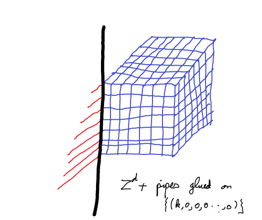

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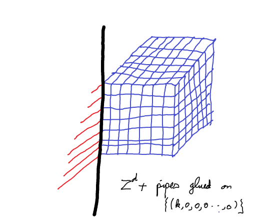

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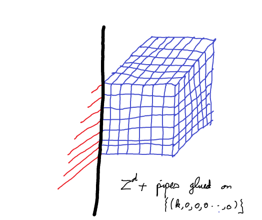

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Further general facts

- β_c is NOT monotone wrt adding edges Could have helped for $d = 2$ percolation cluster.

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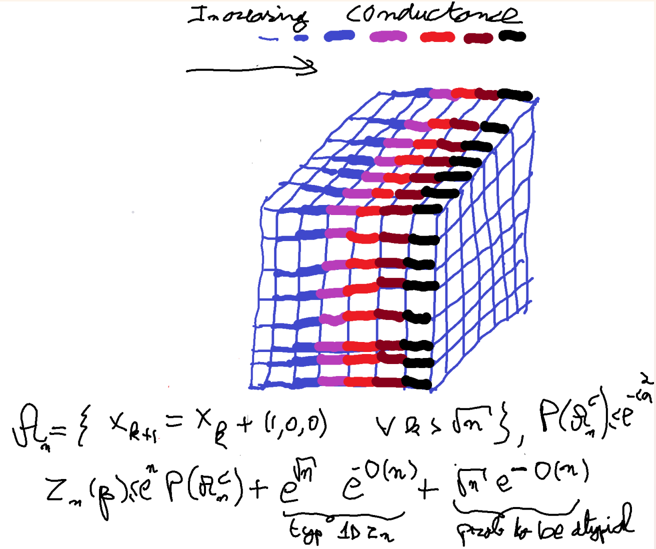

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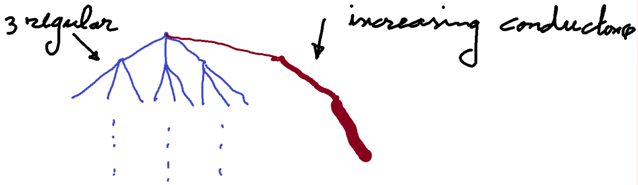

$\mathcal{E} = \{$ no visit to right tree $\}$, $P(\mathcal{E}) > 0$

$$Z_n = Z_n^{\ell} + Z_n^{R}\,; \quad Z_n^{R} \xrightarrow{a.s} 0, \quad Z_n^{\ell} \to Z_\infty^{\ell} > 0$$

But $E Z_\infty = E Z_\infty^{\ell} = P(\mathcal{E}) < E Z_0 = E Z_n$

Hence, $\{Z_n\}$ not UI

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Polymers on trees: initiated by Derrida and Spohn ('88), for regular trees and directed random walks. Link with branching random walks, BBM, KPP equation.

Setup: Galton-Watson tree, i.e. each vertex has a random number of children with distribution $\{p_k\}$, i.i.d.; mean offspring number is $m = \sum kp_k$.

To allow for different behaviors, random walk will be λ-biased: at a vertex v with k children, probability to go to parent is

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Facts Lyons-Pemantle-Peres, ...

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Random trees

Polymers on trees: initiated by Derrida and Spohn ('88), for regular trees and directed random walks. Link with branching random walks, BBM, KPP equation.

Setup: Galton-Watson tree, i.e. each vertex has a random number of children with distribution $\{p_k\}$, i.i.d.; mean offspring number is $m = \sum kp_k$.
To allow for different behaviors, random walk will be λ-biased: at a vertex v with k children, probability to go to parent is

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Polymers on trees

Very strong disorder means $\limsup n^{-1} \log Z_n < 0$, a.s. For a regular tree and $\lambda > m$, always holds by a general result of Carmona, Guerra, Hu and Mejane ('04), since return times to root have exponential tails.

Theorem (Cosco, Seroussi, Z.)

1. *If $\lambda < m$ then can have $\beta_2 > 0$ or $\beta_2 = 0$ depending on $\min\{k : p_k > 0\}$ and λ, m. If $p_0 = 0$ and $p_1 < 1$, we have that $\beta_c > 0$.*
2. *If $\lambda > m$ then $\beta_c = 0$. In fact, if support of $\{p_k\}$ large enough, then there is $\bar{\beta}_c < \infty$ so that very strong disorder holds only for $\beta > \bar{\beta}_c$.*

We believe that if $\lambda = m > 1$ then $\beta_2 = 0$. What about β_c? And in case $\lambda < m$ but $p_0 > 0$, what about β_c?
Toy model for percolation on $\mathbb{Z}^d$.

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