

Scaling Limits and Stochastic Homogenization

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The Royal Society

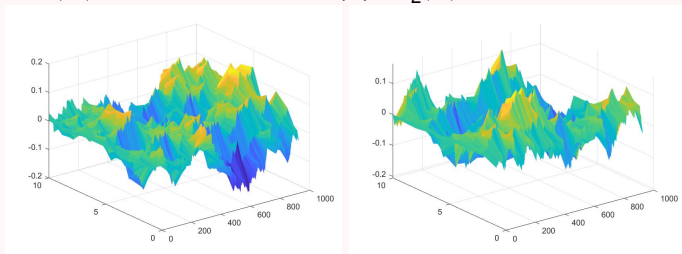
Introduction

Hydrodynamic Limit for Interacting Diffusions

T. Funaki, H. Spohn *Motion by mean curvature from the Ginzburg-Landau interface model*. CMP, 185(1) (1997), pp. 1-36.

$$d\Phi_t(x) = - \sum_{|x-y|_1=1} V'(\Phi_t(x) - \Phi_t(y)) + \sqrt{2}dB_t(x) \text{ for } x \in \Lambda \subset \mathbb{Z}^d.$$

The fields Φ live on a discrete lattice and take values in \mathbb{R}^d , $B_t(x)$ are i.i.d. Brownian motions, V' is the derivative of a strictly convex symmetric function, and $|\cdot|_1$ is the L^1 -norm. ($V(x) = \frac{1}{2}|x|^2$: Laplace)



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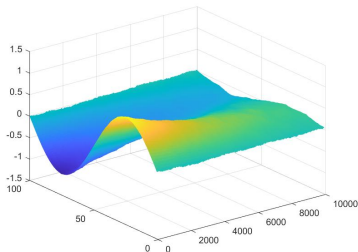
Funaki-Spohn 1999

$$\Phi^\varepsilon(r, t) = \varepsilon \Phi_{\varepsilon^{-2}t}(x) \text{ for } r \in [x - \varepsilon/2, x + \varepsilon/2]^d \text{ with } N = \lceil \varepsilon^{-1} \rceil$$

under **diffusive rescaling** convergence to the solution h of

$$\partial_t h(r, t) = \operatorname{div}((D\sigma)(\nabla h))$$

(Hydrodynamic limit)



Invariant Measure and limit

Invariant measure on finite lattice:

$$\frac{1}{Z} e^{\beta \sum_{x \in \Lambda} V(\nabla_i \Phi(x))} \prod_{x \in \Lambda} d\Phi(x).$$

$\nabla_i \Phi$: Discrete gradients.

Limit points as $\Lambda \rightarrow \mathbb{Z}^d$: DLR states (Dobrushin-Lanford-Ruelle)

Funaki-Spohn: **Unique** DLR state μ_P for each "tilt" $P : V(y + P)$, periodic b.c. by coupling and "PDE methods"

Technique for limit: **Local equilibrium states** (Guo, Papanicolaou Varadhan):

- macroscopic quantity $P \sim \nabla h$ varies slowly on microscale.
- near $\epsilon^{-1}x$, process in law close to invariant measure $\mu_{P(x)}$.
- replace nonlinearities by expectation under $\mu_{P(x)}$
- μ_P stationary (up to tilt) and **ergodic**

See book by Kipnis and Landim

Important fact: For $d = 1, 2$ no invariant measure for fields Φ exists, only for discrete gradients $\nabla_i \Phi$.

What about scaling SPDEs?

$$d_t U_t^\epsilon = \operatorname{div} \mathcal{A} \left(D U_t^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega_1 \right) dt + \frac{1}{\epsilon} \sum_{k \in \mathbb{Z}^d} \mathcal{A} \left(\frac{x-k}{\epsilon} \right) dB_t^k \quad \text{in } \mathbb{R}^d \times (0, +\infty),$$

$$U_0^\epsilon = u_0. \quad \text{in } \mathbb{R}^d$$

$(B^k)_{k \in \mathbb{Z}^d}$ i.i.d. BM, \mathcal{A} stat. ergodic in space-time (\mathbb{Z}^d), ind. of BM's.

Note: Similar to Funaki-Spohn, but different model. (Scale of correlation vs. scale of discretization)

AIM: Use homogenization techniques

Scaled (linear) SPDEs:

A. Dunlap, Y. Gu, L. Ryzik and O. Zeitouni

M. Hairer, E. Pardoux, A. Piatnitski

Here: Nonlinear (divergence form)

Scaling limit for SPDE by homogenization techniques

Homogenization:

Law of microstructure fixed, stationary ergodic.

Scaling limits for interacting particle systems:

Self-organised ergodic structure on microscale,
changing on macroscale (local equilibrium)

invariant measure μ_P corresponds to law of gradient corrector

Theorem (Cardaliaguet-D.-Souganidis 2020):

Solutions of

$$d_t U_t^\epsilon = \operatorname{div} \mathcal{A}(DU_t^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega_1) dt + \frac{1}{\epsilon} \sum_{k \in \mathbb{Z}^d} A(\frac{x-k}{\epsilon}) dB_t^k = f(x, t),$$

$$U_0^\epsilon = u_0.$$

converge a.s. and in expectation in suitably weighted L^2 spaces to unique solutions of

$$\bar{u}_t - \operatorname{div} \bar{a}(D\bar{u}) = f$$

\mathcal{A} strictly monotone

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Strategy of Proof

$$U_t^\epsilon(x) = \epsilon \tilde{V}_t\left(\frac{x}{\epsilon}\right) + \tilde{W}_t^\epsilon(x),$$

$$d\tilde{V}_t = \Delta \tilde{V}_t dt + \sum_{k \in \mathbb{Z}^d} A(x-k) dB_t^k \quad \text{in } \mathbb{R}^d \times \mathbb{R}, \quad (1)$$

$$\partial_t \tilde{W}_t^\epsilon = \operatorname{div} \left(\tilde{a} \left(D\tilde{W}_t^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega \right) \right) \quad \text{in } \mathbb{R}^d \times (0, +\infty) \quad (2)$$

$$\tilde{W}_0^\epsilon = u_0 \quad \text{in } \mathbb{R}^d,$$

with the random nonlinearity

$$a(p, x, t, \omega) = \mathcal{A}(p + D\tilde{V}_t(x, \omega_0), t, x, \omega_1) - D\tilde{V}_t(x, \omega_0)$$

(Simplification under structural assumptions as in F-S)

Tasks:

1. Existence of eternal, stationary, attracting solution for **gradient** of (1)
 $\Rightarrow D\tilde{V}$ space-time stationary ergodic
2. Homogenize nonlinear divergence form PDEs as (2) with space-time stat. erg. coefficients with **low time regularity**.

Heat Equation

Eternal solution for heat equation with additive noise

Space time stationary solutions for

$$dV_t = \Delta V_t dt + \sum_{k \in \mathbb{Z}^d} A(x - k) dB_t^k.$$

via

$$dV_t = \Delta V_t dt + \sum_{k \in \mathbb{Z}^d} A(x - k) dB_t^k \quad \text{on } [-n^2, \infty) \times \mathbb{R}^n$$

$$V_{-n^2}(x) = 0$$

and $n \rightarrow \infty$.

$A(x)$ smooth and compact support

- moment bounds
- correlation decay
- convergence as $n \rightarrow \infty$

Moment Bounds

$$\mathbb{E}[|V_t(x)|^2] \leq Ct \wedge \begin{cases} 1 & \text{if } d \geq 3, \\ \ln(t+1) & \text{if } d = 2, \\ t^{1/2} & \text{if } d = 1. \end{cases}$$

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}[|DV_t(x)|^2] + \mathbb{E}[|D^2V_t(x)|^2] \leq C(t \wedge 1),$$

Method: Heat kernel and Ito formula

Space-time stationary limit expected only for gradients!

Decorrelation

For $l \in \mathbb{Z}$ and $R \geq 1$, let $V^{l,R}$ be the solution to

$$\begin{cases} dV_t^{l,R} = \Delta V_t^{l,R} dt + \sum_{k \in \mathbb{Z}^d, |k-l| \leq R} A(x-k) dB_t^k & \text{in } \mathbb{R}^d \times (0, +\infty), \\ V_0^{l,R} = 0 & \text{in } \mathbb{R}^d. \end{cases}$$

(Switch off noise outside $B_R(l)$)

$$\mathbb{E} \left[|DV_t(0) - DV_t^{0,R}(0)|^2 \right] \leq C \begin{cases} R^{-d} & \text{if } R^2/t \leq 1, \\ \exp\{-R^2/(5t)\} & \text{otherwise,} \end{cases},$$

Note \mathbb{Z}^d invariance in space.

Note: $V^{l_1,R}$ and $V^{l_2,R}$ are **independent** if $|l_1 - l_2| \gg R$.

Convergence

$$dV_t^n = \Delta V_t^n + \sum_{k \in \mathbb{Z}^d} A(x - k) dB_t^k \quad \text{on } [-n^2, \infty) \times \mathbb{R}^n$$

$$V_{-n^2}^n(x) = 0$$

$(DV^n)_{n \in \mathbb{N}}$ is Cauchy sequence in $L^2(B_r \times [-T, T] \times \Omega)$ for any fixed r and T .

Method: Suppose $m \ll n$.

$V^n - V^m$ solves deterministic heat equation.

Solution is heat kernel applied to difference in initial values at time $-m$, i.e. to

$$V_{-m^2}^n.$$

Heat kernel: "Averaging"

Replace V by $V^{l,R}$, use $m, n \gg 1$. (Recall $\mathbb{E} \left[|DV_t^n(0) - DV_t^{0,R}(0)|^2 \right] \leq CR^{-d}$)

Attractor Similar principle: Difference of two solutions satisfies deterministic heat equation. Use gradient estimate by averaging for heat equation

Self-organized ergodicity

Lemma

1. There exists a unique process $Z : \Omega_0 \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ (eternal solution) with

$$\mathbb{E} \left[\int_{\tilde{Q}_1} |Z_t(x)|^2 dx dt \right] < \infty, \quad dZ_{i,t}(x) = \Delta Z_{i,t}(x) dt + \sum_{k \in \mathbb{Z}^d} D_{x_i} A(x-k) dB_t^k.$$

2. Z is an attractor in the sense that, if V is a solution of stoch. heat eq. in $\mathbb{R}^d \times (0, \infty)$ such that $V(\cdot, 0) = 0$, then

$$\lim_{t \rightarrow +\infty} \mathbb{E} \left[\int_{Q_1} |DV_t(x) - Z_t(x)|^2 dx \right] = 0.$$

3. Higher dimensions: For $d \geq 3$, there exists a unique up to constants space-time stationary adapted process $V : \Omega \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ solving the stoch. heat eq. in $\mathbb{R}^d \times \mathbb{R}$ s.t. $\mathbb{E} \left[\int_{\tilde{Q}_1} |V_t(x)|^2 dx dt \right] < \infty$.

Consequence:

$$a(p, x, t, \omega) = \mathcal{A}(p + D\tilde{V}_t(x, \omega_0), t, x, \omega_1) - D\tilde{V}_t(x, \omega_0)$$

stationary ergodic w.r.t $\mathbb{R} \times \mathbb{Z}^d$.

Nonlinear divergence form

Qualitative homogenization for nonlinear divergence form parabolic PDE

$$u_t^\epsilon - \operatorname{div} a \left(Du^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega \right) = f \text{ in } \mathbb{R}^d \times (0, \infty) \quad u^\epsilon(\cdot, 0) = u_0,$$

$a : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ is strongly monotone, Lipschitz continuous and space-time stationary and ergodic with respect to $\mathbb{Z}^d \times \mathbb{R}$ -action.

Main result: Theorem on existence of corrector. (Cardaliaguet-D.-Souganidis 2020) For each P in \mathbb{R}^d there ex. $\chi^P(x, t, \omega)$ s.t.

$$\int_{\tilde{Q}_1} \chi^P(x, t, \omega) dx dt = 0 \quad P\text{-a.s.}, \quad D\chi^P \in \mathbf{L}_{\text{pot}}^2, \quad \partial_t \chi^P \in \mathbf{H}_x^{-1},$$

$$\partial_\tau \chi^P - \operatorname{div}(a(P + D\chi^P, y, \tau, \omega)) = 0 \text{ in } \mathbb{R}^{d+1},$$

$$\epsilon \chi^P \left(\frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega \right) \rightarrow 0$$

in $L_{loc}^2(\mathbb{R}^{d+1})$, \mathbb{P} -a.s. and in expectation. (**Parabolic Sublinearity**)

Effective Nonlinearity

$$u_t^\epsilon - \operatorname{div} a \left(Du^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega \right) = f \text{ in } \mathbb{R}^d \times (0, \infty) \quad u^\epsilon(\cdot, 0) = u_0,$$

$a : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ is strongly monotone, Lipschitz continuous and space-time stationary and ergodic with respect to $\mathbb{Z}^d \times \mathbb{R}$ -action.

$$\partial_\tau \chi^P - \operatorname{div}(a(P + D\chi^P, y, \tau, \omega)) = 0 \text{ in } \mathbb{R}^{d+1},$$

Effective nonlinearity

$$\bar{a}(p) = \mathbb{E} \left[\int_{\tilde{Q}_1} a(p + D\chi^p, y, \tau, \omega) dy d\tau \right]$$

is monotone and Lipschitz continuous.

Qualitative homogenization for nonlinear divergence form parabolic PDE: Related work

$$u_t^\epsilon - \operatorname{div} a \left(Du^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega \right) = f \text{ in } \mathbb{R}^d \times (0, \infty) \quad u^\epsilon(\cdot, 0) = u_0,$$

$a : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}^d$ is strongly monotone, Lipschitz continuous and space-time stationary and ergodic with respect to $\mathbb{Z}^d \times \mathbb{R}$ -action.

Related work:

- Efendiev and Panov
- Efendiev, Jiang and Pankov
- Zhikov, Kozlov and Oleinik
- Landim, Olla and Yau
- Fannjiang and Komorowski
- Komorowski and Olla
- Rhodes
- Delarue and Rhodes
- Lin, Smart and Lin
- Armstrong, Bordas and Mourrat

Existence of corrector: Strategy

Space-time cube \tilde{Q}_L , (finite domain), regularize with $\lambda > 0$.

$$\lambda u_L - \lambda \partial_{tt} u_L + \partial_t u_L - \operatorname{div}(\mathbf{a}(Du_L + \mathbf{p}, \omega)) = 0 \text{ in } \tilde{Q}_L \quad u_L = 0 \text{ in } \partial\tilde{Q}_L.$$

A-priori estimates independent of L lead to existence of

$$\lambda \chi^{\lambda, p} - \lambda \partial_{tt} \chi^{\lambda, p} + \partial_t \chi^{\lambda, p} - \operatorname{div}(\mathbf{a}(D\chi^{\lambda, p} + \mathbf{p}, \omega)) = 0 \text{ in } \mathbb{R}^{d+1}.$$

Estimates uniform in λ for $\partial_t \chi^{\lambda, p}$ in \mathbf{H}^{-1} and $D\chi^{\lambda, p}$ in \mathbf{L}^2 .

Use monotonicity of nonlinearity and lemmas on reconstruction of stationary functions from derivatives.

Lemmas on vector fields

$\theta \in \mathbf{H}_x^{-1}$, $w \in \mathbf{L}_{\text{pot}}^2$ satisfy for $i = 1, \dots, d$,

$$\langle \theta, \partial_{x_i} \phi \rangle_{\mathbf{H}_x^{-1}, \mathbf{H}_x^1} = \mathbb{E} [w_i \partial_t \phi]$$

Then there exists a measurable map $u : \mathbb{R}^{d+1} \times \Omega \rightarrow \mathbb{R}$ such that $Du = w$ and $\partial_t u = \theta$

If in addition $\xi \in \mathbf{L}^2$ satisfies

$$\theta - \operatorname{div}(\xi) = 0 \text{ in } \mathbf{H}_x^{-1},$$

then

$$\mathbb{E} \left[\int_{\tilde{Q}_1} w \cdot \xi \right] = 0.$$

Heuristics

$$\int_{\tilde{Q}_1} Du \cdot \xi \int_{\tilde{Q}_1} = - \int_{\tilde{Q}_1} u \cdot \operatorname{div}(\xi) = - \int_{\tilde{Q}_1} u \partial_t u = - \int_{\tilde{Q}_1} \partial_t \left(\frac{1}{2} u^2 \right).$$

Further ingredients

Lemma for sublinearity \mathbb{P} -a.s. and in expectation,

$$\lim_{R \rightarrow \infty} R^{-(d+3)} \int_{\tilde{Q}_R} |u(x, t)|^2 dx dt = \lim_{R \rightarrow \infty} R^{-(d+1)} \int_{\tilde{Q}_R} \left| \frac{u(x, t)}{R} \right|^2 dx dt = 0$$

Equivalent: $u^\epsilon(x, t, \omega) = \epsilon u(x/\epsilon, t/\epsilon, \omega)$.

$$\lim_{\epsilon \rightarrow 0} \int_{\tilde{Q}_R} |u^\epsilon(x, t)|^2 dx dt = 0.$$

Technical Difficulties:

Time derivative only in H^{-1} , ergodicity along lines (Lemma by Kosygina-Varadhan)

Homogenization: For perturbed test function method, approximate gradient of test function by piecewise constant function

Summary

Cardaliaguet-D.-Souganidis 2020

Result 1 (Scaling limit of SPDE by homogenization techniques)

$$d_t U_t^\epsilon = \operatorname{div} \mathcal{A} \left(D U_t^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega_1 \right) dt + \frac{1}{\epsilon} \sum_{k \in \mathbb{Z}^d} A \left(\frac{x-k}{\epsilon} \right) dB_t^k = f(x, t),$$

converge a.s. and in expectation in suitably weighted L^2 spaces to unique solutions of

$$\bar{u}_t - \operatorname{div} \bar{a}(D\bar{u}) = f$$

Result 2 (Homogenization/correctors)

Problem $u_t^\epsilon - \operatorname{div} a \left(D u^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega \right) = f$ in $\mathbb{R}^d \times (0, \infty)$

Corrector $\partial_\tau \chi^P - \operatorname{div}(a(P + D\chi^P, y, \tau, \omega)) = 0$ in \mathbb{R}^{d+1} ,

Effective nonlinearity

$$\bar{a}(p) = \mathbb{E} \left[\int_{\tilde{Q}_1} a(p + D\chi^p, y, \tau, \omega) dy d\tau \right]$$

Outlook

$$d_t U_t^\epsilon = \operatorname{div} \mathcal{A} \left(D U_t^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon^2}, \omega_1 \right) dt + \frac{1}{\epsilon} \sum_{k \in \mathbb{Z}^d} A \left(\frac{x-k}{\epsilon} \right) dB_t^k = f(x, t),$$

Scaling Limits and Stochastic Homogenization for some Nonlinear Parabolic Equations. P. Cardaliaguet, N. Dirr and P. E. Souganidis
arxiv:2004.03857

- Precise connection with Funaki-Spohn
- More degenerate operators (MCF approach by v. Renesse, Es-Sarhir)
- Multiplicative noise
- Allen-Cahn type problems
- Kac-type interactions and dynamic Lebowitz-Penrose limit

Thank you for your attention!