

# Large deviations for conservative, stochastic PDE and non-equilibrium fluctuations

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High Dimensional Hamilton-Jacobi PDEs

Workshop IV: Stochastic Analysis Related to Hamilton-Jacobi PDEs

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May 2020

joint work with: Ben Fehrman.

[G., Fehrman; arxiv, 2020].

## Introduction: Large deviations in zero range process

- 1 Introduction: Large deviations for the zero range process
  - Fluctuations in the zero range process
  - Link to stochastic PDE
- 2 Two ways to the LDP
  - Scaling and criticality for the skeleton equation
  - Well-posedness of the skeleton equation

## The zero range process

(could also consider simple exclusion, independent particles).

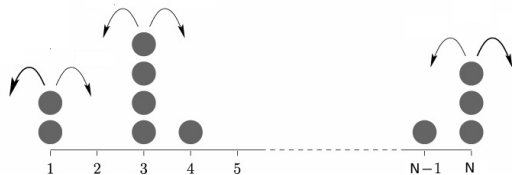


Figure: Harris, Rákos, Schütz; 2005

- Discrete  $d$ -dim. torus  $\mathbb{T}_N^d := (\mathbb{Z}/(N\mathbb{Z}))^d$
- State space  $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N^d}$ , i.e. configurations  $\eta : \mathbb{T}_N^d \rightarrow \mathbb{N}_0$  : System in state  $\eta$  if container  $x$  contains  $\eta(x)$  particles.
- Local jump rate function  $g : \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$ , with  $g(k) = 0$  iff  $k = 0$ ,  $g$  Lipschitz continuous.

- Translation invariant, asymmetric, zero mean transition probability

$$p(x, y) = p(x - y), \quad \sum_k kp(k) = 0.$$

- Zero range process with jump rate  $g$  is the Markov jump process  $\eta_t$  on  $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N^d}$  with generator

$$L_N f(\eta) = \sum_{x, y \in \mathbb{T}_N^d} (f(\eta^{x, y}) - f(\eta)) g(\eta(x)) p(x, y),$$

where  $f : \mathbb{M}_N \rightarrow \mathbb{R}$ ,  $\eta^{x, y} = \eta - 1_{\{x\}} + 1_{\{y\}}$ .

- Invariant probability measures  $\nu_\rho$  (product structure and explicit), indexed by density  $\rho \geq 0$ .

- Hydrodynamic limit?

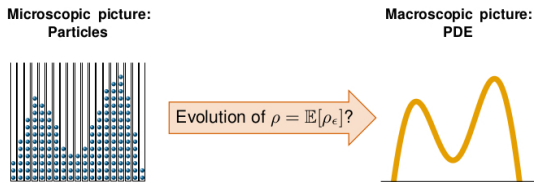


Figure: see Zimmer et. al.

- Empirical density field

$$\mu_t^N(x) := \frac{1}{N^d} \sum_k \delta_{\frac{k}{N}}(x) \eta_{tN^2}(k).$$

- Assume that  $\mathcal{L}(\eta_0^N)$  is associated to a smooth profile  $\rho_0$  with  $0 < \rho_- \leq \rho_0 \leq \rho_+ < \infty$ , that is,  $\mathcal{L}(\eta_0^N)$  around  $Nx$  converges to  $\nu_{\rho_0(x)}$  in probability.

## Theorem (Hydrodynamic limit - Ferrari, Presutti, Vares; 1987)

For  $\delta > 0$ ,  $G$  continuous, bounded,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[ \left| \langle \mu_t^N, G \rangle - \langle \bar{\rho}_t dx, G \rangle \right| > \delta \right] = 0$$

where  $\bar{\rho}(t, x)$  is the unique weak solution to

$$\begin{aligned} \partial_t \bar{\rho} &= \frac{1}{2} \partial_{xx} \Phi(\bar{\rho}) \\ \bar{\rho}(0) &= \rho_0 \end{aligned}$$

with  $\Phi$  the mean local jump rate

$$\Phi(\rho) = \mathbb{E}_{\nu_\rho} [g(\eta(0))].$$

E.g.  $\Phi(\rho) = \rho |\rho|^{m-1}$ .

- E.g. for constant intensity ( $g(k) = 1_{k>0}$ ) get  $\Phi(\rho) = \frac{\rho}{1+\rho}$ .
- Independent particles, simple exclusion process:  $\Phi(\rho) = \rho$ .

Theorem (Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988)

Let  $\eta$  be the zero range process with constant intensity with  $\eta_0^N$  in local equilibrium with smooth profile  $\rho_0$ . Define the fluctuation density field

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_k \delta_{\frac{k}{N}}(x) [\eta_{tN^2}(k) - \langle \eta_{tN^2}(k) \rangle]$$

for  $t \geq 0$ ,  $\langle \cdot \rangle$  the expectation. Let  $\mathcal{P}^N$  the law of  $Y^N$  on  $D([0, \infty), \mathcal{S}'(\mathbb{T}))$ . Then,

$$\mathcal{P}^N \xrightarrow{*} \mathcal{P} \text{ for } N \rightarrow \infty$$

with  $\mathcal{P}$  a Gaussian, supported on  $C([0, \infty), \mathcal{S}'(\mathbb{T}))$ , the martingale solution to

$$dY_t = \partial_{xx}(\Phi'(\bar{\rho}_t(x))Y_t) dt + \partial_x(\sqrt{\Phi(\bar{\rho}_t(x))}dW_t)$$

with initial condition  $Y_0$  is Gaussian, centered and covariance

$$\mathcal{X}(\rho_0(x)) = \mathbb{E}_{\nu_{\rho_0(r)}}(\eta_0(\eta_0 - \rho_0(x))).$$

- Let now  $\rho_0$  constant.
- For  $\mu \in D([0, t_0]; \mathcal{M}_+)$ ,  $\rho_t = d\mu_t/dx$  let

$$I_0(\mu) = \inf \left\{ \int_0^{t_0} \int_{\mathbb{T}} |g|^2 dx ds : g \in L^2_{t,x}, \underbrace{\partial_t \rho = \partial_{xx} \Phi(\rho) + \partial_x (\Phi^{\frac{1}{2}}(\rho) g)}_{\text{"skeleton equation"}} \right\}.$$

- E.g. independent particles:  $\Phi(\rho) = \rho$ .

Theorem (Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995)

For every measurable  $A \subseteq D([0, t_0], \mathcal{M}_+)$ ,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}[\mu^N \in \bar{A}] \leq - \inf_{\mu \in \bar{A}} I_0(\mu) \leq - \inf_{\mu \in A^\circ} I_0(\mu) \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}[\mu^N \in A^\circ].$$

Informally, take  $A = \{\rho dx\}$ ,

$$\mathbb{P}[\mu^N \approx \rho dx] \approx \exp\{-N I_0(\rho dx)\}.$$



## Link to stochastic PDE

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**Aim:** Continuum model reflecting not only mean behavior but also fluctuations

**Ansatz:** Langevin dynamics

$$\partial_t \rho = \partial_{xx} (\Phi(\rho)) + \text{"fluctuations"}.$$

Concretely

$$\partial_t \rho^N = \partial_{xx} (\Phi(\rho^N)) + \frac{1}{\sqrt{N}} \partial_x \left( \sqrt{\Phi(\rho^N)} dW_t \right). \quad (*)$$

**Informal justification:**

- 1 Physics: Fluctuation-dissipation relation, detailed balance, "fluctuating hydrodynamics"
- 2 Mean behavior / law of large numbers

$$\rho^N \rightarrow \bar{\rho} \quad \text{as } N \rightarrow \infty.$$

- 3 Central limit fluctuations:  $Y^N := \sqrt{N}(\rho^N - \bar{\rho})$ . Then, informally  $\mathcal{L}(Y^N) \rightarrow^* \mathcal{L}(Y)$  with

$$\partial_t Y = \partial_{xx} (\Phi'(\bar{\rho}) Y) + \partial_x \left( \sqrt{\Phi(\bar{\rho})} dW_t \right).$$

- 4 Large deviations: See below, large deviations of (\*) are the same as for  $\mu^N$ .

## Large deviations:

- Zero range process [Dirr, Stamatakis, Zimmer; 2016] as special case of macroscopic fluctuation theory (MFT) [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015],

$$\partial_t \rho = \partial_{xx}(\Phi(\rho)) + \frac{1}{\sqrt{N}} \partial_x \left( \sqrt{\Phi(\rho)} dW_t \right).$$

- Model case: Dean-Kawasaki, independent particles,  $\Phi(\rho) = \rho$ , i.e.

$$\partial_t \rho = \partial_{xx} \rho + \frac{1}{\sqrt{N}} \partial_x (\sqrt{\rho} dW_t).$$

*Informally same rate function as the zero range process:*

- Informally applying the contraction principle to the solution map

$$F : \frac{1}{\sqrt{N}} dW \mapsto \rho$$

yields as a rate function

$$I(\rho) = \inf \{ I_{dW}(g) : F(g) = \rho \}.$$

- Schilder's theorem for Brownian sheet suggests

$$I_{dW}(g) = \int_0^T \int_{\mathbb{T}} |g|^2 dx dt.$$

- Get

$$I(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx dt : \partial_t \rho = \partial_{xx}(\Phi(\rho)) + \partial_x(\sqrt{\Phi(\rho)}g) \right\}.$$

- Obstacle

$$\partial_t \rho = \partial_{xx}(\Phi(\rho)) + \frac{1}{\sqrt{N}} \partial_x(\sqrt{\Phi(\rho)} dW_t)$$

- ① not well-posed, supercritical  $\rightarrow$  no regularity structures
  - ② Renormalization? Does renormalization appear in rate function? E.g. compare  $\Phi_{2/3}^4$  [Hairer, Weber; 2014].
- Ansatz: joint limit “small noise, ultraviolet cutoff”

$$\partial_t \rho^{N,K} = \partial_{xx}(\Phi(\rho^{N,K})) + \frac{1}{\sqrt{N}} \partial_x(\sqrt{\Phi(\rho^{N,K})} \circ dW_t^K)$$

where  $W^K = \sum_{k=1}^K e_k \beta^k$  is a spectral (smooth) approximation of  $W = \sum_{k=1}^{\infty} e_k \beta^k$ .

- Gives the correct rate function for  $\frac{1}{N} \ll \frac{1}{K}$ .

**Note:** This is a particular case in which the link between *Macroscopic fluctuation theory* [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and *fluctuating hydrodynamics* [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.

## Two ways to the LDP

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- In the following concentrate on the case

$$\Phi(\rho) = \rho^m, \quad m \geq 1.$$

- We consider stochastic PDE of the type

$$\partial_t \rho^{N,K} = \Delta \left( (\rho^{N,K})^m \right) + \frac{1}{\sqrt{N}} \operatorname{div} \left( (\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K \right), \quad (*)$$

on  $\mathbb{T}^d \times (0, \infty)$ , where  $W^K = \sum_{k=1}^K e_k \beta^k$ .

- Pathwise well-posedness of (\*): [Lions, Souganidis; 1998ff], [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017], [Dareiotis, G.; 2019].

## Two ways to the LDP:

- $\Gamma$ -convergence of the rate functional:  $N \uparrow \infty$  yields LDP for (\*) with rate function

$$I^K(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx dt : \partial_t \rho = \partial_{xx} (\Phi(\rho)) + \partial_x \left( \sqrt{\Phi(\rho)} P^K g \right) \right\}.$$

Then consider  $K \uparrow \infty$ .

- Joint scaling: Weak convergence approach to LDP ( $\frac{1}{N} \ll \frac{1}{K}$ ).

- Both approaches crucially depend on understanding the skeleton PDE.
- The skeleton equation

$$\begin{aligned}\partial_t \rho &= \Delta \rho^m + \operatorname{div} \left( \rho^{\frac{m}{2}} g(t, x) \right) \\ \rho(0, x) &= \rho_0(x),\end{aligned}\tag{*}$$

with  $g \in L^2_{t,x}$ ?

- This leads to the key problem

## Problem

- 1 Existence and uniqueness of solutions to (\*).
- 2 Stability of solutions: Let  $g^n \rightarrow g$  in  $L^2_{t,x}$  with corresponding solutions  $\rho^n, \rho$ . Then

$$\rho^n \rightarrow \rho$$

in  $L^\infty_t L^1_x$ .

- Difficulty: Stable a-priori bound?  $L^p$  framework does not work.
- Do we expect non-concentration of mass / well-posedness?

## Well-posedness of the skeleton equation

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## Scaling and criticality of the skeleton equation

- We consider

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$

with  $g \in L^q(\mathbb{R}_{+,t}; L^p(\mathbb{R}_x^d; \mathbb{R}_x^d))$  and  $\rho_0 \in L^r(\mathbb{R}_x^d)$ .

- Via rescaling (“zooming in”):
  - $p = q = 2$  is critical.
  - $r = 1$  is critical,  $r > 1$  is supercritical.

## Apriori-bounds and energy space

- Consider

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \quad (*)$$

with  $g \in L^2(\mathbb{R}_{+,t}; L^2(\mathbb{R}_x^d; \mathbb{R}_x^d))$ .

- $L^1$  estimate only gives

$$\int_{\mathbb{T}^d} \rho(t, x) dx = \int_{\mathbb{T}^d} \rho_0(x) dx.$$

- Use entropy-entropy dissipation: Evolution of entropy given by  $\int_{\mathbb{T}^d} \log(\rho) \rho$ . Informally gives

$$\int_{\mathbb{T}^d} \log(\rho) \rho dx \Big|_0^t + \int_0^t \int_{\mathbb{T}^d} (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_{\mathbb{T}^d} g^2.$$

- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting (\*) as

$$\partial_t \rho = 2 \operatorname{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d$$

- Conclusion: Have to prove uniqueness within this class of solutions.

**Ansatz for uniqueness:** Show that every weak solution is a renormalized entropy solution (extending the concepts of DiPerna-Lions, Ambrosio to nonlinear PDE).

Let  $\rho$  be a weak solution to

$$\partial_t \rho = 2 \operatorname{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d.$$

Let

$$\chi(t, x, \xi) = 1_{0 < \xi < \rho(x, t)} - 1_{\rho(x, t) < \xi < 0}.$$

Then, informally,

$$\partial_t \chi = m \xi^{m-1} \Delta_x \chi - g(x, t) (\partial_\xi \xi^{\frac{m}{2}}) \nabla_x \chi + (\nabla_x g(x, t)) \xi^{\frac{m}{2}} \partial_\xi \chi + \partial_\xi p$$

with  $p$  parabolic defect measure

$$p = \delta(\xi - \rho) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2.$$

Or on “fluid level”, informally, for  $F$  convex,

$$\partial_t F(\rho) - \Delta_x F^1(\rho) + g(x, t) \nabla_x F^2(\rho) + (\nabla_x g)(x, t) F^3(\rho) = - \int_\xi F''(\xi) p \leq 0$$

where  $(F^m)' = mf(\xi)\xi^{m-1}$ ,  $(F^2)' = f(\xi)(\partial_\xi \xi^{\frac{m}{2}})$ ,  $F^3 = f(\xi)\xi^{\frac{m}{2}}$ .

- How to make that rigorous? Take convolution

$$\rho^\varepsilon = \varphi^\varepsilon *_x \rho.$$

- Commutator errors,

$$\begin{aligned} \partial_t \rho^\varepsilon &= \varphi^\varepsilon * \partial_t \rho = \varphi^\varepsilon * (\Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g)) \\ &= \Delta(\varphi^\varepsilon * \rho^m) + \operatorname{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) \\ &= \Delta(\rho^\varepsilon)^m + \operatorname{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\ &\quad + \Delta(\varphi^\varepsilon * \rho^m) - \Delta(\rho^\varepsilon)^m \\ &\quad + \operatorname{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g) - \operatorname{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\ &\quad + \operatorname{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) - \operatorname{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g). \end{aligned}$$

- Note: Additional commutator errors by commuting convolution and nonlinearities!
- Commutator estimate using non-standard (optimal) regularity  $\rho^{\frac{m}{2}} \in L_t^2 \dot{H}_x^1$
- Additional renormalization step to compensate low time integrability  $\rho^{\frac{m}{2}} g \in L_t^1 L_x^1$ .

## Theorem

A function  $\rho \in L_t^\infty L_x^1$  is a weak solution to

$$\partial_t \rho = 2 \operatorname{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \operatorname{div}(\rho^{\frac{m}{2}} g)$$

if and only if  $\rho$  is a renormalized entropy solution.

### Uniqueness for renormalized entropy solutions (variable doubling)

- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x, \xi, t) = \delta(\xi - \rho(x, t)) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2$$

does not satisfy

$$\lim_{|\xi| \rightarrow \infty} \int_{t,x} q(x, \xi, t) dx dt = 0.$$

- Established arguments [Chen, Perthame; 2003] not applicable.

### Theorem (The skeleton equation)

Let  $g \in L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$ ,  $\rho_0 \in L^1(\mathbb{T}^d)$  non-negative and  $\int \rho_0 \log(\rho_0) dx < \infty$ ,  $m \in [1, \infty)$ .

1 There is a unique weak solution

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \quad (*)$$

For two weak solutions  $\rho^1, \rho^2 \in L^\infty([0, T]; L^1(\mathbb{T}^d))$  we have

$$\|\rho^1 - \rho^2\|_{L^\infty([0, T]; L^1(\mathbb{T}^d))} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(\mathbb{T}^d)}.$$

2 Let  $\{g_n\}_{n \in \mathbb{N}} \subseteq L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$  with

$$\lim_{n \rightarrow \infty} g_n = g \text{ weakly in } L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$$

and let  $\rho_n \in L^1([0, T]; L^1(\mathbb{T}^d))$  be the corresponding solutions with control  $g_n$ . Then,

$$\lim_{n \rightarrow \infty} \rho_n = \rho \text{ strongly in } L^1([0, T]; L^1(\mathbb{T}^d))$$

where  $\rho \in L^1([0, T]; L^1(\mathbb{T}^d))$  is the solution with control  $g$ .

Consider

$$d\rho^N = \Delta(\rho^N)^m dt + \frac{1}{\sqrt{N}} \operatorname{div} \left( \Phi_{n(N)}^{\frac{1}{2}}(\rho^N) \circ dW^{K(N)}(t) \right).$$

Theorem (Large deviation principle)

Let  $K(N), n(N) \rightarrow \infty$  with  $\frac{K(N)^3}{N} \rightarrow 0$  for  $N \rightarrow \infty$ . For  $\rho_0 \in L^{m+1}(\mathbb{T}^d)$  and  $\rho \in L^\infty([0, T]; L^1(\mathbb{T}^d))$  let

$$I_{\rho_0}(\rho) := \inf \left\{ \frac{1}{2} \int_0^T \|g(s)\|_{L_x^2}^2 ds : g \in L_{t,x}^2, \partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \right\}.$$

Then,

- 1 For all  $\rho_0 \in L^{m+1}(\mathbb{T}^d)$ ,  $\rho \mapsto I_{\rho_0}(\rho)$  is a good rate function on  $L^\infty([0, T]; L^1(\mathbb{T}^d))$ .
- 2 The family  $\{\rho^N\}$  satisfies the large deviation principle on  $L^\infty([0, T]; L^1(\mathbb{T}^d))$  with rate function  $I_{\rho_0}$ , uniformly on compact subsets of  $L^{m+1}(\mathbb{T}^d)$ .



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