Properties of Effective Hamiltonian

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Based on joint works and ongoing collaborations with C. Cheng, W. Cheng, S. Luo, H. Mitake, J. Qin, H.V.Tran, J. Xin

Homogenization theory of Hamilton-Jacobi equation

Assume $H(p,x) \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is uniformly coercive in the p variable and periodic in the x variable.
For each $\epsilon > 0$, let $u^\epsilon \in C(\mathbb{R}^n \times [0,\infty))$ be the viscosity solution to the following Hamilton-Jacobi equation

$$\begin{cases} u_t^\epsilon + H\left(Du^\epsilon, \frac{x}{\epsilon}\right) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\epsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1}$$

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It was known (Lions-Papanicolaou-Varadhan, 1987), that u^ϵ, as $\epsilon \to 0$, converges locally uniformly to u, the solution of the effective equation,

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$\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is called "effective Hamiltonian" or "α function", a nonlinear averaging of the original H.

Cell problem: for any $p \in \mathbb{R}^n$, there exists a **UNIQUE** number $\overline{H}(p)$ such that

$$H(p + Dv, x) = \overline{H}(p) \quad \text{in } \mathbb{T}^n.$$

has periodic viscosity solutions v ("corrector").

$$u^{\epsilon}(x,t) \approx u(x,t) + \epsilon v(\frac{x}{\epsilon}, Du).$$

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Let us focus on the mechanical Hamiltonian

$$\boxed{H(p,x) = \frac{1}{2}|p|^2 + V(x).}$$

The ultimate question (realization problem):

"Characterize when a convex functions $F : \mathbb{R}^n \to \mathbb{R}$ can be the effective Hamiltonian associated with a potential function V (smooth or continuous)".

Macroscopic Perspective: An Inverse Problem

Consider $H(p,x) = \frac{1}{2}|p|^2 + V(x)$ and the corresponding $\overline{H}$

$$\frac{1}{2}|p + Dw|^2 + V(x) = \overline{H}(p) \quad \text{in } \mathbb{R}^n.$$

Q: Suppose that V_1 and V_2 are two smooth periodic functions. If the associated effective Hamiltonians are the same, i.e.,

$$\overline{H}_1(p) = \overline{H}_2(p) \qquad \textbf{for all } p \in \mathbb{R}^n,$$

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A basic invariant tranformation:

$$V_2(x) = V_1\left(\frac{x}{\lambda} + c\right) \quad \Rightarrow \quad \overline{H}_1(p) = \overline{H}_2(p).$$

This is the ONLY known $\overline{H}$ invariant transformation for non-separable V when $n \geq 2$.

A Rigidity Question

$$\frac{1}{2}|p + Dw|^2 + V(x) = \overline{H}(p) \quad \text{in } \mathbb{R}^n.$$

Q: For $n \geq 2$, for "typical " V, if

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• Homogeneous Case: True if $\overline{H}_1 = \overline{H}_2 = \frac{1}{2}|p|^2$,

Theorem (Luo, Tran, Y., 2015, Homogeneous Case)

$$\overline{H} \equiv \frac{1}{2}|p|^2 \quad \Rightarrow \quad V \equiv 0.$$

Non-homogeneou Cases

Theorem (Tran, Y. 2017, Nonhomogeneous Case)

If $n = 2$ and each of V_1, V_2 contains exactly 3 mutually non-parallel Fourier modes, then

$$\overline{H}_1 \equiv \overline{H}_2 \quad \Longleftrightarrow \quad V_1(x) = V_2\left(\frac{x}{c} + x_0\right).$$

for some $c \in \mathbb{R} \setminus \{0\}$ and $x_0 \in \mathbb{R}^2$.

For example,

$$V(x, y) = \cos x + \cos y + \cos(x + y).$$

3 mutually non-parallel Fourier modes: $\pm e_1$, $\pm e_2$, $\pm(e_1 + e_2)$.

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Main Method: "**Asymptotic Expansion**" $\Rightarrow$ Some combinatorial issues+ Delicate/tedious analysis in plane geometry/linear algebra

Asymptotic Expansion

Let $p = \lambda Q$ and

$$\frac{1}{2}|\lambda Q + Dv|^2 + V(x) = \overline{H}(\lambda Q).$$

Let $\lambda = \frac{1}{\sqrt{\epsilon}}$. Then

$$\frac{1}{2}|Q + Dv_\epsilon|^2 + \epsilon V(x) = \epsilon \overline{H}\left(\frac{Q}{\sqrt{\epsilon}}\right) = \overline{H}_\epsilon(Q).$$

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• If Q satisfies a Diophantine condition: there exists α, $C > 0$ such that

$$|Q \cdot K| \geq \frac{C}{|K|^\alpha} \quad \text{for all } K \in \mathbb{Z}^n \backslash \{0\}.$$

we can get Taylor expansions:

$$(Approximate\ corrector) \quad v_\epsilon = \epsilon v_1 + \epsilon^2 v_2 + \ldots . \epsilon^m v_m + O(\epsilon^{m+1}).$$

and

$$\overline{H}_\epsilon(Q) = \frac{1}{2}|Q|^2 + \epsilon a_1 + \epsilon^2 a_2 + \ldots \epsilon^m a_m + O(\epsilon^m).$$

Comparing singular parts in Coefficients

- The first coefficient $a_1 = \int_{\mathbb{T}^n} V\,dx$.
- If $a_2(V_1) = a_2(V_2)$, then for any Q satisfying the Diophantine condition

$$\sum_{0\neq k\in\mathbb{Z}^n} \frac{|\lambda_{k1}|^2|k|^2}{|Q\cdot k|^2} = \sum_{0\neq k\in\mathbb{Z}^n} \frac{|\lambda_{k2}|^2|k|^2}{|Q\cdot k|^2}.$$

Here λ_{ki} are Fourier coefficients of V_i.

- Formulas for a_k become more and more complicated when k gets large and are very hard to find a reasonable way to extract useful information.

- Our result is based on comparing a_1, a_2, a_3 and a_4 together with

$$\boxed{\min_{\mathbb{R}^1} \overline{H} = \max_{\mathbb{R}^2} V.}$$

Microscopic Perspective: Find more Properties of $\overline{H}$

Two directions:

(1) Identify analytic properties of $\overline{H}$ as much as we can from mainly mathematical point of view; This part is also related to the optimal convergence rate of $|u^\epsilon - u|$ as $\epsilon \to 0$ (Mitake, Tran and Y., 2019)

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(2) Determine the dependence of $\overline{H}$ on **physical parameters** in the original $H(p,x)$ motivated by applications in practical science, e.g joint project with Jack Xin on G-equation where the effective Hamiltonian is a model of the **turbulent flame speed.** Below is a basic case.

$$\boxed{|p + DG| + AW(x) \cdot (p + DG) = \overline{H}(p, A).}$$

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This talk will focus on (1) and smooth potential functions V.

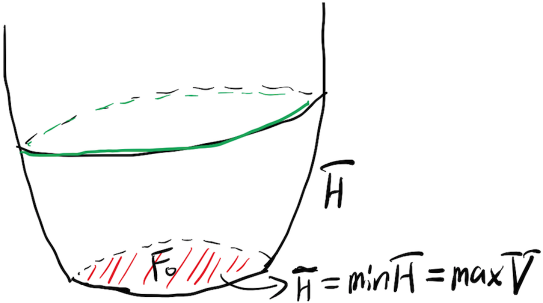
$$\frac{1}{2}|p + Dv|^2 + V(x) = \overline{H}(p).$$

Rough Properties of $\overline{H}$

Below are some well-known properties of $\overline{H}$ in all dimeinsions: $\overline{H}$ is convex, even and grows quadratically:

$$\frac{1}{2}|p|^2 + \min V \leq \overline{H}(p) \leq \frac{1}{2}|p|^2 + \max V.$$

$\star$ For quite general V, the minimum level set F_0 is a *n*-**dimension** convex set.



Some detailed Properties

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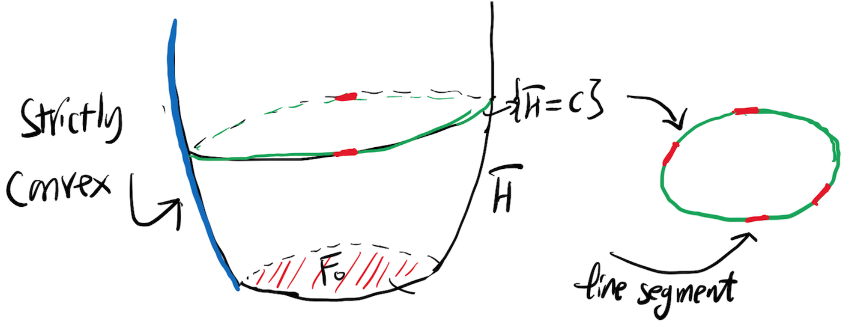
(2) For $n = 2$ and $c > \min \overline{H}$, the level set $\{\overline{H} = c\}$ is C^1 (Dias Carneiro, 1991) and, more interestingly, contains line segments for non-constant V (Bangert 1994)



Sketch of Bangert's Proof

By **Aubry-Mather theory** in 2d, for $n = 2$, if the level set $\{\overline{H} = c\}$ is striclty convex, then every geodesics associated with the periodic metric

$$\boxed{g = \sqrt{2(c - V)}(dx_1^2 + dx_2^2)}$$

is a minimizing geodesics. Then **Hopf's a classical result in 1947** implies the metric g is flat, equivalently, V is constant.

minimizing geodesics = global characteristics of v:

$\frac{1}{2}|p + Dv|^2 + V(x) = c$

Foliation of $\mathbb{R}^2$

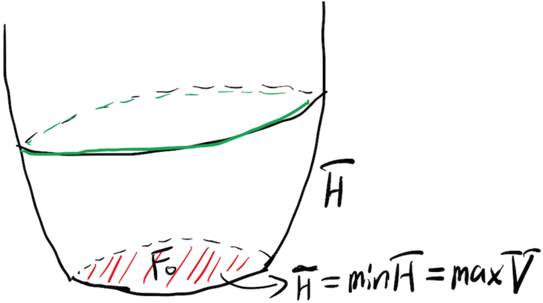
A Generic Property of $\overline{H}$ in 2d: an "Assembly" of 1d Functions

Theorem (In preparation)

For generic V, there exists a dense open set $O \subset \mathbb{R}^2$ which is the union of countably many open subsets $O = \cup_{k=1}^{\infty} O_k$ such that $\overline{H}|_{O_k}$ is a 1d function, i.e., for each $k \in \mathbb{N}$, there exists $q_k \in \mathbb{R}^2$ and $f_k : \mathbb{R} \to \mathbb{R}$ such that

$$\overline{H}(p) = f_k(q_k \cdot p) \quad \text{for } p \in O_k.$$

Geometry Properties of the Flat Part



Assume that V has finiely many non-degenerate maximum points.
Existence of flat part follows easily from the inf-max formula

$$\boxed{\overline{H}(p) = \inf_{\phi \in C^1(\mathbb{T}^n)} \max_{x \in \mathbb{R}^n} \left(\frac{1}{2}|p + D\phi|^2 + V(x) \right).}$$

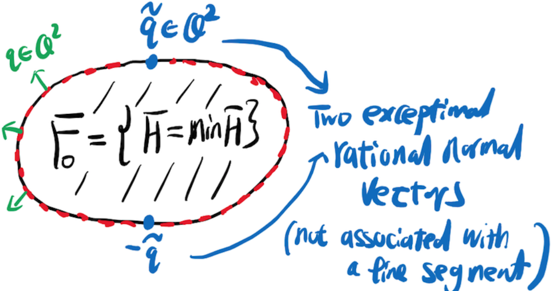
Goal: Understand detailed geometric property of ∂F_0.

Geometry Properties of the Flat Part: Devil's Stair

$$F_0 = \{\overline{H} = \min \overline{H}\}.$$

Theorem (In Preparation, Optimal)

Line segments are dense along ∂F_0. More precisely, there exists at most two rational normal vectors which are not associated with a line segment.



Foliation and Exceptional Rational Normal Vectors

Consider $V(x_1, x_2) = -\frac{x_1^2}{2} - 4x_2^2$.

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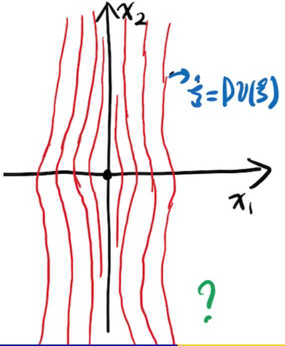
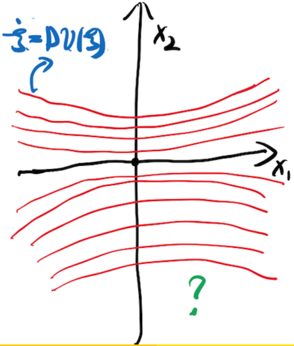
Consider $V(x_1, x_2) = -\frac{x_1^2}{2} - 4x_2^2$.
Question: Does there exist a C^1 solution v to

$$\frac{1}{2}|Dv|^2 + V(x) = 0$$

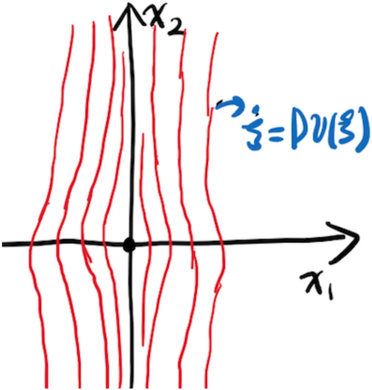
such that its characteristics $\dot{\xi} = Dv(\xi)$ foliate $\mathbb{R}^2$ horizontally or vertically?





Construction of Vertical Foliation

$$\frac{1}{2}|Dv|^2 - \frac{x_1^2}{2} - 4x_2^2 = 0.$$



$$\boxed{v(x_1, x_2) = x_2\sqrt{x_1^2 + x_2^2}.}$$

Horizontal Foliation is Impossible

A characteristics must be a minimizing curve of the action

$$L = \inf \left(\int \frac{1}{2}|\dot{\gamma}|^2 - V(\gamma)\, ds \right).$$

When two points P and Q are very close to the x_1 axis, delicate analysis shows that the orbit passing through the origin has smaller action.