Coarsening Rates for Nonlocal Cahn-Hilliard Equation

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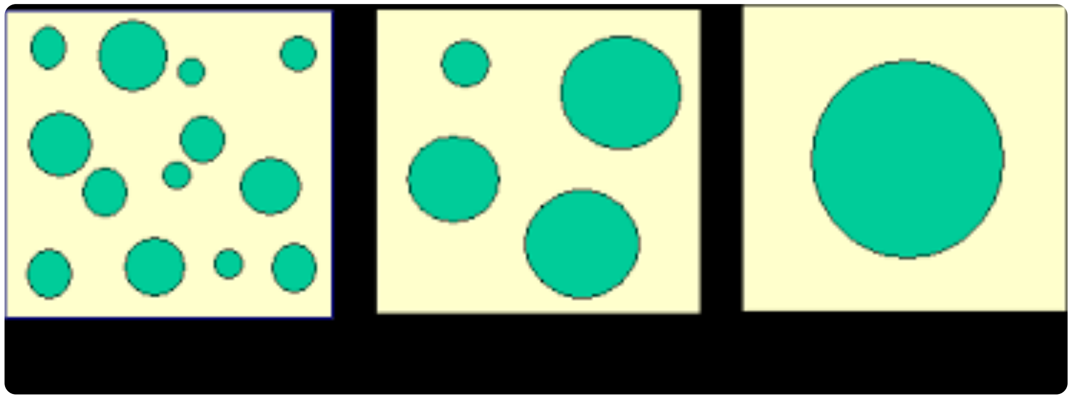
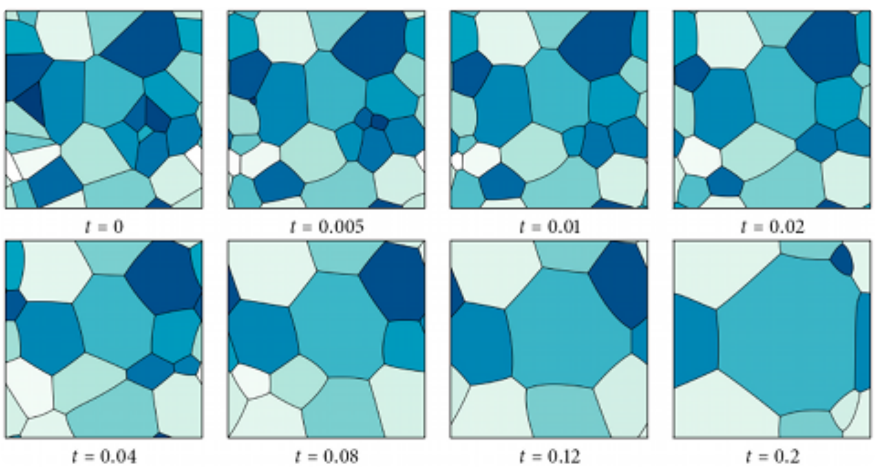
Purdue University

(Joint work with Yuan Gao and Tao Luo)

Evolving Length Scales





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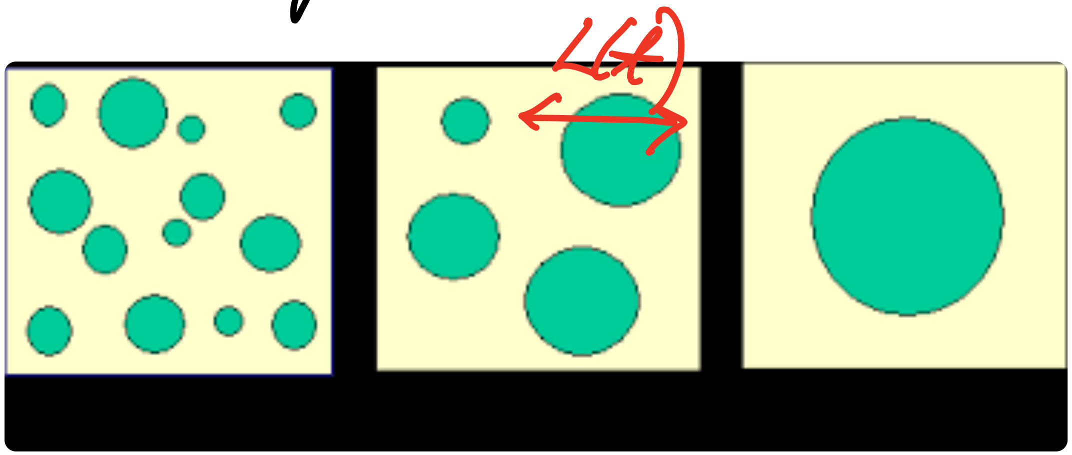
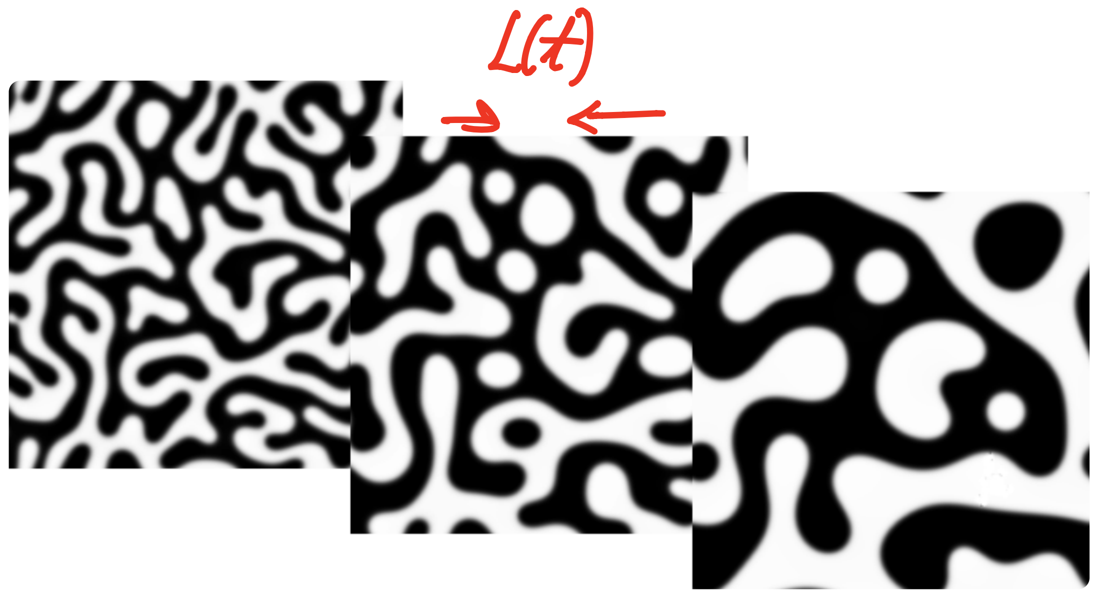
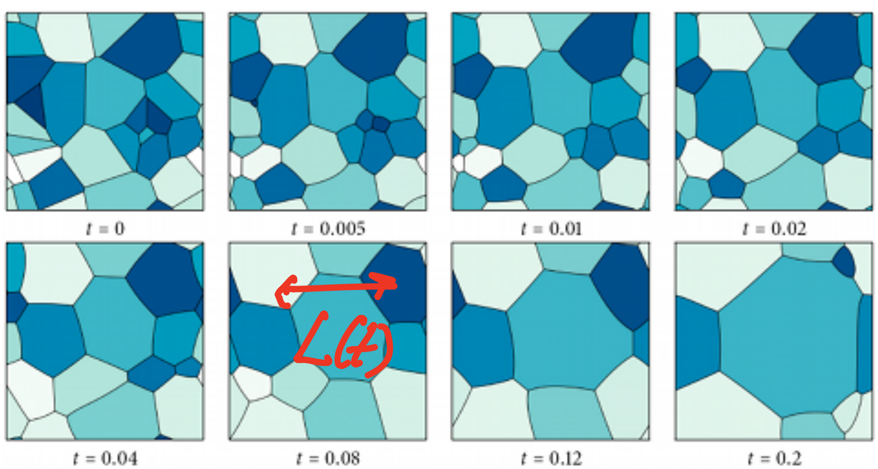


$$L(t) \sim t^{\alpha}$$





A Standard Model for Phase Transition

– Allen Cahn Functional

$$\boxed{E_\varepsilon(u) = \int \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,dx}$$

A Standard Model for Phase Transition

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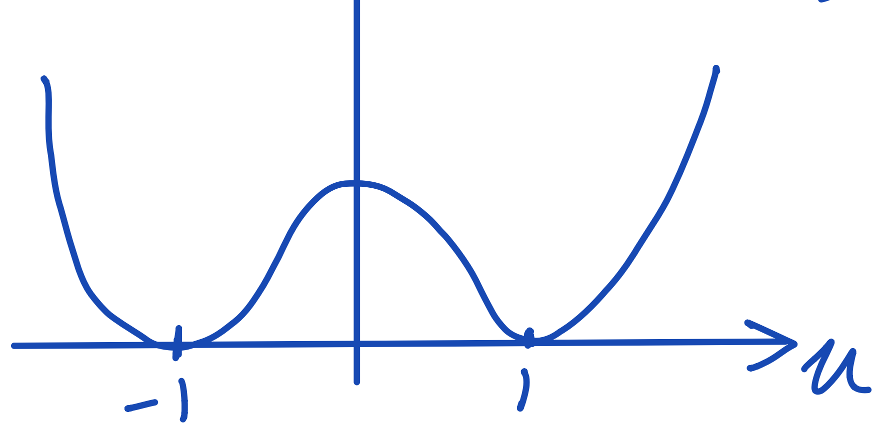
$$\boxed{E_\varepsilon(u) = \int \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} W(u)\, dx}$$

$0 < \varepsilon \ll 1$

$W(u) = (1-u^2)^2$



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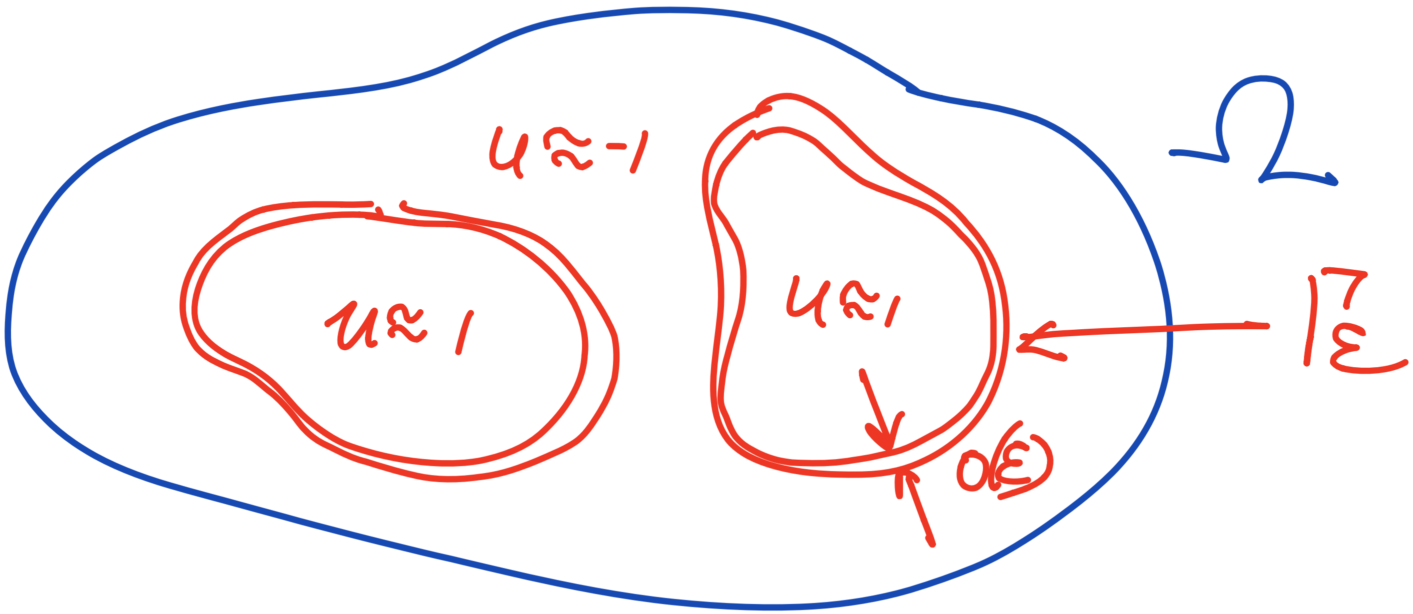

A Standard Model for Phase Transition

– Allen Cahn Functional

$$\boxed{E_\varepsilon(u) = \int \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} W(u)\, dx}$$

$$\Big\downarrow \; \varepsilon \to \quad \Gamma\text{-converges}$$

$$\boxed{E_*(\Gamma) = c\,\mathrm{Per}(\Gamma)}$$

A Standard Model for Phase Transition

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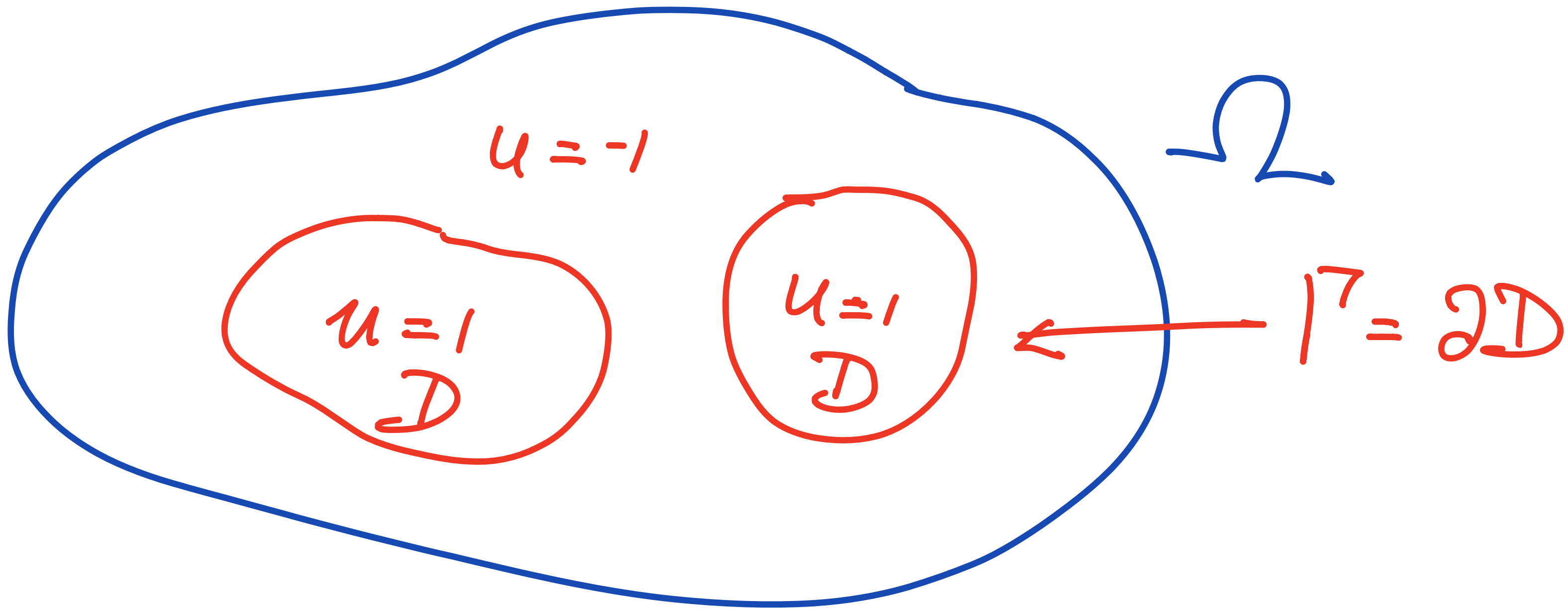


A Standard Model for Phase Transition — Allen Cahn Functional

Dynamics w.r.t. E_ε :

(1) L^2-gradient flow: $\left(u_t = -\nabla_{L^2} E_\varepsilon(u)\right)$

$$\boxed{u_t = \varepsilon^2 \Delta u - W'(u)}$$

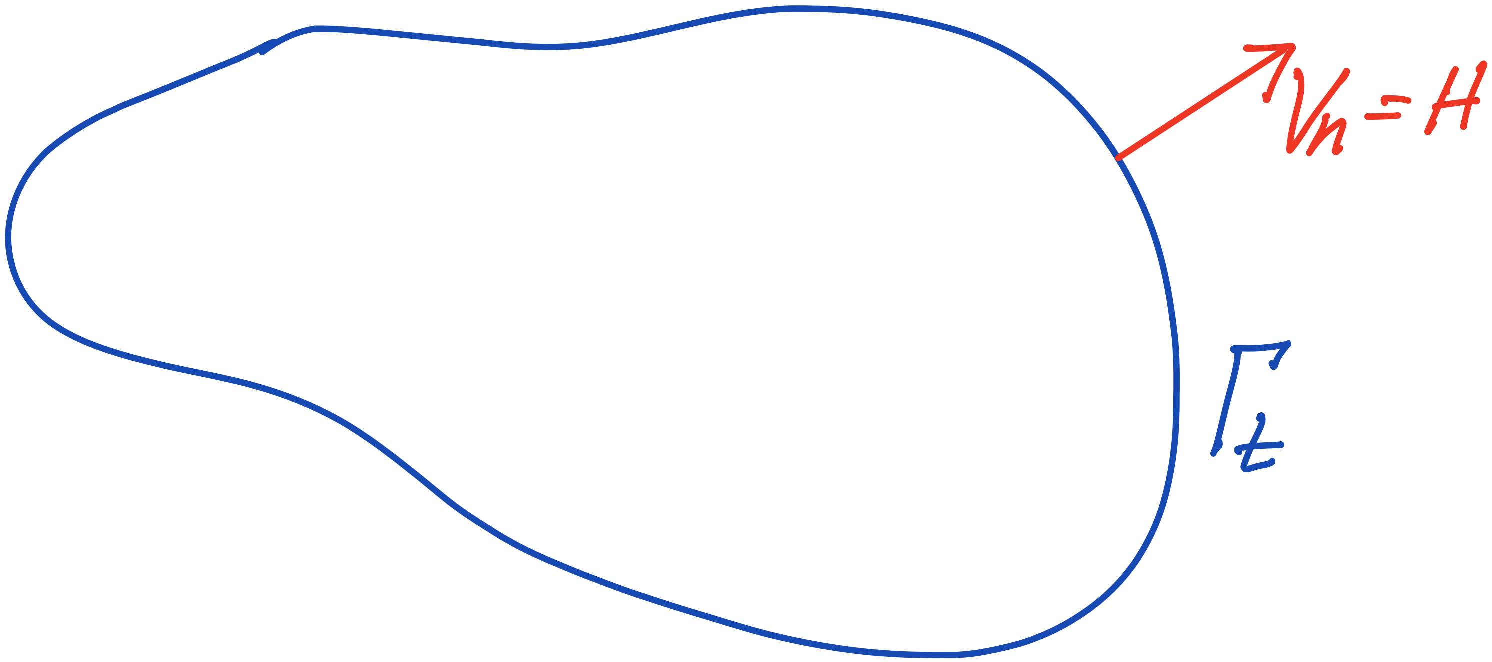
$\varepsilon \to 0$

$$\boxed{V_n = H \quad (\text{motion by mean curvature})}$$

A Standard Model for Phase Transition

— Allen Cahn Functional

$$\boxed{V_n = H \quad \text{(motion by mean curvature)}}$$



A Standard Model for Phase Transition

– Allen Cahn Functional

Dynamics w.r.t. E_ε :

(2) H^{-1}– gradient flow: $\left(u_t = -\nabla_{H^{-1}} E_\varepsilon(u)\right)$

$$\boxed{u_t = \Delta\left(-\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\right)}$$ (Cahn-Hilliard)

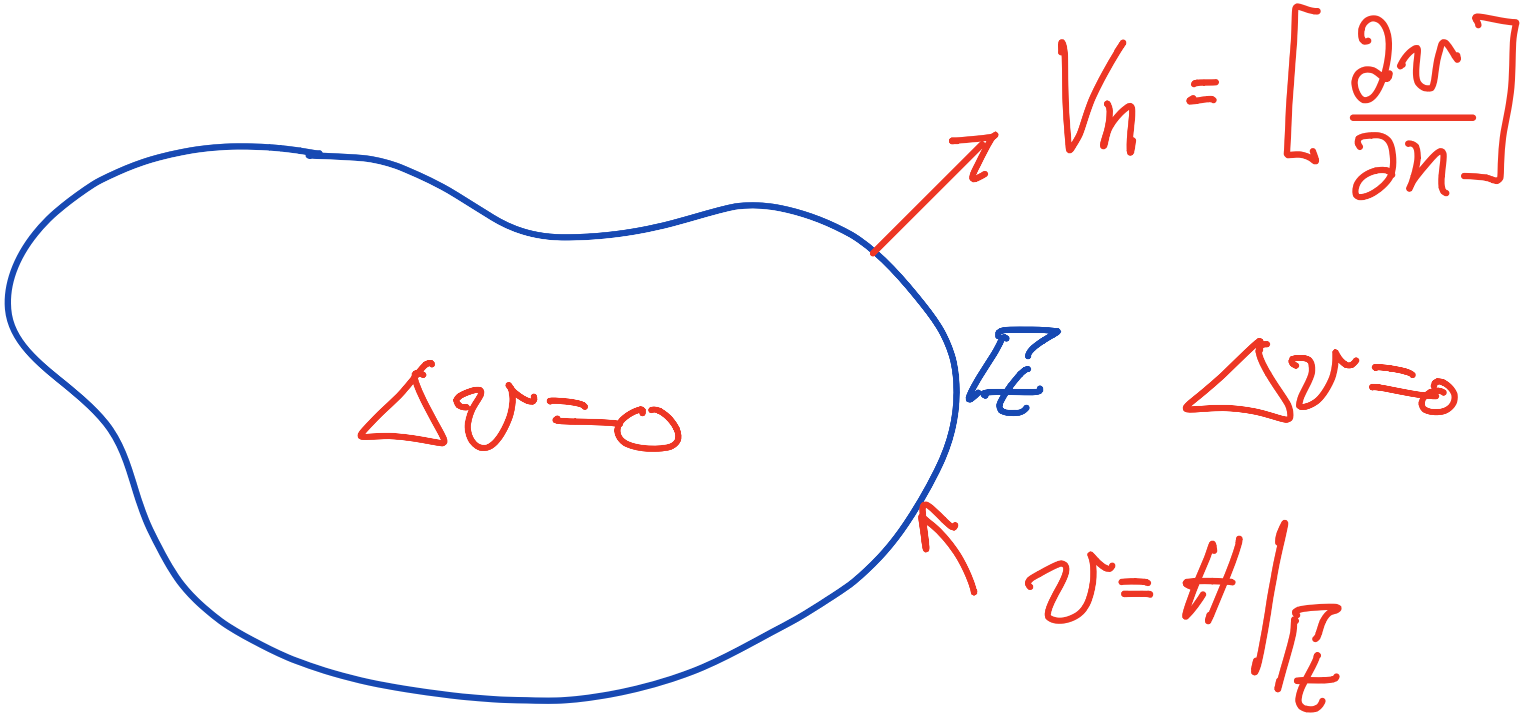
$\varepsilon \to 0$

$$\boxed{\text{Mullins-Sekerka Model}}$$

A Standard Model for Phase Transition

– Allen Cahn Functional

$\boxed{\text{Mullins-Sekerka Model}}$



A Standard Model for Phase Transition

– Allen Cahn Functional

Dynamics w.r.t. E_ε:

(3) H^{-1}-gradient flow: (with degenerate mobility)

$$\boxed{u_t = \operatorname{div}\left[M(u)\nabla\left(-\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\right)\right]}$$

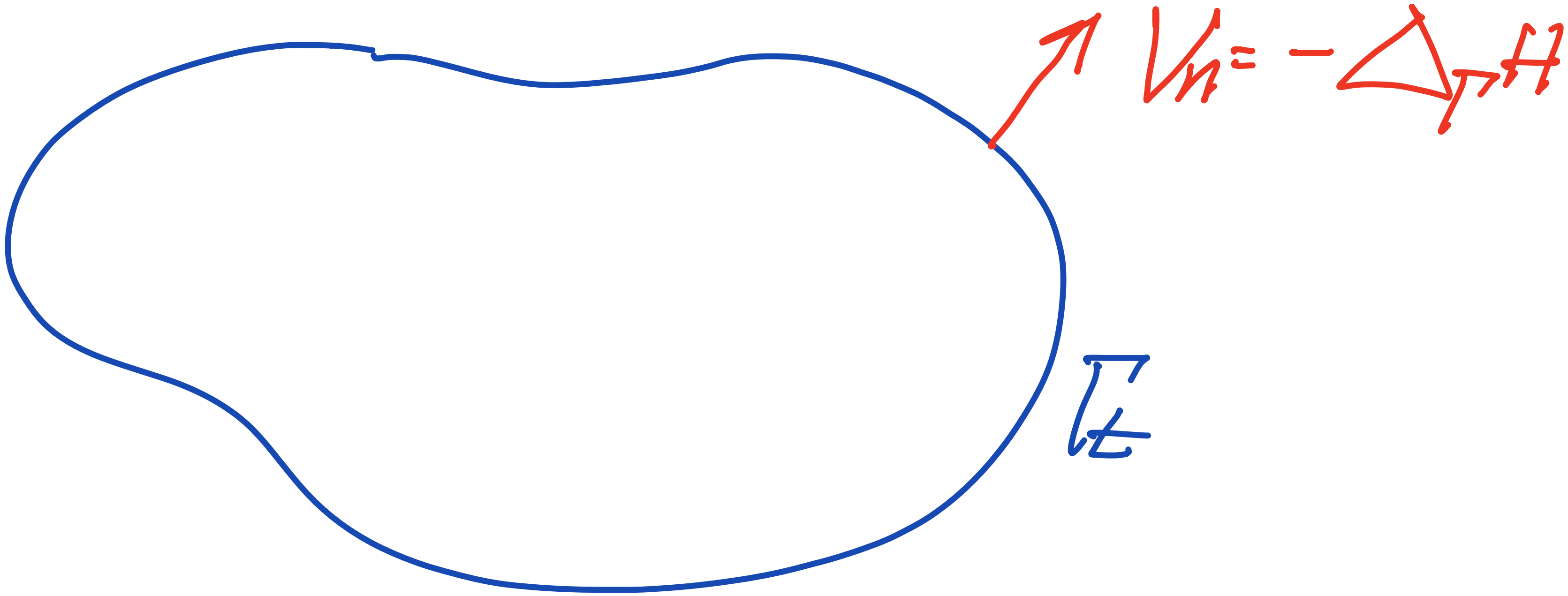
$\varepsilon \to 0$

$$\boxed{V_n = -\Delta_\Gamma H \quad \text{(Surface Diffusion)}}$$

A Standard Model for Phase Transition

– Allen Cahn Functional

$$\boxed{V_n = -\Delta_\Gamma H \text{ (Surface Diffusion)}}$$



Coarsening Rates for (2) and (3)

Typical Length Scale $L(t)$

(2) Cahn-Hilliard Equation/Mullins-Sekerka

$$\boxed{L(t) \sim t^{1/3}}$$

(3) Degenerate Cahn-Hilliard/Surface Diffusion

$$\boxed{L(t) \sim t^{1/4}}$$

(The exponents are sharp due to self similar soln.)

Coarsening Rates for (2) and (3)

Typical Length Scale $L(t)$

(2) Cahn-Hilliard Equation / Mullins-Sekerka

$$\boxed{L(t) \sim t^{1/3}}$$

(3) Degenerate Cahn-Hilliard / Surface Diffusion

$$\boxed{L(t) \sim t^{1/4}}$$

(Kohn-Otto: a weak version of $E(t) \gtrsim t^{-1/3}, t^{-1/4}$)
2002

Non-Local Allen-Cahn Functional

$$E_s(u)$$

$$= C_{n,s} \iint \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dy\, dx + \int W(u)\, dx$$

Non-Local Allen-Cahn Functional

$$\boxed{\begin{aligned} &E_s(u) \\ &= C_{n,s} \int_{T^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dy\, dx + \int_{T^n} W(u)\, dx \end{aligned}}$$

$$\underbrace{C_{n,s} \int_{T^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dy\, dx}_{\|u\|_{H^s}^2}$$

$0 < s < 1$,

$$(-\Delta)^s u = C_{n,s}\, (\text{p.v.}) \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

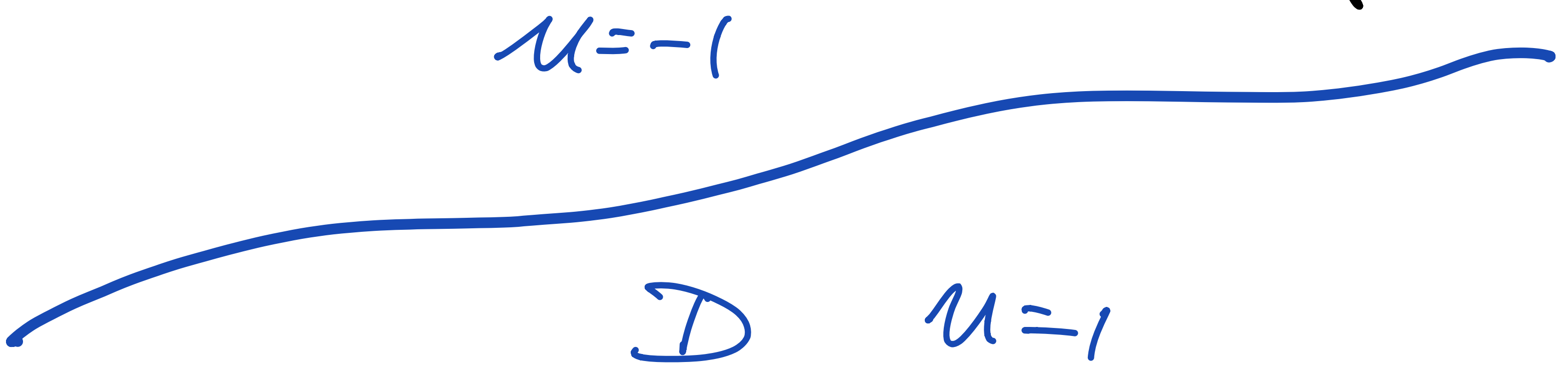
$\left(\frac{1}{2} < s < 1\right)$

Nonlocal Minimal Surface

(Caffarelli–Roquejoffre–Savin) 2010

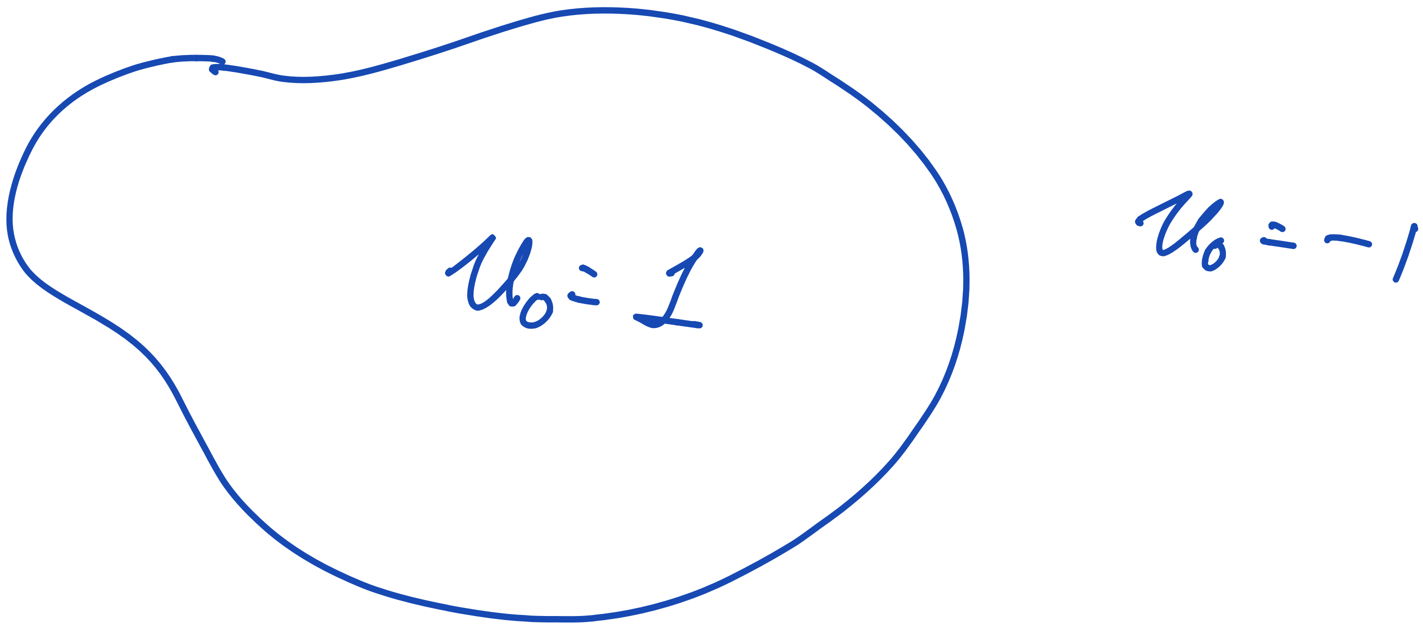
$$E(D) = \|\chi_D\|_{H^s}^2 = \iint \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dx\,dy$$

$(0 < s < \frac{1}{2})$



Nonlocal Thresholding Scheme

(Caffarelli–Souganidis 2010)



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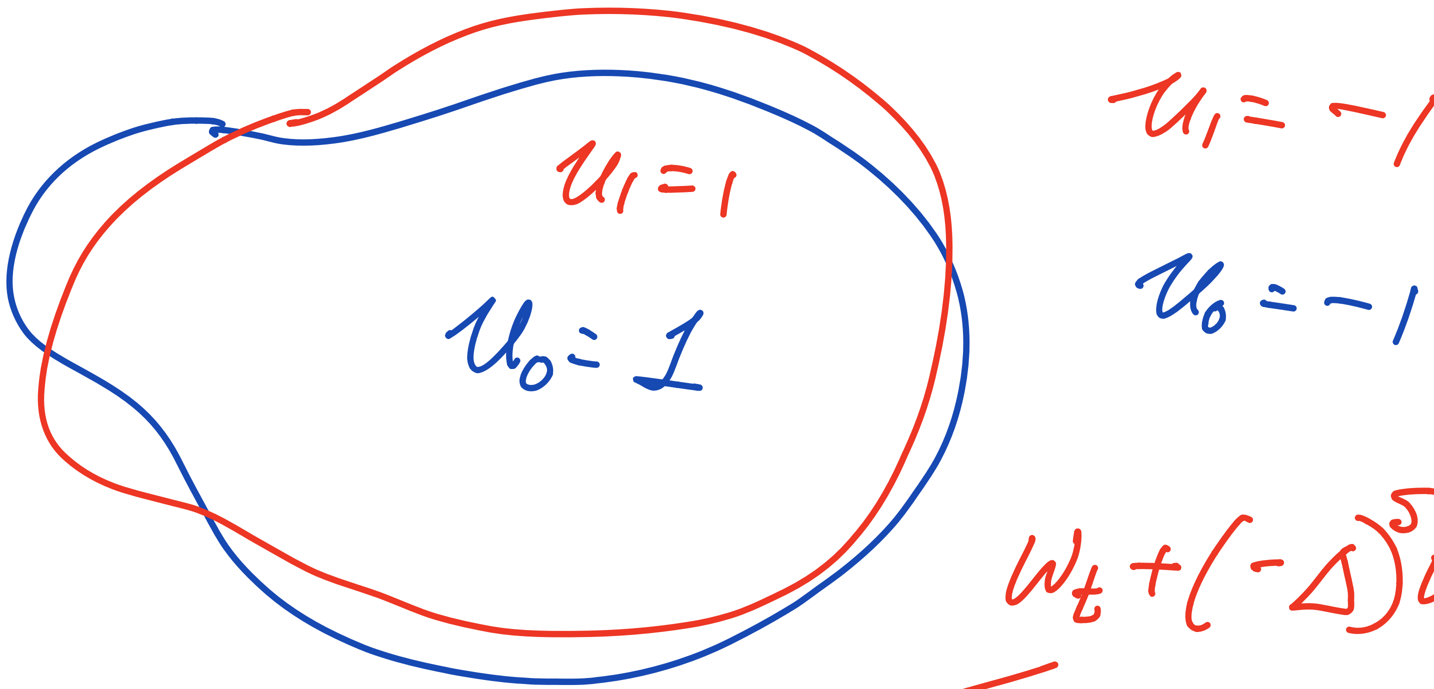
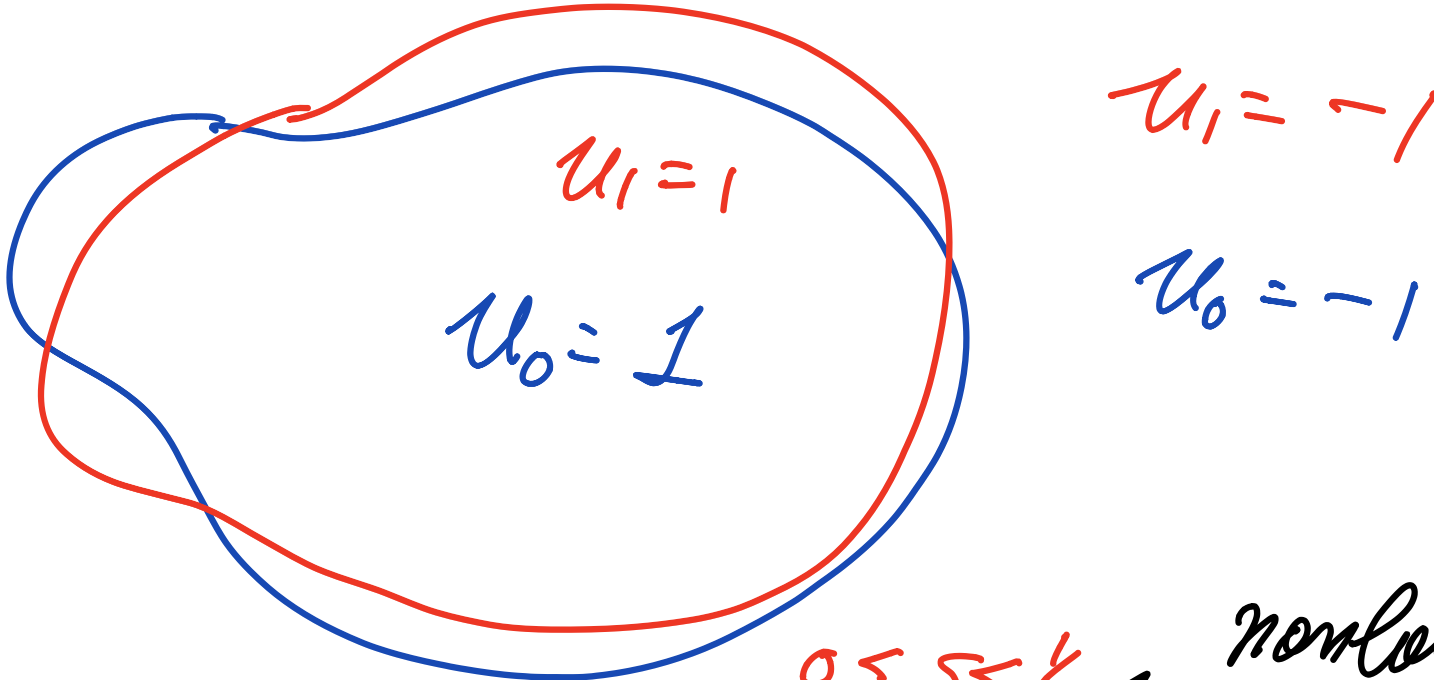


$$w_t + (-\Delta)^s w = 0$$

$$u_1 = \mathrm{sgn}\left(u_0 * p_s,\ \underline{\sigma_s(\Delta t)} \right)$$

Nonlocal Thresholding Scheme

(Caffarelli–Souganidis 2010)



$\{u_0 \to u_1 \to u_2 \to \cdots\cdots\}$ $\Delta t \to 0$

$0 < s < \frac{1}{2}$ → nonlocal MMC

$\frac{1}{2} \leq s < 1$ → local MMC

Γ-Convergence of E_s (Savin-Valdinoci 2012)

Consider:

$$E_{s,\varepsilon}(u) = \begin{cases} \|u\|_{H^s}^2 + \dfrac{1}{\varepsilon^{2s}} \displaystyle\int W(u) & \left(0 < s < \tfrac{1}{2}\right) \\ \dfrac{1}{|\log \varepsilon|} \|u\|_{H^{1/2}}^2 + \dfrac{1}{\varepsilon |\log \varepsilon|} \displaystyle\int W(u) & \left(s = \tfrac{1}{2}\right) \\ \varepsilon^{2s-1} \|u\|_{H^s}^2 + \dfrac{1}{\varepsilon} \displaystyle\int W(u) & \left(\tfrac{1}{2} < s < 1\right) \end{cases}$$

Γ-Convergence of E_s (Savin–Valdinoci 2012)

Then,

$$E_{s,\varepsilon}(u) \xrightarrow{\Gamma\text{-conv}} \begin{cases} \|u\|^2_{H^s} = \mathrm{Per}_s(\partial\{u=1\}) & (0 < s < \frac{1}{2}) \\ \mathrm{Per}(\partial\{u=1\}) = \int |\nabla u| & (\frac{1}{2} \le s < 1) \end{cases}$$

$$(u : \mathbb{R}^n \longrightarrow \{-1, 1\})$$

s-Perimeter (Per_s) and s-Mean Curvature (H_s)

$$u : \mathbb{R}^n \to \{-1, 1\}, \quad D = \{u = 1\}, \quad \Gamma = \partial D$$

$$\|u\|_{H^s}^2 = Per_s(\partial\{u=1\}) \quad (0 < s < 1/2)$$

$$= \iint \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy$$

$$\boxed{H_s(D, p) = \int \frac{u(y)}{|y-p|^{n+2s}} \, dy, \quad p \in \partial D}$$

s-Perimeter (Per_s) and s-Mean Curvature (H_s)

Scaling Property: $\left(0 < s < \frac{1}{2}\right)$

① $Per_s(B_R) = \underline{R^{n-2s}}\, Per_s(B_1)$

② $H_s(R\tilde{p}, B_R) = \underline{\dfrac{1}{R^{2s}}}\, H_s(\tilde{p}, B_1), \quad \tilde{p} \in \partial B_1$

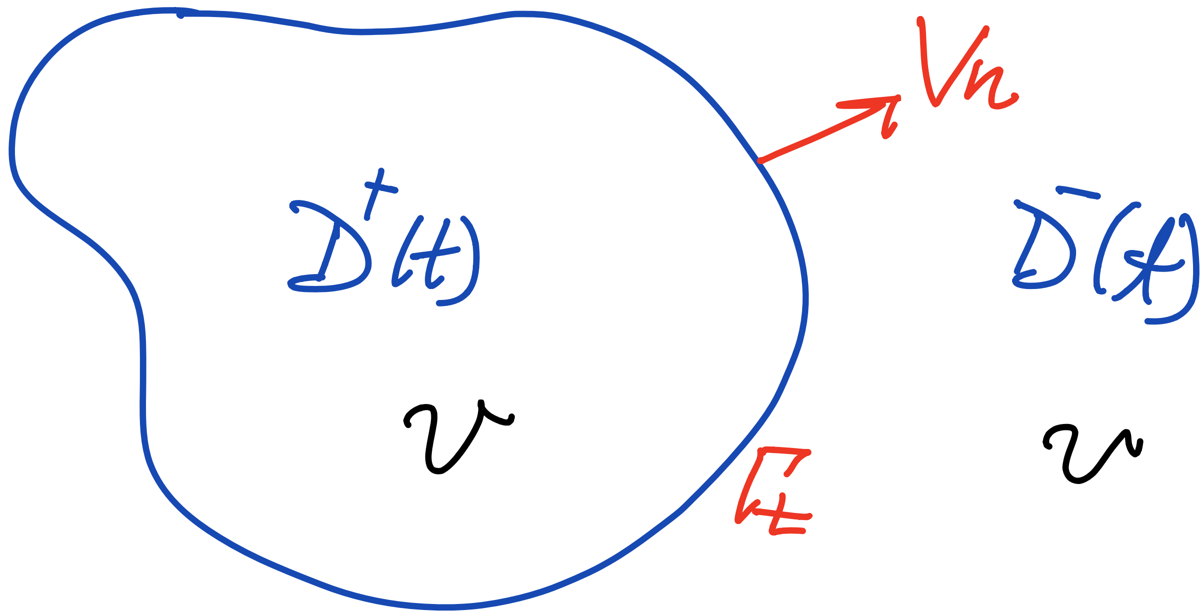
Nonlocal Cahn-Hilliard

$$\boxed{\partial_t u = \Delta\left((-\Delta)^s u + W'(u)\right)}$$

$$\left(\begin{aligned} &\partial_t u + \operatorname{div} J = 0 \\ &J = -\nabla \mu \\ &\mu = \frac{\delta E_s}{\delta u} = (-\Delta)^s u + W'(u) \end{aligned}\right)$$

(Fick's Law)

Nonlocal Mullins-Sekerka (NMS)



$$\Delta v = 0 \quad \text{in } D^+(t) \ \& \ D^-(t)$$

$$v = H_s \quad \text{on } \Gamma_t$$

$$V_n = \left[\frac{\partial v}{\partial n}\right] \quad \text{on } \Gamma_t$$

Nonlocal Cahn–Hilliard

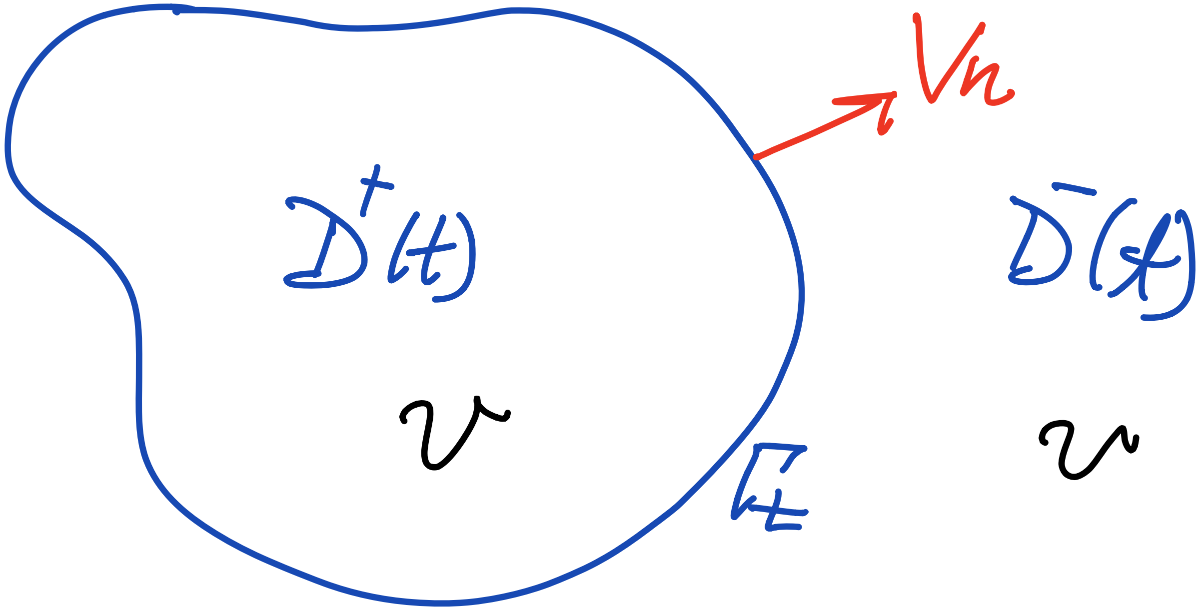
(A somewhat different version:

$$\boxed{u_t = (-\Delta)^s \left(\varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) \right)}$$

)

K.P. Zhu, 2014

Nonlocal Mullins-Sekerka (NMS)



$$
\left(
\begin{array}{ll}
\Delta v = 0 & \text{in } D^+(t) \ \& \ D^-(t) \\
v = H & \text{on } \Gamma_t \\
V_n = \left[\frac{\partial}{\partial n} (-\Delta)^{s-1} v \right] & \text{on } \Gamma_t
\end{array}
\right)
$$

Self-Similarity Scaling of NMS

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^n \setminus \Gamma_t \\ v = H_s & \text{on } \Gamma_t \\ V_n = \left[\frac{\partial v}{\partial n}\right] & \text{on } \Gamma_t \end{cases}$$

Invariant under the change of space & time scales:

$$x \longrightarrow \lambda x, \quad t \longrightarrow \lambda^{2s+2} t$$

Expect: $L(t) \sim t^{\frac{1}{2s+2}}$

Kohn-Otto's Approach to Coarsening Rates

2002

① Length scale of a pattern given by u:

$$L(u) := \fint |\nabla^{-1} u|$$

$$= \sup\left\{ \fint u \zeta \, dx : \ \sup |D\zeta| \le 1 \right\}$$

$$\sim (W^{1,\infty})^*$$

Kohn-Otto's Approach to Coarsening Rates

① Length scale of a pattern given by u:

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$$= \sup\left\{ \fint u \zeta \, dx : \ \sup |D\zeta| \leq 1 \right\}$$

$$\sim (W^{1,\infty})^*$$

$$\boxed{L\left(u\left(\frac{\cdot}{\lambda}\right)\right) = \lambda L(u(\cdot)) \sim O(\lambda)}$$

Kohn-Otto's Approach to Coarsening Rates

② Energy of the system (local case)

$$E(u) = \int_{\Omega} \frac{1}{2}|\nabla u|^2 + W(u)\, dx$$

$$\simeq \text{interfacial area per unit vol}$$

$$\simeq \frac{1}{L(u)}$$

Kohn-Otto's Approach to Coarsening Rates

② Energy of the system (local case)

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u)\, dx$$

$\cong$ interfacial area per unit vol

$$\cong \frac{1}{L(u)}$$

$$\boxed{L(u(t)) \sim t^{1/3} \iff E(u(t)) \sim t^{-1/3}}$$

Kohn-Otto's Approach to Coarsening Rates

③ "Weak" (or integral) formulation:

$$\boxed{\frac{1}{T}\int_0^T E(t)^2\,dt \geq \frac{C}{T}\int_0^T \left(t^{-1/3}\right)^2 dt}$$

$$\left(\begin{array}{l} \text{"} E(t) \gtrsim t^{-1/3} \text{"} \\[1em] \text{"} L(t) \lesssim t^{1/3} \text{"} \end{array}\right)$$

Kohn–Otto's Approach to Coarsening Rates

3 Lemmas:

① Interpolation Inequality:

$$\forall u, \quad E(u)L(u) \gtrsim 1 \quad (\text{if } E(u) \ll 1)$$

② Dissipation Inequality:

$$(\dot{L}(t))^2 \lesssim -(\dot{E}(t))$$

Kohn-Otto's Approach to Coarsening Rates

3 Lemmas:

3 ODE Lemma

1 & 2 $\Longrightarrow$

$$\int_0^T E(t)^2 \, dt \gtrsim \int_0^T \left(t^{-1/3}\right)^2 dt$$

Kohn-Otto's Approach to Coarsening Rates

Related works using Kohn-Otto approach:

(1) Kohn-Yan (2003, 2004)

expitaxial growth, multi-component sys

(2) Brenier-Otto-Seis, Otto-Seis-Slepcev (2011)

demixing in binary viscous liquid

(3) Elsey-Slepcev (2015)

mean curvature flow of Voronoi diagram

Interpolation Inequality (Local Case)

$$\boxed{\left(\fint |\nabla^{-1} u|\right)\left(\fint \frac{1}{2}|\nabla u|^2 + W(u)\right) \gtrsim 1}$$

$$\Downarrow \left(\text{if } u : \Omega \longrightarrow \{-1, 1\}\right)$$

"
$$\boxed{\left(\fint |\nabla^{-1} u|\, dx\right)^{1/2}\left(\fint |\nabla u|\, dx\right)^{1/2} \gtrsim \fint |u|\, dx}$$
"

Interpolation Inequality (Local Case)

$$\boxed{\left(\int |\nabla^{-1} u|\right)\left(\int \frac{1}{2}|\nabla u|^2 + W(u)\right) \gtrsim 1}$$

$$\Downarrow \left(\text{if } u : \Omega \longrightarrow \{-1, 1\}\right)$$

"If we let $v = \nabla^{-1} u$",

$$\boxed{\left(\int |v|\,dx\right)^{1/2}\left(\int |D^2 v|\,dx\right)^{1/2} \gtrsim \left(\int |Dv|\right)}$$

(Gagliardo-Nirenberg Ineq.)

Coarsening Rate for Nonlocal Cahn-Hilliard

(Gao-Luo-Y.)

(L_0, E_0 = initial length scale and energy)

(a) If $0 < s < \frac{1}{2}$

$$\int_0^T E(t)^\sigma \, dt \gtrsim \int_0^T \left(t^{-\frac{s}{s+1}}\right)^\sigma dt \quad \begin{pmatrix} T \gg L_0^{2s+2} \gg 1 \gg E_0 \\ 1 < \sigma < 1 + \frac{1}{s} \end{pmatrix}$$

(b) If $\frac{1}{2} \le s \le 1$,

$$\int_0^T E(t)^\sigma \, dt \gtrsim \int_0^T \left(t^{-\frac{1}{3}}\right)^\sigma dt \quad \begin{pmatrix} T \gg L_0^{3} \gg 1 \gg E_0 \\ 1 < \sigma < 3 \end{pmatrix}$$

Interpolation Inequality (Nonlocal Case)

(a) $0 < s < \frac{1}{2}$

$$E(u) L(u)^{2s} \gtrsim 1 \quad \text{for } E(u) \ll 1$$

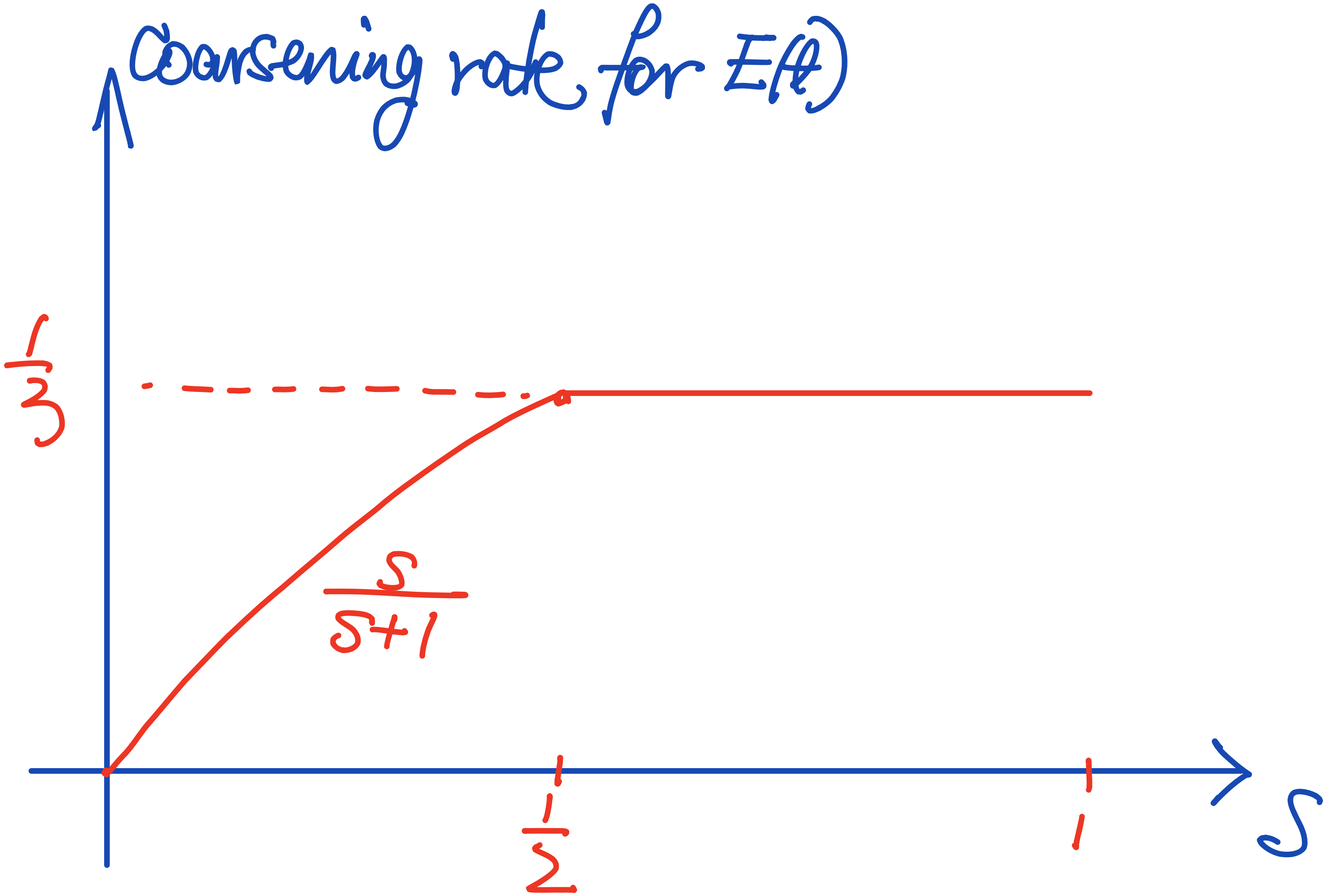
(b) $\frac{1}{2} \leq s < 1$

$$E_*(u) L(u) \gtrsim 1 \quad \text{for } E_*(u) \ll 1$$

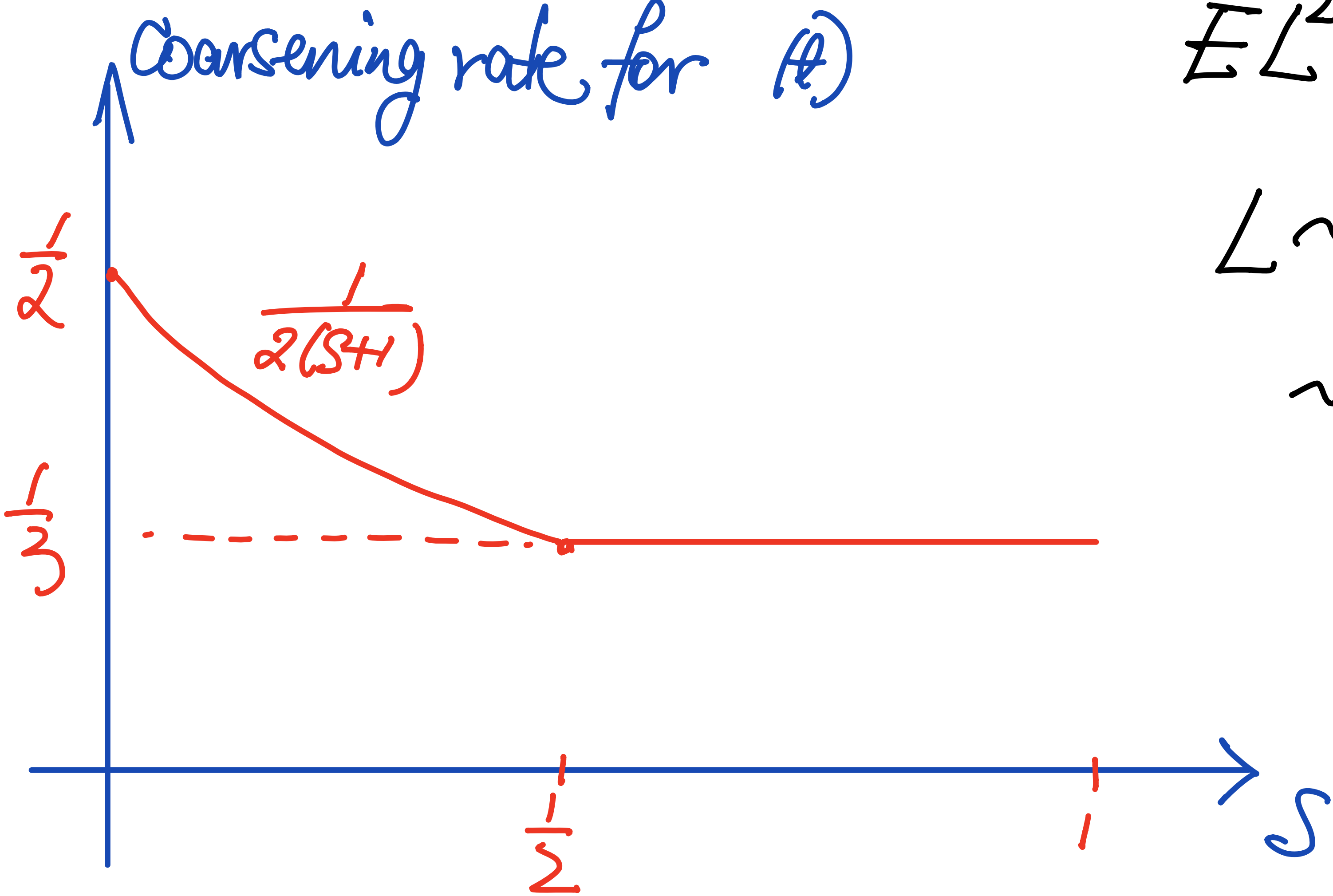
$$\left(E_*(u) = \int |\nabla u| \right)$$

Coarsening Rate for Nonlocal Cahn-Hilliard

(Gao-Luo-Y.)



Coarsening Rate for Nonlocal Cahn-Hilliard



$$EL^{2s} \gtrsim 1$$

$$L \sim E^{-\frac{1}{2s}}$$

$$\sim t^{\frac{1}{2(s+1)}}$$

Proof of Interpolation Inequality.

$$u : \Omega \to \{-1, 1\}, \quad u_\varepsilon = u * \varphi_\varepsilon$$

① $1 - ⨍ u^2 = ⨍ (1 - u^2) \lesssim \left(⨍ (1-u^2)^2 dx \right)^{1/2} \lesssim E^{1/2}$

② $⨍ u^2 \lesssim ⨍ u_\varepsilon^2 + ⨍ (u - u_\varepsilon)^2$

Proof of Interpolation Inequality.

$$u : \Omega \to \{-1, 1\}, \quad u_\varepsilon = u * \varphi_\varepsilon$$

① $$1 - \fint u^2 = \fint (1-u^2) \leq \left(\fint (1-u^2)^2 dx\right)^{1/2} \leq E^{1/2}$$

② $$\fint u^2 \lesssim \fint u_\varepsilon^2 + \fint (u-u_\varepsilon)^2$$

$$\lesssim \frac{1}{\varepsilon}\int |\nabla u| + E$$

Proof of Interpolation Inequality.

$$u : \Omega \to \{-1, 1\}, \quad u_\varepsilon = u * \varphi_\varepsilon$$

$$\int_\Omega (u - u_\varepsilon)^2 \, dx = \int_\Omega \left(\int_{B_\varepsilon(0)} (u(x) - u(x-y)) \varphi_\varepsilon(y) \, dy \right)^2 dx$$

$$\lesssim \int_\Omega \left(\int_{B_\varepsilon(0)} |u(x) - u(x-y)| \, dy \right)^2 dx$$

$$\lesssim \int_\Omega \int_{B_\varepsilon(0)} |u(x) - u(x-y)|^2 \, dy \, dx$$

Proof of Interpolation Inequality.

$$u : \Omega \to \{-1, 1\}, \qquad u_\varepsilon = u * \varphi_\varepsilon$$

$$\fint_\Omega (u - u_\varepsilon)^2 \, dx \lesssim \fint_\Omega \fint_{B_\varepsilon(0)} |u(x) - u(x-y)|^2 \, dy \, dx$$

$$= \fint_\Omega \fint_{B_\varepsilon(x)} |u(x) - u(y)|^2 \, dy \, dx$$

$$\lesssim \varepsilon^{2s} \fint_\Omega \int_{B_\varepsilon(x)} \frac{|u(x) - u(y)|^2}{\varepsilon^{n+2s}} \, dy \, dx \lesssim \varepsilon^{2\varepsilon} E$$

$$\geq |x-y|^{n+2s}$$

Proof of Dissipation Inequality

$$\boxed{(\dot{L})^2 \lesssim -\dot{E}}$$ (diffused interface version)

$$u_t = -\operatorname{div} J, \quad J = -\nabla\mu, \quad \mu = \frac{\delta E}{\delta u}$$

① $$\dot{E} = \int \frac{\delta E}{\delta u} u_t \, dx = -\int |J|^2 dx$$

② $$|L(t_2) - L(t_1)| \leq \int_{t_1}^{t_2} \int |J| \, dx \, dt$$

$$\left(\Longrightarrow \dot{L} \leq \int |J| \lesssim \left(\int |J|^2 dx\right)^{1/2} = (-\dot{E})^{1/2} \right)$$

Proof of Dissipation Inequality

$$\boxed{(\dot{L})^2 \lesssim -\dot{E}}$$

② $$|L(t_2) - L(t_1)| \le \int_{t_1}^{t_2} \int |J| \, dx \, dt$$

pf: $$L(t_2) - L(t_1) \le \int \big(u(t_2) - u(t_1)\big) \zeta_*(t_2) \, dx$$

$$= \int_{t_1}^{t_2} \int \frac{\partial u}{\partial t} \zeta_* \, dx \, dt = \int_{t_1}^{t_2} \int J \cdot \nabla \zeta_* \, dx \, dt \lesssim \int_{t_1}^{t_2} \int |J| \, dt$$

Some Technical Considerations

(1) Interpolation Inequality $\left(\frac{1}{2} < s < 1\right)$

$$\left(\int |\nabla^{-1} u|\right)\left(\|u\|_{H^s}^2 + \int W(u)\right) \gtrsim 1$$

(?)

$$\int |\nabla^{-1} u| \int |\nabla u| \gtrsim 1 \qquad u \in BV$$

(2) Existence and Uniqueness of Solutions of

$$u_t = \Delta\left((-\Delta)^s u + W'(u)\right)$$

(?)

(Garcke-Elliott, local case)

Some Technical Considerations

(3) Sharp Interface Limit (?)

($\Longrightarrow$ nonlocal Mullins-Sekerka)

(i) Imbert-Souganidis (nonlocal Allen-Cahn, $\frac{1}{2}<s<1$ to MMC,)

(ii) Millot-Sire-Wang (2019)

$$(-\Delta)^s u^\varepsilon + \frac{1}{\varepsilon^{2s}} W'(u^\varepsilon) \longrightarrow H_s, \quad \left(0<s<\tfrac{1}{2}\right)$$

$(\delta E_{s,\varepsilon})$

Some Technical Considerations

(3) Sharp Interface Limit (?)

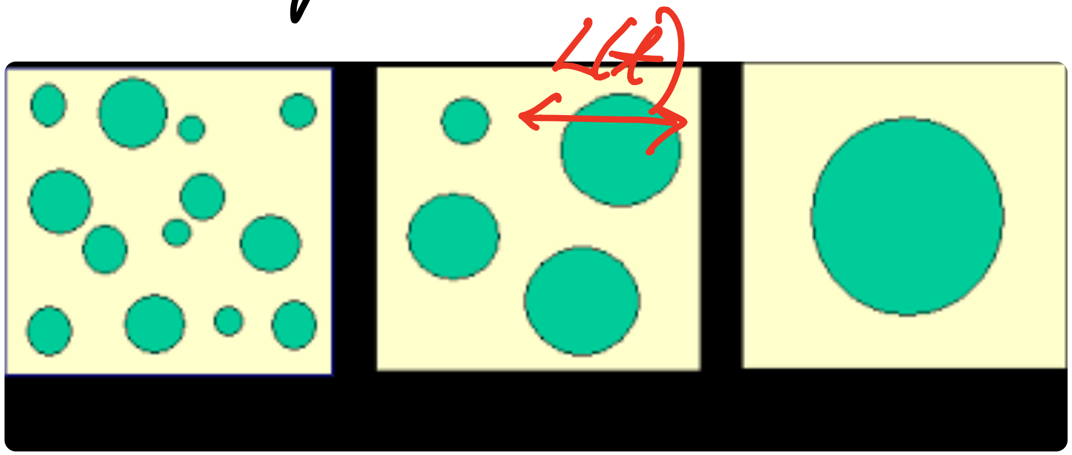
($\Longrightarrow$ nonlocal Mullins-Sekerka)

(iii) Roger (2005): use variation time disc.

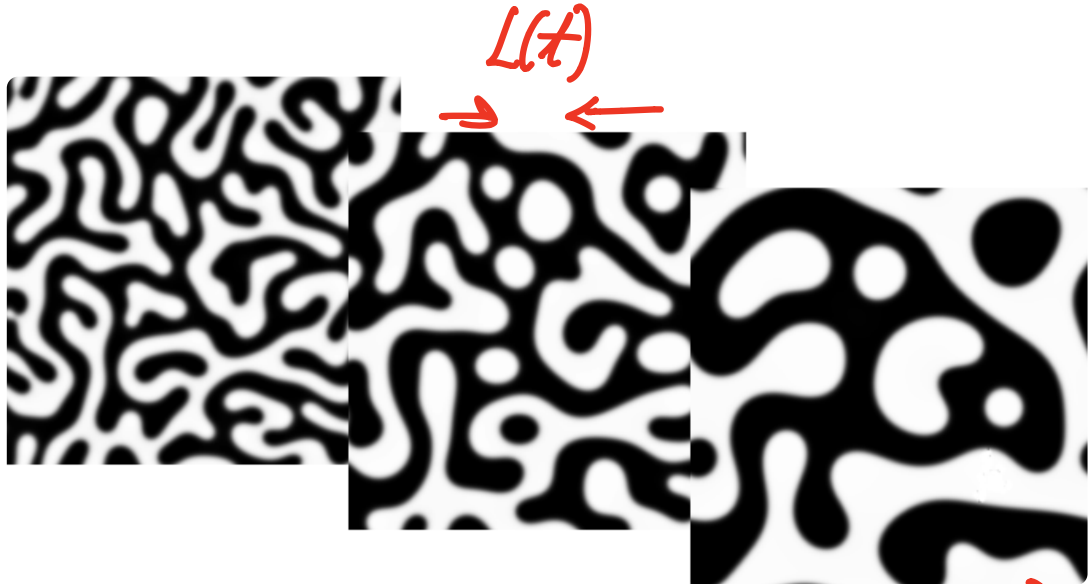
(iv) Le (2008): use Sandier-Serfaty approach

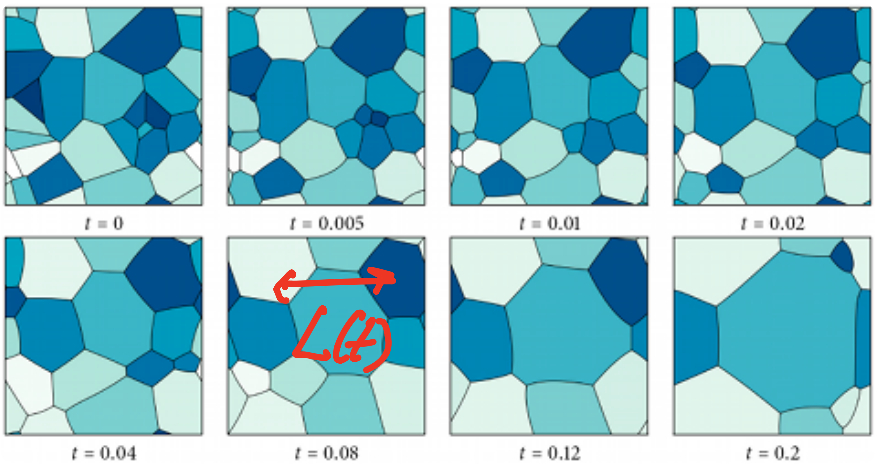
(local) Cahn-Hilliard $\longrightarrow$ (local) MS

Evolving Length Scales



$L(t) \sim t^{\alpha}$





(Self-Similarity of actual patterns?)

Thank You