Set Values for Mean Field Games with Multiple Equilibriums

Jianfeng Zhang (USC)

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with Melih Işeli

Also Feinstein, Rudloff, Z. (2020)

Consider simple setting in the talk, but

$d > 1$ ✓

Common noise ✓

path dependence ✓

Volatility control ??

I. Mean field games.

1. N-player game: $i = 1, \dots, N$

Control: $\vec{\alpha} := (\alpha^1, \dots, \alpha^N)$

state: $X_t^i = x_i + \int_0^t b(s, X_s^i, \mu_s^N, \alpha_s^i) ds + B_t^i$

$$\mu_t^N := \frac{1}{N} \sum_{j=1}^N \delta_{X_t^j}$$

Utility: $J_i^N(x_i, \mu_0^N; \vec{\alpha}) := \mathbb{E}\left[g(X_T^i, \mu_T^N) + \int_0^T f(s, X_s^i, \mu_s^N, \alpha_s^i) ds \right]$

Def. $\vec{\alpha}^*$ is a NE if, for $i = 1, \dots, N$,

$$J_i^N(x_i, \mu_0^N; \vec{\alpha}^*) = \sup_{\alpha^i} J_i^N(x_i, \mu_0^N; \vec{\alpha}^{*,-i}, \alpha^i)$$

2. Mean field game.

"Others": $X_t^\alpha = \xi + \int_0^t b(s, X_s^\alpha, \underset{= \mathcal{L}_{X_s^\alpha}}{\mu_s^\alpha}, \alpha_s) ds + B_t$

"individual": $X_t^{\alpha,\alpha'} = x + \int_0^t b(s, X_s^{\alpha,\alpha'}, \mu_s^\alpha, \alpha_s') ds + B_t'$

Utility: $J(x, \underset{= \mathcal{L}_\xi}{\mu}, \alpha, \alpha') := \mathbb{E}\left[g(X_T^{\alpha,\alpha'}, \mu_T^\alpha) + \int_0^T f(s, X_s^{\alpha,\alpha'}, \mu_s^\alpha, \alpha_s') ds \right]$

Optimal value: $\bar{J}(x, \mu; \alpha) := \sup_{\alpha'} J(x, \mu; \alpha, \alpha')$

Def α^* is an MFE at μ if:

$$J(x, \mu; \alpha^*, \alpha^*) = \bar{J}(x, \mu; \alpha^*), \quad \mu\text{-a.e. } x$$

Note: $\bar{J}(\underset{t}{\cdot}, \underset{x}{\cdot}; \mu, \alpha)$ is the solution to a standard HJB, and thus has desired regularity.

3. Assume α^* is unique,

Denote $V(0, x, \mu) := \bar{J}(x, \mu; \alpha^*)$

Two main problems in the literature:

(i) Master equation for $V(t, x, \mu)$ ← game from $[t, T]$

(ii) $V_N(0, x, \mu_0^N) \overset{?}{\to} V(0, x, \mu)$ if $\mu_0^N \to \mu$

<u>Question</u>: what if α^* is NOT unique?

II. The set value: heuristic discussions

0. Set value: $\mathbb{V}(0, x, \mu) \cancel{:=} \{\bar{J}(x, \mu, \alpha^*) : \text{all MFE } \alpha^*\} \subset \mathbb{R}$

<u>Note</u>: (i) α^* is common for all players, NOT for the individual player "x"

(ii) Master equation is local for μ, but global for x.

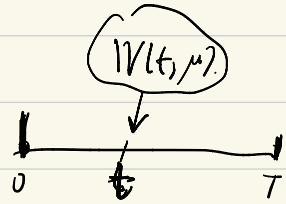
So we should define $\mathbb{V}(t, \mu)$, Not $\mathbb{V}(t, x, \mu)$.

1. N-player: $\mathbb{V}^N(0, \mu_0^N) := \{(J_1(x_1, \mu_0^N; \vec{\alpha}^*), \cdots J_N(x_N, \mu_0^N, \vec{\alpha}^*)) : \text{all } \vec{\alpha}^*\} \subset \mathbb{R}^N$

MFG: $\mathbb{V}(0, \mu) := \{ J(\bullet, \mu; \alpha^*) : \text{all MFE } \alpha^* \} \subset C^0(\mathbb{R}$

Q1. $\mathbb{V}^N(0,\mu_0^N) \to \mathbb{V}(0,\mu)$ if $\mu_0^N \to \mu$?

2. For the dynamic set value $\mathbb{V}(t,\mu)$, do we have DPP?



$$\mathbb{V}(0,\mu) \overset{?}{=} \{ J(t,\psi;\, 0, \overset{x}{\downarrow}{\cdot}, \mu;\, \alpha^*) :$$

$\psi : \mathcal{P}_2 \to C^0(\mathbb{R})$ s.t. $\psi(\nu) \in \mathbb{V}(t,\nu)$, $\forall \nu$

all MFE α^* for the MFG on $[0,t]$ with terminal Condition ψ

Note: If $\mathbb{V}(t,\mu) = \{V(t,\cdot,\mu)\}$ is a singleton, this is exactly the standard DPP.

3. Information of the controls. FRZ (2020

	$\mathbb{V}^N \to \mathbb{V}$	DPP	DPP
(i) $\alpha_t = \alpha_t(B_{[0,t]})$	✓	✗	✗
(ii) $\alpha_t = \alpha_t(X_t) / \alpha_t(X_t, \mathcal{L}_{X_t})$	✓	✓	✗
(iii) $\alpha_t = \alpha_t(X_{[0,t]}) / \alpha_t(X_{[0,t]}, \mathcal{L}_{X_{[0,t]}})$	✓	✓	✓

4. state dependent or path dependent ?

Note: In stochastic control, usually ~~admissible~~ path-dep. controls, but state dependent optimal controls.

However, for games, in general.

$$\mathbb{V}_{state}(t,\mu) \neq \mathbb{V}_{path}(t,\mu).$$

We shall use $\alpha_t(X_t)$ in the rest of the talk.

Note: if we use $\alpha_t(X_{[0,t]})$.

the set value will also be path dependent,

So we need to consider $\mathbb{V}(t, \mathcal{L}_{X_{[0,t]}})$

II. The rigorous result.

0. Note: the above "results" are indeed true in discrete mode

But, for cont model.

for fixed μ, $\mathbb{V}(t,\mu) \subset C^0(\mathbb{R})$ measurable?

Is the mapping $\mu \in \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{V}(t,\mu)$ measurable?

1. ε-MFE.

Def. α^ε is an ε-MFE at μ if:

$$\mu\left(x: J(x,\mu;\alpha^\varepsilon,\alpha^\varepsilon) \geq \bar{J}(x,\mu;\alpha^\varepsilon) - \varepsilon\right) \cancel{= 1}$$

$$\geq 1-\varepsilon.$$

2. The set value:

$$\mathbb{V}_\varepsilon(t,\mu) := \left\{ \varphi \in C^0(\mathbb{R}) : \sup_x \left| \bar{J}(t,x,\mu,\alpha^\varepsilon) - \varphi(x) \right| \leq \varepsilon \text{ for all } \alpha^\varepsilon \right\}$$

$$\overline{\mathbb{V}}(t,\mu) := \bigcap_{\varepsilon>0} \mathbb{V}_\varepsilon(t,\mu).$$

Note: (i) $\overline{\mathbb{V}}(t,\mu) \neq \phi \Leftrightarrow \mathbb{V}_\varepsilon(t,\mu) \neq \phi, \ \forall \varepsilon > 0$

(ii) MFE $\overline{\mathcal{A}}(t,\mu) \neq \phi \underset{\not\Leftarrow}{\Rightarrow} \mathcal{A}_\varepsilon(t,\mu) \neq \phi \ \forall \varepsilon$

set of ε-MFE

3. $\overline{\mathbb{V}}_N \to \overline{\mathbb{V}}$

and $\overline{\mathbb{V}}$ satisfies DPP ✓

$$\overline{\mathbb{V}}^N_\varepsilon(0,\mu^N_0) := \left\{ \varphi \in C^0_K(\mathbb{R}) : \sup_i \left| J(x_i,\mu^N_0,\vec{\alpha}^\varepsilon) - \varphi(x_i) \right| \leq \varepsilon \text{ all } \vec{\alpha}^\varepsilon \right\}$$

mod. of cont.

$$\bigcap_{\varepsilon>0} \varliminf_{N\to\infty} \mathbb{V}^N_\varepsilon(0,\mu^N_0) = \overline{\mathbb{V}}(0,\mu) = \bigcap_{\varepsilon>0} \varlimsup_{N\to\infty} \overline{\mathbb{V}}^N_\varepsilon(0,\mu^N_0)$$

Question: how to pick an "optimal" value in the set?

e.g. $$\sup_{\psi \in \mathbb{V}(0,\mu)} \int \psi(x)\, \mu(dx).$$