Deterministic MFGs with control on the acceleration

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Plan

- Introduction to the problem
- Some non-coercive cases
- Control on the acceleration: the case without constraints
- Work in progress:
  - Control on the acceleration: the case with state-constraint and control-constraint
  - Control on the acceleration: the case with only state-constraint
- Perspectives

Deterministic Mean Field Game

u is the value function of a typical agent following deterministic dynamics;
m is the density distribution of the agents (a density of a probability measure).

$$\begin{cases} (HJ) & -\partial_t u + H(x, Du) = F[m(t)](x), & \mathbb{R}^n \times (0,T) \\ (C) & \partial_t m - \mathrm{div}(m\, D_p H(x, Du)) = 0, & \mathbb{R}^n \times (0,T) \\ & m(x,0) = m_0(x),\ \ u(x,T) = G[m(T)](x), & \mathbb{R}^n \end{cases}$$

- F and G encode the interactions between the agent and the whole population.
- $m_0(x)$ has compact support.
- H is coercive w.r.t p $(H(x,p) = \frac{1}{2}|p|^2)$.

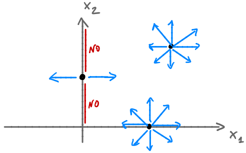
MFG with non-coercive Hamiltonian

- The player has a forbidden direction at some points: degenerate dynamics in $\mathbb{R}^2$;
- The player has a forbidden direction at every point: degenerate dynamics of "Heisenberg type" in $\mathbb{R}^3$;
- The dynamics of the player is controlled by the acceleration;

$x = (x_1, x_2) \in \mathbb{R}^2$, $p = (p_1, p_2)$.

- h bounded, $h(0)=0$, $h(x_1) \neq 0$, if $x_1 \neq 0$:
- Dynamics of the player $\begin{cases} x_1'(s) = \alpha_1(s), \\ x_2'(s) = h(x_1(s))\,\alpha_2(s) \end{cases}$
- Cost $\int_t^T \frac{1}{2}|\alpha^2| + F[m(s)](x(s))ds + G[m(T)](x(T))$;
- Hamiltonian $H(x,p) = \frac{1}{2}(p_1^2 + h^2(x_1)p_2^2)$ is not coercive;

Forbidden direction:

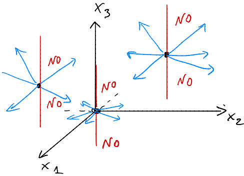


$$(MFG) \begin{cases} -\partial_t u + \frac{1}{2}(\partial^2_{x_1} u + h^2(x_1)\partial^2_{x_2} u) = F[m(t)](x), \\ \partial_t m - \Big(\partial_{x_1}(m\partial_{x_1} u) + h^2(x_1)\partial_{x_2}(m\partial_{x_2} u)\Big) = 0, \\ m(x,0) = m_0(x),\ u(x,T) = G[m(T)](x). \end{cases}$$

Dynamics of "Heisenberg type" (M., Marchi, Tchou, in preparation)

$x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $p = (p_1, p_2, p_3)$

- Dynamics $\begin{cases} x_1'(s) = \alpha_1(s), \\ x_2'(s) = \alpha_2(s), \\ x_3'(s) = -x_2(s)\,\alpha_1(s) + x_1(s)\,\alpha_2(s) \end{cases}$



- Cost: $\int_t^T \frac{1}{2}|\alpha^2| + F[m(s)](x(s))ds + G[m(T)](x(T))$
- Hamiltonian: $H(x,p) = \frac{1}{2}((p_1 - x_2 p_3)^2 + (p_2 + x_1 p_3)^2)$ is not coercive;

Difference with the previous case:

$\begin{cases} x_1' = \alpha_1 \\ x_2' = h(x_1)\alpha_2 \end{cases}$ $\quad B(x) = \begin{pmatrix} 1 & 0 \\ 0 & h(x_1) \end{pmatrix}$ $\quad \det BB^T = h^2(x_1) \neq 0$ a.e. x

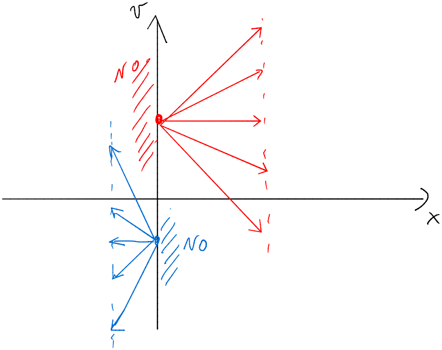
$\begin{cases} x_1' = \alpha_1 \\ x_2' = \alpha_2 \\ x_3' = -x_2\alpha_1 + x_1\alpha_2 \end{cases}$ $\quad C(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x_2 & x_1 \end{pmatrix}$ $\quad \det CC^T = 0$ $\forall x$.

Control on the acceleration (Achdou, M., Marchi, Tchou, NoDEA, 2020)

State variable $y = (x, v) \in \mathbb{R}^{2n}$, x is the position, v is the velocity.

Dynamics of the generic player is controlled by the acceleration (double integrator).

$$\begin{cases} x'(s) = v(s), \\ v'(s) = \alpha(s), \\ x(t) = x, \;\; v(t) = v. \end{cases}$$



Optimal control problem associated to MFG

The player aims to choose the control α so to minimize the cost

$$J_t(x,v,\alpha) = \int_t^T \frac{1}{2}|\alpha(s)|^2 + \frac{1}{2}|v(s)|^2 + F[m(s)](x(s),v(s))ds$$
$$+G[m(T)](x(T),v(T))$$

$m(t)$ is the distribution of the whole population at time t,
$\frac{1}{2}|v|^2$ stands for kinetic energy.

The Hamiltonian $\quad H(x,v,p_x,p_v) = -v \cdot p_x + \frac{|p_v|^2}{2} - \frac{|v|^2}{2}$

- is not coercive with respect to $p = (p_x, p_v)$;
- is unbounded w.r.t. v.

The MFG system

$$u = u(x, v, t),\ m = m(x, v, t),\ (x, v, t) \in \mathbb{R}^{2N} \times (0, T).$$

MFG system

$$\begin{array}{ll} (HJ) & -\partial_t u - v \cdot D_x u + \frac{1}{2}|D_v u|^2 - \frac{1}{2}|v|^2 = F[m(t)](x, v) \\ (C) & \partial_t m - v \cdot D_x m - \mathrm{div}_v(m D_v u) = 0 \\ & m(x, v, 0) = m_0(x, v), u(x, v, T) = G[m(T)](x, v) \end{array}$$

Assumptions

- $F[\cdot]$ and $G[\cdot]$ are regularizing nonlocal coupling with:
  $\|F[m](\cdot, \cdot)\|_{C^2}$, and $\|G[m](\cdot, \cdot)\|_{C^2} \leq C$, $\forall m \in \mathcal{P}_1$
- the initial m_0 has a compactly supported density in $C^\delta(\mathbb{R}^{2N})$.

Goal

Definition

The pair (u, m) is a solution if:

1) $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^{2N} \times [0, T])$.
2) $m \in C([0, T]; \mathcal{P}_1(\mathbb{R}^{2N}))$; $\forall t \in [0, T]$, $m(t)$ is AC w.r.t. Lebesgue measure and m is bounded.
3) (HJ)-equation is satisfied by u in the viscosity sense.
4) (C)-equation is satisfied by m in the sense of distributions.

Main Theorem

- system MFG has a solution (u, m);
- $m(x, v, s) = \Phi(x, v, 0, s)\#m_0$, i.e. is the push-forward of m_0 through the characteristic flow $\Phi(x, v, 0, s)$ associated to
$$\begin{cases} x'(s) = v(s), & x(0) = x, \\ v'(s) = -D_v u(x(s), v(s), s), & v(0) = v. \end{cases}$$

Literature

- *Classical coercive cases:*
  - P.L. Lions' lectures at Collége de France 2012 - Cardaliaguet "Notes on Mean Field Games",
  - Cardaliaguet-Hadikhanloo, ESAIM-COCV 2017
  - Gomes-Pimentel-Voskanyan "Regularity Theory for Mean-Field Game systems" 2016
  - Gomes-Voskanyan, SiamJCO 2016
- *Lagrangian approach:*
  - Benamou-Carlier-Santambrogio, Springer 2016
  - Cannarsa-Capuani, PDE models for multi-agent phenomena, Springer INdAM (2018)
  - Cannarsa-Mendico: similar direction; different techniques. (2019).

Coercive case. PL Lions, notes by P. Cardaliaguet.

$$\begin{cases} (HJ) & -\partial_t u + \frac{1}{2}|Du|^2 = F[m](x) & \mathbb{R}^N \times (0,T) \\ (C) & \partial_t m - \mathrm{div}(m\,Du) = 0 & \mathbb{R}^N \times (0,T) \\ & m(x,0) = m_0(x), u(x,T) = G[m(T)](x) & \mathbb{R}^N. \end{cases}$$

Here $H(x,p) = \frac{1}{2}|p|^2$.

1. The MFG system has a solution (u,m);
2. $m(x,s) = \Phi(x,0,s)\# m_0(x)$, i.e. is the push-forward of m_0 through the characteristic flow $\Phi(x,0,s)$ associated to the dynamics

$$\begin{cases} x'(s) = -D_x u(x(s),s) \\ x(0) = x. \end{cases}$$

Let us go back to our problem

$$\begin{cases} (HJ) & -\partial_t u - v \cdot D_x u + \frac{|D_v u|^2}{2} - \frac{|v|^2}{2} = F[m(t)] \\ (C) & \partial_t m + v \cdot D_x m - \mathrm{div}_v(D_v u m) = 0 \\ & m(x,v,0) = m_0(x,v), u(x,v,T) = G[m(T)](x,v). \end{cases}$$

Theorem

1. MFG system has a solution (u, m),
2. $m(x,s) = \Phi(x,0,s)\#m_0(x)$, i.e. is the push-forward of m_0 through the flow

$$\begin{cases} x'(s) = v(s), & x(0) = x \\ v'(s) = -D_v u(x(s), v(s), s), & v(0) = v. \end{cases}$$

Plan of the proof

1. We fix $\overline{m} \in C([0,T], \mathcal{P}_1)$, we find properties of u unique solution of

$$(HJ): \begin{cases} -\partial_t u - v \cdot D_x u + \frac{1}{2}|D_v u|^2 - \frac{1}{2}|v|^2 = F[\overline{m}](x,v), \\ u(x,v,T) = G[\overline{m}](x,v). \end{cases}$$

2. We find an unique m such that

$$(C): \begin{cases} \partial_t m + v \cdot D_x m - \mathrm{div}_v(m\, D_v u) = 0 \\ m(x,v,0) = m_0(x,v). \end{cases}$$

Fixed point: $\overline{m} \to \mathcal{T}(\overline{m}) = m$.

Difficulties

- Degenerate Hamiltonian due to the dynamics of the generic player $\rightarrow$ we cannot prove a contraction property of the flow associated to the dynamics;
- Unbounded Hamiltonian: some estimates of the value function hold only locally w.r.t v ($|u(x,v,t)| \leq C(1+|v|^2)$);
- Unbounded drift $(-v, D_v u)$ in the continuity equation ($\partial_t m - \mathrm{div}(m\,(-v, D_v u)) = 0$).

STEP1: The optimal control problem associated to HJ

$f(x, v, t) = F[\overline{m}(t)](x, v)$, $g = G[\overline{m}(T)]$, f and g bounded and regular.

$$(HJ): \begin{cases} -\partial_t u - v \cdot D_x u + \frac{1}{2}|D_v u|^2 - \frac{1}{2}|v|^2 = f(x, v, t), \\ u(x, v, T) = g(x, v). \end{cases}$$

The value function

$$u(x, v, t) := \inf_{\alpha} \int_t^T \frac{1}{2}|\alpha|^2 + \frac{1}{2}|v|^2 + f(x(s), v(s), s)\, ds + g(x(T), v(T))$$

$x(s), v(s)$ verify the dynamics $\begin{cases} x'(s) = v(s), & x(t) = x \\ v'(s) = \alpha(s), & v(t) = v \end{cases}$; $\quad \alpha \in L^2$,

is the unique viscosity solution to (HJ) such that $|u(x, v, t)| \leq C(1 + |v|^2)$.

Almost as in the coercive case:

1. no bifurcation of optimal trajectories after the starting time
   (This is an open problem for others non-coercive cases.)
2. $u(x, v, t)$ is Lipschitz continuous in x, locally Lipschitz in (v, t)
3. $u(x, v, t)$ is semiconcave w.r.t. (x, v).
4. Optimal synthesis: $Du(\cdot, \cdot, t)$ exists in (x, v) iff there exists an unique solution of

$$\begin{cases} x'(s) = v(s) \\ v'(s) = -D_v u(x(s), v(s), s), \\ x(t) = x, v(t) = v \end{cases}$$

   which moreover is the optimal trajectory for $u(x, v, t)$.
   The system has a unique solution for a.e. starting point (x, v).
   The optimal control law is $\alpha(s) = -D_v u(x(s), v(s), s)$.

STEP 2: the continuity equation (C)

$$(C) \quad \begin{cases} \partial_t m + v \cdot D_x m - \mathrm{div}_v(m\, D_v u) = 0 & \mathbb{R}^{2N} \times (0,T) \\ m(x,v,0) = m_0(x,v) & \mathbb{R}^{2N}, \end{cases}$$

where u solves (HJ) with $\overline{m}$ prescribed.

Theorem

$\forall\ \overline{m} \in C([0,T];\mathcal{P}_1)$, $\exists!$ solution m to (C) in $C^{\frac{1}{2}}([0,T];\mathcal{P}_1) \cap L^\infty([0,T];\mathcal{P}_2)$. Moreover, m is the push forward of m_0 by the flow

$$\begin{cases} x'(s) = v(s), & x(0) = x, \\ v'(s) = -D_v u(x(s), v(s), s), & v(0) = v. \end{cases} \tag{1}$$

STEP 2: Existence of a solution m of the continuity equation

Vanishing viscosity method to the whole MFG system (σ viscosity)

$(u^\sigma, m^\sigma) \to (u, m)$, and m satisfies the continuity equation with drift $D_v u$ of the STEP 1.

Tools: the representation formula from stochastic optimal control problem and uniform estimates for (u^σ, m^σ) (especially on the semiconcavity of u^σ, on the regularity of m^σ).

Uniqueness and the representation of m

Let m be any solution to the continuity equation. ($e_t(\gamma) = \gamma(t)$ is the evaluation map). Then, by

- superposition principle (Ambrosio, Gigli, Savarè)
- disintegration theorem
- the drift is $(v, -D_v u)$ (and it has a linear growth at infinity)
- optimal synthesis

$$\begin{aligned}
\int_{\mathbb{R}^{2N}} \psi \, dm_t &= \int_\Gamma \psi(e_t(\gamma)) d\eta(\gamma) \\
&= \int_{\mathbb{R}^{2N}} \left(\int_{e_0^{-1}(x,v)} \psi(e_t(\gamma)) d\eta_{(x,v)}(\gamma) \right) dm_0(x,v) \\
&= \int_{\mathbb{R}^{2N}} \psi(\gamma(t)) dm_0(x,v) \quad \forall \psi \in C_0^0(\mathbb{R}^{2N})
\end{aligned}$$

where $\gamma(\cdot) = (x(\cdot), v(\cdot))$ solves

$$\begin{cases} x'(s) = v(s), & x(0) = x, \\ v'(s) = -D_v u(x(s), v(s), s), & v(0) = v. \end{cases}$$

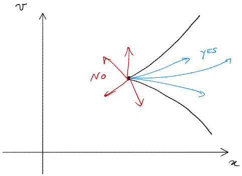
State constraint + bounded set of controls: 1-d case

x is the position, v is the velocity of the agent.

Assume

1. the position of the agent is constrained in $[-1, 1]$
2. the control α is bounded: $\alpha \in [-1, 1]$.

$$\begin{cases} x'(s) = v(s) & s \in (t, T) \\ v'(s) = \alpha(s) & s \in (t, T) \\ x(t) = x, \ v(t) = v. \end{cases}$$



Study of the (HJ) equation

The value function is

$$u(x,v,t) := \inf \left\{ J_t(x(\cdot), v(\cdot), \alpha(\cdot)) : \|x(\cdot)\|_\infty \le 1, \|\alpha(\cdot)\|_\infty \le 1 \right\}.$$

where $x(\cdot), v(\cdot)$ satisfy
$$\begin{cases} x'(s) = v(s) & s \in (t,T) \\ v'(s) = \alpha(s) & s \in (t,T) \\ x(t) = x, \ v(t) = v. \end{cases}$$

The viability set is

$$\Omega_t := \{(x,v) \ : u(x,v,t) < +\infty\}.$$

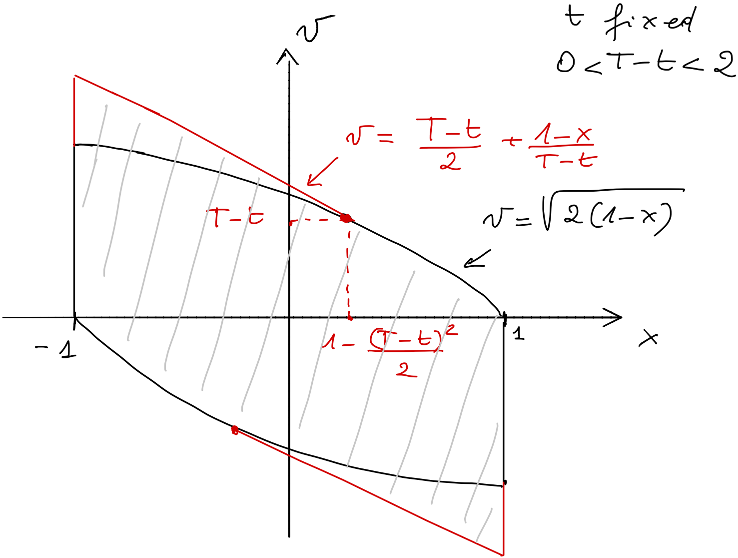
Difficulties:

- viability set Ω_t depending on t;
- uniqueness of optimal trajectories;
- Regularity of the value function:

The function $u(x,v,t)$ is Hölder continuous with exponent $1/2$:

$$|u(x,v,t) - u(y,w,s)| \le C(|x-y|^2 + |v-w|^2 + |t-s|^2)^{1/4}.$$

Viability set Ω_t



Mild solution

$(e_t(\gamma) = \gamma(t))$.

Definition: Cannarsa-Capuani, 2018

A probability measure η on the set of admissible trajectories is a MFG equilibrium for m_0 if

- $e_0 \# \eta = m_0$
- $\operatorname{supp} \eta \subset \cup_{(x,v)} \{\text{opt. trajectories for } u(x,v,0) \text{ with } m(t) = e_t \# \eta\}$.

Theorem

Assume $\operatorname{supp} m_0 \subset \Omega_0$ (Ω_0 is the viability set at $t=0$). Then

1. There exists a MFG equilibrium η.
2. There exists a mild solution to the MFG, namely (u, m) where
   - $m(t) = e_t \# \eta$
   - u is the value function associated to m.

Proof following [Cannarsa-Capuani].

Proposition (closed graph property)

The map

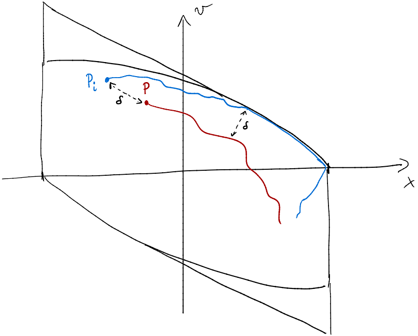
$$\Gamma : (x,v) \mapsto \{(x(\cdot),v(\cdot)) : \ (x(\cdot),v(\cdot)) \text{ optimal for } u(x,v,0)\}$$

has closed graph.

Key ingredient for the closed graph property: approximation of admissible trajectories

Fix (x, v). Let (x_i, v_i) be any sequence $(x_i, v_i) \to (x, v)$. Let $(x(\cdot), v(\cdot))$ be any admissible path for (x, v). Then, $\exists (x_i(\cdot), v_i(\cdot))$ admissible for (x_i, v_i), such that

$$(x_i(s), v_i(s)) \to (x(s), v(s)) \text{ uniformly in } [t, T];$$
$$v_i'(s) \to v'(s) \text{ in } L^2([t, T]).$$

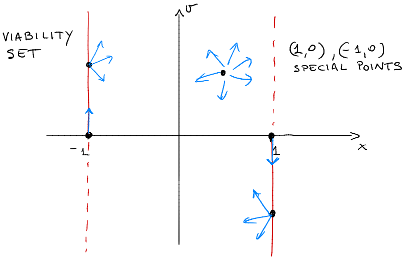


State constraint: 1-d case

Assume that the position of the agent is constrained in $[-1, 1]$.

Value function $\quad u(x, v, t) := \inf \{J_t(x(\cdot), v(\cdot), \alpha(\cdot)) : \|x(\cdot)\|_\infty \leq 1\}$

where $x(\cdot), v(\cdot)$ satisfy
$$\begin{cases} x'(s) = v(s) & s \in (t, T) \\ v'(s) = \alpha(s) & s \in (t, T) \\ x(t) = x, \ v(t) = v. \end{cases}$$



Closed graph property $\rightarrow$ Existence of a mild solution

Perspectives

1) General MFG with control on the acceleration and state constraints.

2) More general type of degenerate Hamiltonian also with unbounded coefficients.

Thank You!