Global well-posedness of master equations for deterministic displacement convex potential mean field games

Alpár R. Mészáros

(based on a joint work with W. Gangbo)

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Content of the talk

→ Mean field games.

→ Formal derivation of master equations.

→ Potential mean filed games linked to optimal control problems in infinite dimensions.

→ Vectorial vs. scalar master equations.

→ Displacement convexity, regularity estimates and well-posedness of master equations.
On Mean Field Games


Optimal control problem of a typical agent: they predict the evolution of the whole population's density, $\rho$: $[0, T_0]\to P_2(\mathbb{R}^d)$ and for $(t, x)\in [0, T_0]\times \mathbb{R}^d$ solve

$$\tilde{u}(t, x) := \inf_{Q_t=\rho} \left\{ \int_0^t L(Q_s, \dot{Q}_s) + f(Q_s, \rho_s) \, ds + u_0(Q_0, \rho_0) \right\}, \tag{1}$$

Data:

- $L: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ Lagrangian function;
- $T_0 > 0$: time horizon;
- $f, u_0: \mathbb{R}^d \times P_2(\mathbb{R}^d) \to \mathbb{R}$ running- and the initial costs of the agents;
- $\mu \in P_2(\mathbb{R}^d)$: distribution of the agents at time $T_0$.

Notations:

- $P_2(\mathbb{R}^d) := \{ \mu \text{ Borel probability measure on } \mathbb{R}^d: \int_{\mathbb{R}^d} |x|^2 \, d\mu(x) < +\infty \}$;
- $\mathcal{B} := \{ \mu \in P_2(\mathbb{R}^d): \int_{\mathbb{R}^d} |x|^2 \, d\mu(x) \leq r^2 \}$. 

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On Mean Field Games


→ Optimal control problem of a typical agent: they predict the evolution of the whole population’s density, $\rho : [0, T_0] \to \mathcal{P}_2(\mathbb{R}^d)$ and for $(t, x) \in [0, T_0] \times \mathbb{R}^d$ solve

$$\tilde{u}(t, x) := \inf_{Q; Q_t = x} \left\{ \int_0^t L(Q_s, \dot{Q}_s) + f(Q_s, \rho_s) \, ds + u_0(Q_0, \rho_0) \right\},$$  \hspace{1cm} (1)
On Mean Field Games


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$$\tilde{u}(t, x) := \inf_{\mathcal{Q} : Q_t = x} \left\{ \int_0^t L(Q_s, \dot{Q}_s) + f(Q_s, \rho_s) \, ds + u_0(Q_0, \rho_0) \right\}, \quad (1)$$

Data: $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ Lagrangian function; $T_0 > 0$: time horizon; $f, u_0 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ running- and the initial costs of the agents; $\mu \in \mathcal{P}_2(\mathbb{R}^d)$: distribution of the agents at time $T_0$.

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The value function formally solves a **Hamilton-Jacobi-Bellman equation**.

The density of the population is transported by the velocity field given by the optimal control $\alpha^* := D_p H(\cdot, D\tilde{u})$ in the above problem.

One arrives to the coupled system

\[
\begin{aligned}
\partial_t \tilde{u} + H(x, D\tilde{u}) &= f(x, \rho) \quad \text{in} \ (0, T_0) \times \mathbb{R}^d \\
\partial_t \rho + \nabla \cdot (\rho D_p H(x, D\tilde{u})) &= 0 \quad \text{in} \ (0, T_0) \times \mathbb{R}^d \\
\tilde{u}(0, x) &= u_0(x, \rho_0), \quad \rho(T_0, \cdot) = \mu \quad \text{in} \ \mathbb{R}^d.
\end{aligned}
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The MFG system

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\end{align*}
\]

(MFG)

→ The Hamiltonian \( H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) is defined as \( H(x, \cdot) = L^*(x, \cdot) \).

→ A solution \((\tilde{u}, \rho)\) of the above system characterizes equilibrium situations.
The master equation associated to (MFG)

Introduced by Lions in his lectures, this an nonlocal Hamilton-Jacobi equation set on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. It contains all information from (MFG), hence it fully characterizes Nash equilibria.

Let $T > 0$ be given. Define $u: [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ as $u(t, x, \rho) := \tilde{u}(t, x)$, $\forall (t, x, \rho) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, where $(\tilde{u}, \rho)$ is the solution of (MFG) with time horizon $T$ and final agent distribution $\rho$.

We must have $u(t, x, \rho_t) = \tilde{u}(t, x)$, $\forall (t, x) \in [0, T] \times \mathbb{R}^d$.

Question: which equation does $u$ satisfy?

Answer: 

$$\partial_t u(t, x, \rho_t) = \partial_t \tilde{u}(t, x) = f(x, \rho_t) - H(x, D\tilde{u}) = f(x, \rho_t) - H(x, Dx u).$$

Need to identify $\partial_t u(t, x, \rho_t)$! We use the theory of optimal transport.
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$$u(T_0, x, \mu) := \tilde{u}(T_0, x), \ \forall (T_0, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d),$$

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→ Need to identify \(\partial_t(u(t, x, \rho_t))\)! We use the theory of optimal transport.
For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ we define the 2-Wasserstein distance $W_2$ as

$$W_2^2(\mu, \nu) := \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\}$$

$$= \inf \left\{ \int_{\Omega} |X(\omega) - Y(\omega)|^2 \, d\omega : X, Y \in H, X_\# \mathcal{L}^d \llcorner \Omega = \mu, Y_\# \mathcal{L}^d \llcorner \Omega = \nu \right\}.$$ 

Notations:

$\Pi(\mu, \nu) := \{ \gamma \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d), (\pi^x)_\# \gamma = \mu, (\pi^y)_\# \gamma = \nu \}$,

$\pi^x, \pi^y : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are the canonical projections.

$\Pi_o(\mu, \nu) \subseteq \Pi(\mu, \nu)$: set of optimal plans.

For $T : \mathcal{X} \to \mathcal{Y}$ Borel function $T_\# \rho_0 = \rho_1$ means that $\rho_1(A) = \rho_0(T^{-1}(A))$ for any $A \subseteq \mathcal{Y}$ Borel set.

$\Omega := [0, 1]^d$, $H := L^2(\Omega; \mathbb{R}^d)$. 

Revolutionary result from [Otto, CPDE, 2001] and [Ambrosio-Gigli-Savaré, Birkhäuser, Springer, 2005]: $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ has a differential geometric structure.

$\mathrm{Tan} \mu \mathcal{P}_2(\mathbb{R}^d) = \nabla C^\infty_c(\mathbb{R}^d) L^2 \mu$ and $\mathrm{T} \mathcal{P}_2(\mathbb{R}^d) = \bigcup_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} \{ \mu \} \times \mathrm{Tan} \mu \mathcal{P}_2(\mathbb{R}^d)$. 

\[\text{Brenier, CPAM, 1991}\]: if $\mu \in \mathcal{P}^{\text{ac}}_2(\mathbb{R}^d)$, then $\gamma_{\text{opt}} = (\text{id}, T)_\# \mu$, where $T = \nabla \Psi$, with $\Psi : \mathbb{R}^d \to \mathbb{R}$ convex.
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- $\Omega := [0, 1)^d$, $\mathbb{H} := L^2(\Omega; \mathbb{R}^d)$.
- [Brenier, CPAM, 1991]: if $\mu \in \mathcal{P}_2^{ac}(\mathbb{R}^d)$, then $\gamma_{opt} = (\text{id}, T)\# \mu$, where $T = \nabla \Psi$, with $\Psi : \mathbb{R}^d \to \mathbb{R}$ convex.
- Revolutionary result from [Otto, CPDE, 2001] and [Ambrosio-Gigli-Savaré, Birkhäuser, Springer, 2005]): $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ has a differential geometric structure.
OT toolbox

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\[
= \inf \left\{ \int_{\Omega} |X(\omega) - Y(\omega)|^2 \, d\omega : X, Y \in \mathbb{H}, X_#\mathcal{L}^d \subset \Omega = \mu, Y_#\mathcal{L}^d \subset \Omega = \nu \right\}.
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→ Revolutionary result from [Otto, CPDE, 2001] and [Ambrosio-Gigli-Savaré, Birkhäuser, Springer, 2005]): \( (\mathcal{P}_2(\mathbb{R}^d), W_2) \) has a differential geometric structure.

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Wasserstein gradients

Definition (Ambrosio-Gigli-Savaré, 2005; Gangbo-Tudorascu, JMPE, 2019)

Let $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. We say that $\xi \in L^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$ belongs to the subdifferential of $\mathcal{U}$ at $\mu$, and we denote $\xi \in \partial^- \mathcal{U}(\mu)$, if for all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$

$$\mathcal{U}(\nu) \geq \mathcal{U}(\mu) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi(x) \cdot (y - x) \, d\gamma(x, y) + o(W_2(\mu, \nu)), \quad \forall \gamma \in \Pi_o(\mu, \nu).$$

The superdifferential of $\mathcal{U}$ at $\mu$ is defined in a similar way and we have in particular that $\partial^+ \mathcal{U}(\mu) = -\partial^- \mathcal{U}(\mu)$.

We say that $\mathcal{U}$ is differentiable at $\mu$, if $\partial^- \mathcal{U}(\mu) \cap \partial^+ \mathcal{U}(\mu) \neq \emptyset$. In this case there exists a unique element $\xi \in \partial^- \mathcal{U}(\mu) \cap \partial^+ \mathcal{U}(\mu) \cap \text{Tan} \mu \mathcal{P}_2(\mathbb{R}^d)$ that we denote by $\nabla w \mathcal{U}(\mu)$.

Chain rule:

If $(\sigma_t)_{t \in (0,1)}$ is a geodesic curve (i.e. $\partial_t \sigma_t + \nabla \cdot (v \sigma_t) = 0$ with $\int_0^1 \int_{\mathbb{R}^d} |v|_2^2 \, d\sigma_t \, dt < +\infty$ and with $\|v_t\|_{\sigma_t}$ minimal for a.e. $t$) along which $\mathcal{U}$ is differentiable, then

$$\frac{d}{dt} \mathcal{U}(\sigma_t) = \int_{\mathbb{R}^d} \nabla w \mathcal{U}(\sigma_t)(x) \cdot v_t(x) \, d\sigma_t(x),$$

$L^1 - a.e.$.
Wasserstein gradients

Definition (Ambrosio-Gigli-Savaré, 2005; Gangbo-Tudorascu, JMPE, 2019)

Let $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. We say that $\xi \in L^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$ belongs to the subdifferential of $U$ at $\mu$, and we denote $\xi \in \partial^- U(\mu)$, if for all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$

$$U(\nu) \geq U(\mu) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi(x) \cdot (y - x) \, d\gamma(x, y) + o(W_2(\mu, \nu)), \ \forall \gamma \in \Pi_0(\mu, \nu).$$

$\rightarrow$ The superdifferential of $U$ at $\mu$ is defined in a similar way and we have in particular that $\partial^+ U(\mu) = -\partial^- (-U)(\mu)$. 
Wasserstein gradients

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U(\nu) \geq U(\mu) + \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi(x) \cdot (y - x) \, d\gamma(x, y) + o(W_2(\mu, \nu)), \quad \forall \gamma \in \Pi_o(\mu, \nu).
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→ The superdifferential of \( U \) at \( \mu \) is defined in a similar way and we have in particular that \( \partial^+ U(\mu) = -\partial^- (-U)(\mu) \).

→ We say that \( U \) is differentiable at \( \mu \), if \( \partial^- U(\mu) \cap \partial^+ U(\mu) \neq \emptyset \). In this case there exists a unique element \( \xi \in \partial^- U(\mu) \cap \partial^+ U(\mu) \cap \text{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d) \) that we denote by \( \nabla_w U(\mu) \).
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Chain rule:
→ If $(\sigma_t)_{t \in (0,1)}$ is a geodesic curve (i.e. $\partial_t \sigma + \nabla \cdot (v\sigma) = 0$ with $\int_0^1 \int_{\mathbb{R}^d} |v|^2 \, d\sigma_t \, dt < +\infty$ and with $\|v_t\|_{\sigma_t}$ minimal for a.e. $t$ ) along which $\mathcal{U}$ is differentiable, then

$$\frac{d}{dt} \mathcal{U}(\sigma_t) = \int_{\mathbb{R}^d} \nabla_w\mathcal{U}(\sigma_t)(x) \cdot v_t(x) \, d\sigma_t(x), \quad \mathcal{L}^1 - \text{a.e. } t \in (0,1).$$
Let $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ be defined as

$$\mathcal{U}(\mu) = \int_{\mathbb{R}^d} \varphi_0(x) \, d\mu(x) + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_1(x - y) \, d\mu(x) \, d\mu(y),$$

where for $i = 0, 1$, $\varphi_i \in C^1(\mathbb{R}^d)$, has at most quadratic growth at infinity, with a gradient which has at most linear growth at infinity. Then $\mathcal{U}$ is differentiable at any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Let $\varphi$ be even. We have

$$\nabla_w \mathcal{U}(\mu)(x) = D\varphi_0(x) + (D\varphi_1 * \mu)(x).$$
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$\nabla_w \mathcal{U}(\mu)(\cdot)$ is defined only on $\text{spt}(\mu)$! So, if we would like to speak about its value at generic $x \in \mathbb{R}^d$, we need to perform an extension first (if we can)!
Examples

Let $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be defined as

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In the previous example, if $\varphi_1 \equiv 0$, and if $\varphi_0$ is differentiable on a bounded open set $B \subset \mathbb{R}^d$ and elsewhere not, $\nabla_w \mathcal{U}(\mu)(x) = D\varphi_0(x)$, provided $\text{spt}(\mu) \subseteq B$. Clearly, this object makes sense only for $x \in B$. If $\text{spt}(\mu) \setminus B \neq \emptyset$, $\mathcal{U}$ is not differentiable at $\mu$. 

Back to the deterministic master equation

Now we can formally derive

$$\partial_t (u(t, x, \rho_t)) = \partial_t u(t, x, \rho_t) + \int_{\mathbb{R}^d} \nabla_w u(t, x, \rho_t)(z) \cdot D_p H(z, D_x u(t, z, \rho_t)) \, d\rho_t(z)$$

The master equation reads as

$$\begin{cases} 
\partial_t u(t, x, \mu) + \int_{\mathbb{R}^d} \nabla_w u(t, x, \mu)(z) \cdot D_p H(z, D_x u(t, z, \mu)) \, d\mu(z) + H(x, D_x u(t, x, \mu)) = f(x, \mu), \\
u(0, x, \mu) = u_0(x, \mu), \quad \text{in } \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d). 
\end{cases}$$

(Master)
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→ An important application of master equations: serve as a tool to show the convergence of Nash equilibria of N–player differential game to solutions of MFGs, as \( N \to +\infty \). (cf. [Cardaliaguet-Delarue-Lasry-Lions, 2019]).
Back to the deterministic master equation

Now we can formally derive
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Our objective:

Describe a class of data \( u_0, f, H \) for which one can find a classical solution \( u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) to (Master) for arbitrary large \( T > 0 \) (independent of the data)!
Literature on the problem

Deterministic case:
→ [Gangbo-Swiech, JDE, 2015]: potential games (to be described in a moment);
  \( f, g \) smooth \( H(x, p) = \frac{1}{2}|p|^2 \) \( \rightarrow \) short time existence \((T \text{ depends on the data, and cannot be arbitrary large})\).
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In presence of individual and/or common noise:
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→ [Bensoussan-Graber-Yam, arXiv, 2019]: (only individual noise), Hilbert space techniques via optimal control; $H(x, p) = \frac{1}{2}|p|^2$; short time existence;
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Well-posedness of (Master) in the potential case

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The lack of non-degenerate diffusion in (MFG) makes impossible to use PDE techniques to show the well-posedness of deterministic master equations.

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Potential game setting

Suppose that \( \exists F, U_0 : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) of class \( C^1 \) such that

\[
D_x f(x, \mu) = \nabla_w F(\mu)(x) \quad \text{and} \quad D_x u_0(x, \mu) = \nabla_w U_0(\mu)(x), \quad \forall (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).
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$$D_x f(x, \mu) = \nabla_w F(\mu)(x) \text{ and } D_x u_0(x, \mu) = \nabla_w U_0(\mu)(x), \forall (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).$$

→ Define $\mathcal{H}, \mathcal{L} : \bigcup_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} \{\mu\} \times L^2_{\mu}(\mathbb{R}^d; \mathbb{R}^d)$ as

$$\mathcal{H}(\mu, \xi) := \int_{\mathbb{R}^d} H(x, \xi(x)) \, d\mu(x) \text{ and } \mathcal{L}(\mu, \zeta) = \int_{\mathbb{R}^d} L(x, \zeta(x)) \, d\mu(x).$$
The optimal control problem

\[ \mathcal{U}(t, \mu) := \inf \left\{ \mathcal{U}_0(\sigma_0) + \int_0^t \mathcal{L}(\sigma_s, v_s) + \mathcal{F}(\sigma_s) \, ds : \partial_s \sigma + \nabla \cdot (v \sigma) = 0, \, \sigma_t = \mu \right\} \] (HL-\(P_2\))

→ Solve the optimal control problem

GOAL: show that \( \mathcal{U} \) is a classical solution to the corresponding HJB equation, and there exists \( u : [0, T] \times \mathbb{R}^d \times P_2(\mathbb{R}^d) \to \mathbb{R} \) regular enough such that

\[ \nabla w \mathcal{U}(t, \mu)(x) = D_x u(t, x, \mu) \] (formally)

Then \( u \) is a candidate for the solution to (Master).

→ The equation satisfied formally by \( \mathcal{U} \) reads as

\[ \begin{align*}
\partial_t \mathcal{U}(t, \mu) + H(\mu, \nabla w \mathcal{U}(t, \mu)) &= F(\mu), \\
\mathcal{U}(0, \mu) &= \mathcal{U}_0(\mu), \\
\text{in } (0, T) \times P_2(\mathbb{R}^d), \\
\text{in } P_2(\mathbb{R}^d).
\end{align*} \] (HJB-\(P_2\))

→ It is not hard to show that \( \mathcal{U} \), as the value function is locally Lipschitz and locally displacement semi-concave (see [Gangbo-Swiech, 2015], [Gangbo-Nguyen-Tudorascu, 2008]).

→ Question: how do we get further regularity? Since we are aiming for \( u \) to be differentiable w.r.t. \( \mu \), it is necessary to have \( \mathcal{U} \) twice differentiable w.r.t. \( \mu \).
The optimal control problem

→ Solve the optimal control problem

\[ \mathcal{U}(t, \mu) := \inf \left\{ \mathcal{U}_0(\sigma_0) + \int_0^t \mathcal{L}(\sigma_s, v_s) + \mathcal{F}(\sigma_s) \, ds : \partial_s \sigma + \nabla \cdot (v \sigma) = 0, \sigma_t = \mu \right\} \]  \hspace{1cm} (\text{HL-} \mathcal{P}_2)

→ \textbf{GOAL:} show that \( \mathcal{U} \) is a \textbf{classical solution} to the corresponding HJB equation,
The optimal control problem

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\[ U(t, \mu) := \inf \left\{ U_0(\sigma_0) + \int_0^t L(\sigma_s, v_s) + F(\sigma_s) \, ds : \partial_s \sigma + \nabla \cdot (v \sigma) = 0, \sigma_t = \mu \right\} \]  \hspace{1cm} \text{(HL-\(\mathcal{P}_2\))}

→ GOAL: show that \(U\) is a classical solution to the corresponding HJB equation, and there exists \(u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) regular enough
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The optimal control problem

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The optimal control problem

→ Solve the optimal control problem
\[ U(t, \mu) := \inf \left\{ U_0(\sigma_0) + \int_0^t L(\sigma_s, v_s) + F(\sigma_s) \, ds : \partial_s \sigma + \nabla \cdot (v \sigma) = 0, \, \sigma_t = \mu \right\} \] (HL-\(P_2\))

→ GOAL: show that \( U \) is a classical solution to the corresponding HJB equation, and there exists \( u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R} \) regular enough such that \( \nabla_w U(t, \mu)(x) = D_x u(t, x, \mu) \) (formally “\( \delta_\mu U(t, \mu)(x) = u(t, x, \mu) \)”). Then \( u \) is a candidate for the solution to (Master).

→ The equation satisfied formally by \( U \) reads as
\[
\begin{cases}
\partial_t U(t, \mu) + H(\mu, \nabla_w U(t, \mu)) = F(\mu), & \text{in } (0, T) \times \mathcal{P}_2(\mathbb{R}^d), \\
U(0, \mu) = U_0(\mu), & \text{in } \mathcal{P}_2(\mathbb{R}^d).
\end{cases}
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The optimal control problem

→ Solve the optimal control problem

$$U(t, \mu) := \inf \left \{ U_0(\sigma_0) + \int_0^t \mathcal{L}(\sigma_s, v_s) + \mathcal{F}(\sigma_s) \, ds : \partial_s \sigma + \nabla \cdot (v \sigma) = 0, \sigma_t = \mu \right \} \quad \text{(HL-} \mathcal{P}_2)$$

→ **GOAL**: show that $U$ is a classical solution to the corresponding HJB equation, and there exists $u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ regular enough such that

$$\nabla_w U(t, \mu)(x) = D_x u(t, x, \mu)$$

(formally “$\delta_{\mu} U(t, \mu)(x) = u(t, x, \mu)$”). Then $u$ is a candidate for the solution to (Master).

→ The equation satisfied formally by $U$ reads as

$$\begin{cases}
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    U(0, \mu) = U_0(\mu), & \text{in } \mathcal{P}_2(\mathbb{R}^d).
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→ It is not hard to show that $U$, as the value function is locally Lipschitz and locally displacement semi-concave (see [Gangbo-Swiech, 2015], [Gangbo-Nguyen-Tudorascu, 2008]).
The optimal control problem

→ Solve the optimal control problem

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→ **GOAL:** show that \( U \) is a classical solution to the corresponding HJB equation, and there exists \( u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) regular enough such that

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→ **Question:** how do we get further regularity? Since we are aiming for \( u \) to be differentiable w.r.t. \( \mu \), it is necessary to have \( U \) twice differentiable w.r.t. \( \mu \).
Candidate for the solution to (Master):

→ Suppose that \((\sigma_s)_{s \in [0, t]}\) with \(\sigma_t = \mu\) is the unique minimizer in (HL-\(\mathcal{P}_2\)).

→ Define \(u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\) as

\[
    u(t, x, \mu) := \inf \left\{ u_0(\gamma_0, \sigma_0) + \int_0^t L(\gamma_s, \dot{\gamma}_s) + f(\gamma_s, \sigma_s) \, ds : \gamma \in W^{1,2}([0, t]; \mathbb{R}^d), \, \gamma_t = x \right\}.
\]
Master equations from (HL-$\mathcal{P}_2$) – the scalar case

Candidate for the solution to (Master):

→ Suppose that $(\sigma_s)_{s \in [0,t]}$ with $\sigma_t = \mu$ is the unique minimizer in (HL-$\mathcal{P}_2$).

→ Define $u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ as

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→ Notice that $u(t, \cdot, \mu)$ is defined for all $x \in \mathbb{R}^d$ (not only for $x \in \text{spt}(\mu)$).
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→ Notice that $u(t, \cdot, \mu)$ is defined for all $x \in \mathbb{R}^d$ (not only for $x \in \text{spt}(\mu)$).

→ The regularity of $u(\cdot, \cdot, \mu)$ can be studied by classical methods.
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→ The regularity of $u(\cdot, \cdot, \mu)$ can be studied by classical methods.

→ Suppose that $u(s, \cdot, \sigma_s)$ is differentiable. Then $\dot{\gamma}_s = D_pH(\gamma_s, D_xu(s, \gamma_s, \sigma_s))$. 
Master equations from (HL-$\mathcal{P}_2$) – the scalar case

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→ Suppose that $u(s, \cdot, \sigma_s)$ is differentiable. Then $\dot{\gamma}_s = D_pH(\gamma_s, D_xu(s, \gamma_s, \sigma_s))$.

→ Suppose that $U(s, \cdot)$ is differentiable. Then $\partial_s \sigma_s + \nabla \cdot (\sigma_s D_pH(\cdot, \nabla_w U(s, \sigma_s)(\cdot))) = 0$. 

→ Suppose that $\mathcal{U}(s, \cdot)$ is differentiable. Then $\partial_s \sigma_s + \nabla \cdot (\sigma_s D_pH(\cdot, \nabla_w \mathcal{U}(s, \sigma_s)(\cdot))) = 0$. 

→ Thus, if $x \in \text{spt}(\mu)$, the strict convexity of $H(y, \cdot)$ yields $D_xu(s, \gamma_s, \sigma_s) = \nabla_w U(s, \sigma_s)(\gamma_s)$. 

→ So, one formally has $D_xu(s, \cdot, \mu) = \nabla_w U(s, \mu)(\cdot)$ on $\text{spt}(\mu)$.

→ Therefore, $D_xu(s, \cdot, \mu)$ would produce a natural extension for $\nabla_w U(s, \mu)(\cdot)$ to the whole $\mathbb{R}^d$!
Master equations from (HL-$\mathcal{P}_2$) – the scalar case

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→ Suppose that \((\sigma_s)_{s\in[0,t]}\) with \(\sigma_t = \mu\) is the unique minimizer in (HL-$\mathcal{P}_2$).

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\begin{align*}
u(t, x, \mu) &:= \inf \left\{ u_0(\gamma_0, \sigma_0) + \int_0^t L(\gamma_s, \dot{\gamma}_s) + f(\gamma_s, \sigma_s) \, ds : \gamma \in W^{1,2}([0, t]; \mathbb{R}^d), \gamma_t = x \right\}.
\end{align*}
\]

→ Notice that \(u(t, \cdot, \mu)\) is defined for all \(x \in \mathbb{R}^d\) (not only for \(x \in \text{spt}(\mu)\)).

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→ Suppose that \(\mathcal{U}(s, \cdot)\) is differentiable. Then \(\partial_s \sigma_s + \nabla \cdot (\sigma_s D_pH(\cdot, \nabla_w \mathcal{U}(s, \sigma_s)(\cdot))) = 0\).

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Master equations from (HL-$\mathcal{P}_2$) – the scalar case

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$$u(t, x, \mu) := \inf \left\{ u_0(\gamma_0, \sigma_0) + \int_0^t L(\gamma_s, \dot{\gamma}_s) + f(\gamma_s, \sigma_s) \, ds : \gamma \in W^{1,2}([0, t]; \mathbb{R}^d), \gamma_t = x \right\}.$$  

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→ Suppose that $u(s, \cdot, \sigma_s)$ is differentiable. Then $\dot{\gamma}_s = D_pH(\gamma_s, D_xu(s, \gamma_s, \sigma_s))$.

→ Suppose that $\mathcal{U}(s, \cdot)$ is differentiable. Then $\partial_s \sigma_s + \nabla \cdot (\sigma_s D_pH(\cdot, \nabla \mathcal{U}(s, \sigma_s)(\cdot))) = 0$.

→ Thus, if $x \in \text{spt}(\mu)$, the strict convexity of $H(y, \cdot)$ yields

$$D_xu(s, \gamma_s, \sigma_s) = \nabla \mathcal{U}(s, \sigma_s)(\gamma_s).$$

→ So, one formally has $D_xu(s, \cdot, \mu) = \nabla \mathcal{U}(s, \mu)(\cdot)$ on $\text{spt}(\mu)$. 
Candidate for the solution to (Master):

→ Suppose that $(\sigma_s)_{s \in [0, t]}$ with $\sigma_t = \mu$ is the unique minimizer in (HL-$\mathcal{P}_2$).

→ Define $u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ as

\[
\begin{aligned}
 u(t, x, \mu) := \inf \left\{ u_0(\gamma_0, \sigma_0) + \int_0^t L(\gamma_s, \dot{\gamma}_s) + f(\gamma_s, \sigma_s) \, ds : \gamma \in W^{1,2}([0, t]; \mathbb{R}^d), \, \gamma_t = x \right\}.
\end{aligned}
\]

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→ The regularity of $u(\cdot, \cdot, \mu)$ can be studied by classical methods.

→ Suppose that $u(s, \cdot, \sigma_s)$ is differentiable. Then $\dot{\gamma}_s = D_pH(\gamma_s, D_xu(s, \gamma_s, \sigma_s))$.

→ Suppose that $U(s, \cdot)$ is differentiable. Then $\partial_s \sigma_s + \nabla \cdot (\sigma_s D_pH(\cdot, \nabla wU(s, \sigma_s)(\cdot))) = 0$.

→ Thus, if $x \in \text{spt}(\mu)$, the strict convexity of $H(y, \cdot)$ yields

\[
D_xu(s, \gamma_s, \sigma_s) = \nabla wU(s, \sigma_s)(\gamma_s).
\]

→ So, one formally has $D_xu(s, \cdot, \mu) = \nabla wU(s, \mu)(\cdot)$ on $\text{spt}(\mu)$.

→ Therefore, $D_xu(s, \cdot, \mu)$ would produce a natural extension for $\nabla wU(s, \mu)(\cdot)$ to the whole $\mathbb{R}^d$!
Master equations from (HL-\(\mathcal{P}_2\)) – the vectorial case

→ Take \(\nabla_w\) of (HJB-\(\mathcal{P}_2\)) to get

\[
\partial_s \nabla_w U(s, \mu)(x) + D_x H(x, \nabla_w U(s, \mu)(x)) + D_x \nabla_w U(s, \mu)(x) D_p H(x, \nabla_w U(s, \mu)(x)) \\
+ \int_{\mathbb{R}^d} D^2_{ww} U(s, \mu)(x, y) D_p H(y, \nabla_w U(s, \mu)(y)) \, d\mu(y) = \nabla_w F(\mu)(x).
\]
Master equations from (HL-$\mathcal{P}_2$) – the vectorial case

$\rightarrow$ Take $\nabla_w$ of (HJB-$\mathcal{P}_2$) to get

$$
\partial_s \nabla_w U(s, \mu)(x) + D_x H(x, \nabla_w U(s, \mu)(x)) + D_x \nabla_w U(s, \mu)(x)D_p H(x, \nabla_w U(s, \mu)(x)) \\
+ \int_{\mathbb{R}^d} D^2_{ww} U(s, \mu)(x, y)D_p H(y, \nabla_w U(s, \mu)(y)) \, d\mu(y) = \nabla_w F(\mu)(x).
$$

$\rightarrow$ By setting $\mathcal{V}(s, x, \mu) := \nabla_w U(s, \mu)(x)$, this would correspond to the so-called ‘vectorial master equation’.

GOAL: obtain the necessary regularity on both $U$ and $u$ which let us justify the previous heuristic arguments!
Master equations from (HL-$\mathcal{P}_2$) – the vectorial case

→ Take $\nabla_w$ of (HJB-$\mathcal{P}_2$) to get

$$
\partial_s \nabla_w \mathcal{U}(s, \mu)(x) + D_x H(x, \nabla_w \mathcal{U}(s, \mu)(x)) + D_x \nabla_w \mathcal{U}(s, \mu)(x)D_p H(x, \nabla_w \mathcal{U}(s, \mu)(x))
$$

$$
+ \int_{\mathbb{R}^d} D_{ww} \mathcal{U}(s, \mu)(x, y)D_p H(y, \nabla_w \mathcal{U}(s, \mu)(y)) \, d\mu(y) = \nabla_w \mathcal{F} (\mu)(x).
$$

→ By setting $\mathcal{V}(s, x, \mu) := \nabla_w \mathcal{U}(s, \mu)(x)$, this would correspond to the so-called ‘vectorial master equation’.

→ Notice that $\mathcal{V}(s, \cdot, \mu)$ is only defined on spt(\mu).
Master equations from (HL-$\mathcal{P}_2$) – the vectorial case

→ Take $\nabla_w$ of (HJB-$\mathcal{P}_2$) to get

$$\partial_s \nabla_w U(s, \mu)(x) + D_x H(x, \nabla_w U(s, \mu)(x)) + D_x \nabla_w U(s, \mu)(x) D_p H(x, \nabla_w U(s, \mu)(x))$$

$$+ \int_{\mathbb{R}^d} D_{ww}^2 U(s, \mu)(x, y) D_p H(y, \nabla_w U(s, \mu)(y)) \, d\mu(y) = \nabla_w F(\mu)(x).$$

→ By setting $\mathcal{V}(s, x, \mu) := \nabla_w U(s, \mu)(x)$, this would correspond to the so-called ‘vectorial master equation’.

→ Notice that $\mathcal{V}(s, \cdot, \mu)$ is only defined on spt($\mu$).

→ By the previous connection between $D_x u$ and $\nabla_w U$, $D_x u$ would define a solution to the vectorial master equation which is defined for all $x \in \mathbb{R}^d$. 
Master equations from (HL-$\mathcal{P}_2$) – the vectorial case

→ Take $\nabla_w$ of (HJB-$\mathcal{P}_2$) to get

$$\partial_s \nabla_w \mathcal{U}(s, \mu)(x) + D_x H(x, \nabla_w \mathcal{U}(s, \mu)(x)) + D_x \nabla_w \mathcal{U}(s, \mu)(x)D_p H(x, \nabla_w \mathcal{U}(s, \mu)(x))$$

$$+ \int_{\mathbb{R}^d} D_{ww} \mathcal{U}(s, \mu)(x, y)D_p H(y, \nabla_w \mathcal{U}(s, \mu)(y)) \, d\mu(y) = \nabla_w \mathcal{F}(\mu)(x).$$

→ By setting $\mathcal{V}(s, x, \mu) := \nabla_w \mathcal{U}(s, \mu)(x)$, this would correspond to the so-called ‘vectorial master equation’.

→ Notice that $\mathcal{V}(s, \cdot, \mu)$ is only defined on $\text{spt}(\mu)$.

→ By the previous connection between $D_x u$ and $\nabla_w \mathcal{U}$, $D_x u$ would define a solution to the vectorial master equation which is defined for all $x \in \mathbb{R}^d$.

**GOAL:** obtain the necessary regularity on both $\mathcal{U}$ and $u$ which let us justify the previous heuristic arguments!
Recall, $\Omega = [0, 1)^d$ and $H = L^2(\Omega; \mathbb{R}^d)$. We define $\tilde{F}, \tilde{U}_0 : H \rightarrow \mathbb{R}$ and $\tilde{H}, \tilde{L} : H \times H \rightarrow \mathbb{R}$ as

$$\tilde{F}(X) := F(X_\# \mathcal{L}^d \downarrow \Omega) \text{ and } \tilde{U}_0(X) := U_0(X_\# \mathcal{L}^d \downarrow \Omega), \ \forall X \in H,$$
Lift of HJB from \((\mathcal{P}_2(\mathbb{R}^d), W_2)\) to \(\mathbb{H}\)

Recall, \(\Omega = [0, 1)^d\) and \(\mathbb{H} = L^2(\Omega; \mathbb{R}^d)\). We define \(\tilde{F}, \tilde{U}_0 : \mathbb{H} \to \mathbb{R}\) and \(\tilde{H}, \tilde{L} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}\) as

\[
\tilde{F}(X) := F(X#L^d \upharpoonright \Omega) \quad \text{and} \quad \tilde{U}_0(X) := U_0(X#L^d \upharpoonright \Omega), \quad \forall X \in \mathbb{H},
\]

and

\[
\tilde{H}(X, \xi) := \int_\Omega H(X(\omega), \xi(\omega)) \, d\omega \quad \text{and} \quad \tilde{L}(X, \zeta) = \int_\Omega L(X(\omega), \zeta(\omega)) \, d\omega.
\]

In fact, we have \(\mathcal{P}_2(\mathbb{R}^d) = \mathbb{H}/\sim\), where \(X \sim Y\), if \(X#L^d \upharpoonright \Omega = Y#L^d \upharpoonright \Omega\).
Lift of HJB from \( \mathcal{P}_2(\mathbb{R}^d), W_2 \) to \( \mathbb{H} \)

→ Recall, \( \Omega = [0, 1)^d \) and \( \mathbb{H} = L^2(\Omega; \mathbb{R}^d) \). We define \( \tilde{F}, \tilde{U}_0 : \mathbb{H} \to \mathbb{R} \) and \( \tilde{H}, \tilde{L} : \mathbb{H} \times \mathbb{H} \to \mathbb{R} \) as

\[
\tilde{F}(X) := F(X \# \mathcal{L}^d \sqsubset \Omega) \quad \text{and} \quad \tilde{U}_0(X) := U_0(X \# \mathcal{L}^d \sqsubset \Omega), \quad \forall \, X \in \mathbb{H},
\]

and

\[
\tilde{H}(X, \xi) := \int_{\Omega} H(X(\omega), \xi(\omega)) \, d\omega \quad \text{and} \quad \tilde{L}(X, \zeta) = \int_{\Omega} L(X(\omega), \zeta(\omega)) \, d\omega.
\]

→ In fact, we have \( \mathcal{P}_2(\mathbb{R}^d) = \mathbb{H}/\sim \), where \( X \sim Y \), if \( X \# \mathcal{L}^d \sqsubset \Omega = Y \# \mathcal{L}^d \sqsubset \Omega \).

→ Solve the optimal control problem

\[
\tilde{U}(t, X) := \inf \left\{ \tilde{U}_0(X_0) + \int_0^t \tilde{L}(X_s, \dot{X}_s) + \tilde{F}(X_s) \, ds : X_t = X \right\} \quad (\text{HL-}\mathbb{H})
\]
Lift of HJB from \((\mathcal{P}_2(\mathbb{R}^d), W_2)\) to \(\mathcal{H}\)

→ Recall, \(\Omega = [0, 1)^d\) and \(\mathcal{H} = L^2(\Omega; \mathbb{R}^d)\). We define \(\tilde{\mathcal{F}}, \tilde{U}_0 : \mathcal{H} \to \mathbb{R}\) and \(\tilde{\mathcal{H}}, \tilde{\mathcal{L}} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}\) as

\[
\tilde{\mathcal{F}}(X) := \mathcal{F}(X_\# \mathcal{L}^d \subseteq \Omega) \quad \text{and} \quad \tilde{U}_0(X) := U_0(X_\# \mathcal{L}^d \subseteq \Omega), \quad \forall X \in \mathcal{H},
\]

and

\[
\tilde{\mathcal{H}}(X, \xi) := \int_{\Omega} H(X(\omega), \xi(\omega)) \, d\omega \quad \text{and} \quad \tilde{\mathcal{L}}(X, \zeta) = \int_{\Omega} L(X(\omega), \zeta(\omega)) \, d\omega.
\]

→ In fact, we have \(\mathcal{P}_2(\mathbb{R}^d) = \mathcal{H}/\sim\), where \(X \sim Y\), if \(X_\# \mathcal{L}^d \subseteq \Omega = Y_\# \mathcal{L}^d \subseteq \Omega\).

→ Solve the optimal control problem

\[
\tilde{U}(t, X) := \inf \left\{ \tilde{U}_0(X_0) + \int_0^t \tilde{\mathcal{L}}(X_s, \dot{X}_s) + \tilde{\mathcal{F}}(X_s) \, ds : X_t = X \right\}
\]  

(\text{HL-\mathcal{H}})

→ Remark: since the data \(\tilde{\mathcal{F}}, \tilde{U}_0, \tilde{\mathcal{L}}\) are rearrangement invariants, so is \(\tilde{U}(t, \cdot)\) (this means \(\tilde{\mathcal{F}}(X) = \tilde{\mathcal{F}}(Y)\), whenever \(X \sim Y\)).
Important links between the control problems and HJB equations

→ Under reasonable assumptions ($\tilde{\mathcal{L}}$ is convex in the second variable, regular enough, bounded from below, $\tilde{F}, \tilde{U}_0$ are bounded below and regular), the control problem (HL-$\mathcal{H}$) has a solution (at least for short time).
Important links between the control problems and HJB equations

→ Under reasonable assumptions (\(\tilde{\mathcal{L}}\) is convex in the second variable, regular enough, bounded from below, \(\tilde{\mathcal{F}}, \tilde{\mathcal{U}}_0\) are bounded below and regular), the control problem (HL-H) has a solution (at least for short time).

→ The theory of Crandall-Lions ensures that \(\tilde{\mathcal{U}}\) is the unique locally Lipschitz continuous viscosity solution to the corresponding HJB equation

\[
\begin{align*}
\partial_t \tilde{\mathcal{U}}(t, X) + \tilde{\mathcal{H}}(X, \nabla \tilde{\mathcal{U}}(t, X)) &= \tilde{\mathcal{F}}(X), & \text{in } (0, T) \times \mathbb{H}, \\
\tilde{\mathcal{U}}(0, X) &= \tilde{\mathcal{U}}_0(X), & \text{in } \mathbb{H}.
\end{align*}
\]

(HJB-H)

→ Furthermore, we have the correspondence \(\tilde{\mathcal{U}}(t, X) = \mathcal{U}(t, \tilde{L}d \Omega)\).

→ And so, [Gangbo-Tudorascu, 2018] implies that \(\mathcal{U}\) is a viscosity solution to (HJB-H).

→ Moreover, \(\mathcal{U}(t, \cdot)\) is differentiable at \(\mu\) if and only if \(\tilde{\mathcal{U}}(t, \cdot)\) is differentiable at \(X\), for any \(X\) s.t. \(X \tilde{L}d \Omega\).

→ In this case \(\nabla \tilde{\mathcal{U}}(t, X) = \nabla \mathcal{U}(t, \mu) \circ X\).

→ A similar observation was made by Lions in his lectures.
Important links between the control problems and HJB equations

→ Under reasonable assumptions (\(\tilde{L}\) is convex in the second variable, regular enough, bounded from below, \(\tilde{F}, \tilde{U}_0\) are bounded below and regular), the control problem (HL-\(\mathbb{H}\)) has a solution (at least for short time).

→ The theory of Crandall-Lions ensures that \(\tilde{U}\) is the unique locally Lipschitz continuous viscosity solution to the corresponding HJB equation

\[
\begin{aligned}
\partial_t \tilde{U}(t, X) + \tilde{H}(X, \nabla \tilde{U}(t, X)) &= \tilde{F}(X), \quad \text{in } (0, T) \times \mathbb{H}, \\
\tilde{U}(0, X) &= \tilde{U}_0(X), \quad \text{in } \mathbb{H}.
\end{aligned}
\]

(HJB-\(\mathbb{H}\))

→ Under further suitable assumptions on the data, we have also that \(\tilde{U}(t, \cdot)\) is locally semi-concave (see for instance [Gomes-Nurbekyan, 2015]).
Important links between the control problems and HJB equations

→ Under reasonable assumptions (\(\mathcal{L}\) is convex in the second variable, regular enough, bounded from below, \(\tilde{\mathcal{F}}, \tilde{\mathcal{U}}_0\) are bounded below and regular), the control problem (HL-\(\mathcal{H}\)) has a solution (at least for short time).

→ The theory of Crandall-Lions ensures that \(\tilde{\mathcal{U}}\) is the unique locally Lipschitz continuous viscosity solution to the corresponding HJB equation

\[
\begin{cases}
\partial_t \tilde{U}(t, X) + \tilde{\mathcal{H}}(X, \nabla \tilde{U}(t, X)) = \tilde{\mathcal{F}}(X), & \text{in } (0, T) \times \mathcal{H}, \\
\tilde{U}(0, X) = \tilde{U}_0(X), & \text{in } \mathcal{H}.
\end{cases}
\]

(HJB-\(\mathcal{H}\))

→ Under further suitable assumptions on the data, we have also that \(\tilde{U}(t, \cdot)\) is locally semi-concave (see for instance [Gomes-Nurbekyan, 2015]).

→ Furthermore, we have the correspondence \(\tilde{U}(t, X) = U(t, X \# \mathcal{L}^d \sqcap \Omega)\).
Important links between the control problems and HJB equations

→ Under reasonable assumptions ($\tilde{\mathcal{L}}$ is convex in the second variable, regular enough, bounded from below, $\tilde{\mathcal{F}}, \tilde{\mathcal{U}}_0$ are bounded below and regular), the control problem (HL-$\mathbb{H}$) has a solution (at least for short time).

→ The theory of Crandall-Lions ensures that $\tilde{\mathcal{U}}$ is the unique locally Lipschitz continuous viscosity solution to the corresponding HJB equation

$$
\begin{aligned}
\partial_t \tilde{\mathcal{U}}(t, X) + \tilde{\mathcal{H}}(X, \nabla \tilde{\mathcal{U}}(t, X)) = \tilde{\mathcal{F}}(X), & \quad \text{in } (0, T) \times \mathbb{H}, \\
\tilde{\mathcal{U}}(0, X) = \tilde{\mathcal{U}}_0(X), & \quad \text{in } \mathbb{H}.
\end{aligned}
$$

(HJB-$\mathbb{H}$)

→ Under further suitable assumptions on the data, we have also that $\tilde{\mathcal{U}}(t, \cdot)$ is locally semi-concave (see for instance [Gomes-Nurbekyan, 2015]).

→ Furthermore, we have the correspondence $\tilde{\mathcal{U}}(t, X) = \mathcal{U}(t, X \# \mathcal{L}^d \subset \Omega)$.

→ And so, [Gangbo-Tudorascu, 2018] implies that $\mathcal{U}$ is a viscosity solution to (HJB-$\mathcal{P}_2$).
Important links between the control problems and HJB equations

→ Under reasonable assumptions ($\tilde{L}$ is convex in the second variable, regular enough, bounded from below, $\tilde{F}, \tilde{U}_0$ are bounded below and regular), the control problem (HL-\(H\)) has a solution (at least for short time).

→ The theory of Crandall-Lions ensures that $\tilde{U}$ is the unique locally Lipschitz continuous viscosity solution to the corresponding HJB equation

\[
\begin{aligned}
\partial_t \tilde{U}(t, X) + \tilde{\mathcal{H}}(X, \nabla \tilde{U}(t, X)) &= \tilde{F}(X), & \text{in } (0, T) \times \mathbb{H}, \\
\tilde{U}(0, X) &= \tilde{U}_0(X), & \text{in } \mathbb{H}.
\end{aligned}
\]  

(HJB-\(H\))

→ Under further suitable assumptions on the data, we have also that $\tilde{U}(t, \cdot)$ is locally semi-concave (see for instance [Gomes-Nurbekyan, 2015]).

→ Furthermore, we have the correspondence $\tilde{U}(t, X) = U(t, X \# L^d \sqsubset \Omega)$.

→ And so, [Gangbo-Tudorascu, 2018] implies that $U$ is a viscosity solution to (HJB-\(P_2\)). Moreover, $U(t, \cdot)$ is differentiable at $\mu$ if and only if $\tilde{U}(t, \cdot)$ is differentiable at $X$, for any $X$ s.t. $X \# L^d \sqsubset \Omega$. 
Important links between the control problems and HJB equations

→ Under reasonable assumptions ($\tilde{\mathcal{L}}$ is convex in the second variable, regular enough, bounded from below, $\tilde{\mathcal{F}}, \tilde{\mathcal{U}}_0$ are bounded below and regular), the control problem (HL-$\mathcal{H}$) has a solution (at least for short time).

→ The theory of Crandall-Lions ensures that $\tilde{\mathcal{U}}$ is the unique locally Lipschitz continuous viscosity solution to the corresponding HJB equation

$$\begin{cases}
\partial_t \tilde{\mathcal{U}}(t, X) + \tilde{\mathcal{H}}(X, \nabla \tilde{\mathcal{U}}(t, X)) = \tilde{\mathcal{F}}(X), & \text{in } (0, T) \times \mathcal{H}, \\
\tilde{\mathcal{U}}(0, X) = \tilde{\mathcal{U}}_0(X), & \text{in } \mathcal{H}.
\end{cases}$$

(HJB-$\mathcal{H}$)

→ Under further suitable assumptions on the data, we have also that $\tilde{\mathcal{U}}(t, \cdot)$ is locally semi-concave (see for instance [Gomes-Nurbekyan, 2015]).

→ Furthermore, we have the correspondence $\tilde{\mathcal{U}}(t, X) = \mathcal{U}(t, X \# \mathcal{L}^d \subset \Omega)$.

→ And so, [Gangbo-Tudorascu, 2018] implies that $\mathcal{U}$ is a viscosity solution to (HJB-$\mathcal{P}_2$). Moreover, $\mathcal{U}(t, \cdot)$ is differentiable at $\mu$ if and only if $\tilde{\mathcal{U}}(t, \cdot)$ is differentiable at $X$, for any $X$ s.t. $X \# \mathcal{L}^d \subset \Omega$. In this case

$$\nabla \tilde{\mathcal{U}}(t, X) = \nabla_w \mathcal{U}(t, \mu) \circ X.$$
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\end{array} \right. 
\end{aligned}$$

in $(0, T) \times \mathbb{H}, \tilde{\mathcal{U}}(0, X) = \tilde{\mathcal{U}}_0(X)$, in $\mathbb{H}$. (HJB-$\mathbb{H}$)

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$$\nabla \tilde{\mathcal{U}}(t, X) = \nabla_w \mathcal{U}(t, \mu) \circ X.$$

→ A similar observation was made by Lions in his lectures.
Further regularity of $\tilde{U}(t, \cdot)$ for arbitrary time horizon

→ Innocent observation: if in addition $\tilde{u}_0$ and $\tilde{L} + \tilde{F}$ are convex, then so is $\tilde{U}(t, \cdot)$ for all $t \in [0, T]$. 
Further regularity of $\tilde{U}(t, \cdot)$ for arbitrary time horizon

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→ Fact: if $\tilde{V} : H \to \mathbb{R}$ is both locally semi-concave and locally semi-convex, then it is of class $C_{\text{loc}}^{1,1}(H)$. 

Semi-convexity: $\exists \lambda \in \mathbb{R}$ such that for all $s \in [0, 1]$ 

$$\tilde{V}((1-s)X + sY) \leq (1-s)\tilde{V}(X) + s\tilde{V}(Y) - \lambda s(1-s)\|X - Y\|^2.$$ 

$C_{\text{loc}}^{1,1}(H)$: $\tilde{V}$ is Fréchet differentiable and there exists $\lambda > 0$ such that for all $X, Y \in H$ 

$$\left|\left|\tilde{V}(Y) - \tilde{V}(X) - \int_{\Omega} \nabla \tilde{V}(X)(\omega) \cdot (Y(\omega) - X(\omega))\right|\right| \leq \lambda \|X - Y\|^2$$ 

or equivalently 

$$\|\nabla \tilde{V}(X) - \nabla \tilde{V}(Y)\| \leq \lambda \|X - Y\|.$$ 

And so, in this setting one can obtain $\tilde{U}(t, \cdot) \in C_{\text{loc}}^{1,1}(H)$. 

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Further regularity of $\tilde{U}(t, \cdot)$ for arbitrary time horizon

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→ And so, in this setting one can obtain $\tilde{U}(t, \cdot) \in C_{\text{loc}}^{1,1}(\mathbb{H})$. 

Further regularity of $U(t, \cdot)$

Theorem (Gangbo-M., 2020)

Let $V : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and let $\tilde{V} : \mathbb{H} \to \mathbb{R}$ be its lift. Then $\tilde{V} \in C^{1,1}_{loc}(\mathbb{H})$ if and only if $V \in C^{1,1}_{loc}(\mathcal{P}_2(\mathbb{R}^d))$. 

Here, we say that $V \in C^{1,1}_{loc}(\mathcal{P}_2(\mathbb{R}^d))$, if $V$ is differentiable and there exists $\lambda > 0$ such that

\[(1) \quad \left| \left| \left| \left| V(\nu) - V(\mu) - \int_{\mathbb{R}^d} \nabla w V(\mu)(x) \cdot (y-x) \, d\gamma(x,y) \right| \right| \right| \leq \lambda \left( W^2(\mu,\nu) \right), \quad \forall \gamma \in \Pi^{o}(\mu,\nu), \]

and

\[(2) \quad \text{spt}(\mu) \ni x \mapsto \nabla w V(\mu)(x) \text{ is } \lambda \text{-Lipschitz (independently of } \mu). \]

So, in this convex setting, one can obtain that $U(t, \cdot) \in C^{1,1}_{loc}(\mathcal{P}_2(\mathbb{R}^d))$. 

Further regularity of $\mathcal{U}(t, \cdot)$

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(1) \[ \left| V(\nu) - V(\mu) - \int_{\mathbb{R}^{2d}} \nabla_w V(\mu)(x) \cdot (y - x) \, d\gamma(x, y) \right| \leq \frac{\lambda}{2} W_2^2(\mu, \nu), \ \forall \gamma \in \Pi_o(\mu, \nu) \]

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Further regularity of $U(t, \cdot)$

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Let $V : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and let $\tilde{V} : \tilde{\mathbb{H}} \to \mathbb{R}$ be its lift. Then $\tilde{V} \in C^{1,1}_{\text{loc}}(\tilde{\mathbb{H}})$ if and only if $V \in C^{1,1}_{\text{loc}}(\mathcal{P}_2(\mathbb{R}^d))$.

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\text{spt}(\mu) \ni x \mapsto \nabla_w V(\mu)(x) \text{ is } \lambda\text{-Lipschitz (independently of } \mu). \quad \rightarrow \quad \text{So, in this convex setting, one can obtain that } U(t, \cdot) \in C^{1,1}_{\text{loc}}(\mathcal{P}_2(\mathbb{R}^d)).
\end{equation}
Correspondence of convexities

Theorem (Gangbo-M., 2020)

Let $V : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and let $\tilde{V} : \mathbb{H} \to \mathbb{R}$ be its lift. Then $\tilde{V}$ continuous is locally semi-convex if and only if $V$ is locally displacement semi-convex.

Typical examples of coupling functions in MFG:

$$F(\mu) = \int_{\mathbb{R}^d} \phi_0(x) \, d\mu(x) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_1(x-y) \, d\mu(x) \, d\mu(y),$$

for $\phi_0, \phi_1 : \mathbb{R}^d \to \mathbb{R}$ smooth. If $\phi_0$ and $\phi_1$ are $\lambda$–convex, then $F$ is displacement semi-convex.
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Here we say that $V$ is displacement (or geodesically) semi-convex (see [McCann, 1997], [Ambrosio-Gigli-Savaré, 2005]) if there exists $\lambda \in \mathbb{R}$ such that for any $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $[0, 1] \ni s \mapsto \mu_s := ((1 - s)x + sy)_\# \gamma$, ($\gamma \in \Pi_o(\mu, \nu)$)

$$V(\mu_s) \leq (1 - s)V(\mu) + sV(\nu) - \frac{\lambda}{2}s(1 - s)W_2^2(\mu, \nu).$$
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$$\mathcal{F}(\mu) = \int_{\mathbb{R}^d} \varphi_0(x) \, d\mu(x) + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_1(x - y) \, d\mu(x) \, d\mu(y),$$

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for \( \varphi_0, \varphi_1 : \mathbb{R}^d \to \mathbb{R} \) smooth.

→ If \( \varphi_0 \) and \( \varphi_1 \) are \( \lambda \)-convex, then \( \mathcal{F} \) is displacement semi-convex.
Displacement convexity vs monotonocity à la Lasry-Lions

Previously, in all works on the global well-posedness of master equation in the literature, it was assumed the monotonocity condition

$$\int_{\mathbb{R}^d} [f(x, \mu) - f(x, \nu)] \, d(\mu - \nu)(x) \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^d} [u_0(x, \mu) - u_0(x, \nu)] \, d(\mu - \nu)(x) \geq 0.$$
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→ In the potential game case, i.e. if \( \nabla_w \mathcal{F}(\mu)(x) = D_x f(x, \mu) \) and \( \nabla_w \mathcal{U}_0(\mu)(x) = D_x u_0(x, \mu) \) for some \( \mathcal{F}, \mathcal{U}_0 : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \), this monotonicity is equivalent to the convexity of \( \mathcal{F}, \mathcal{U}_0 \) along classical convex interpolations of measures, i.e. \([0, 1] \ni s \mapsto (1 - s)\mu + s\nu\).
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Let \( F(\mu) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_1(x - y) \, d\mu(x) \, d\mu(y) \) such that \( \varphi_1 \) is even. Clearly, if we set \( f(x, \mu) := \delta_\mu F(\mu)(x) = (\varphi_1 * \mu)(x) \), we have that
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D_x f(x, \mu) = \nabla_w F(\mu)(x) = (D\varphi_1 * \mu)(x).
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Lemma

Let \( \varphi_1 \in L^1(\mathbb{R}^d) \). \( f \) is monotone in the sense of Lasry-Lions if and only if the Fourier transform of \( \varphi_1 \) is nonnegative.
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Let \( F(\mu) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_1(x - y) \, d\mu(x) \, d\mu(y) \) such that \( \varphi_1 \) is even. Clearly, if we set \( f(x, \mu) := \delta_\mu F(\mu)(x) = (\varphi_1 * \mu)(x) \), we have that
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\]

**Lemma**

Let \( \varphi_1 \in L^1(\mathbb{R}^d) \). \( f \) is monotone in the sense of Lasry-Lions if and only if the Fourier transform of \( \varphi_1 \) is nonnegative. As a consequence, there are \( \varphi_1 \) such that \( F \) is displacement convex, but \( f \) fails to be monotone.
Higher regularity of $\mathcal{U}(t, \cdot)$?

$\rightarrow$ The $C^{1,1}$ regularity of $\tilde{\mathcal{U}}(t, \cdot)$ or $\mathcal{U}(t, \cdot)$ is **not enough** to obtain classical well-posedness of master equations.
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→ Using purely Hilbert space calculus, one would be able to obtain higher regularity of $\tilde{\mathcal{U}}$, by using its representation via the Hamiltonian flow.
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→ However, this would require to impose $\tilde{\mathcal{H}}, \tilde{\mathcal{F}}, \tilde{\mathcal{U}}_0$ to be of class $C^2$ (in the Fréchet sense) or better.
Higher regularity of $U(t, \cdot)$?

$\rightarrow$ The $C^{1,1}$ regularity of $\tilde{U}(t, \cdot)$ or $U(t, \cdot)$ is not enough to obtain classical well-posedness of master equations.

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$\rightarrow$ However, this would require to impose $\tilde{H}, \tilde{F}, \tilde{U}_0$ to be of class $C^2$ (in the Fréchet sense) or better.

$\rightarrow$ Surprisingly, such regularity assumption might be too restrictive.
Hilbert space regularity is too restrictive for the study of MFG

Let $\Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and let $\tilde{\Phi} \in C^2(\mathbb{H})$ be its lift.
Hilbert space regularity is too restrictive for the study of MFG

Let \( \Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) and let \( \tilde{\Phi} \in C^2(\mathbb{H}) \) be its lift. In this case, we have the special representation of the Hessian:

\[
\nabla^2 \tilde{\Phi}(X)(h, h_*) = \int_{\Omega} D_x(\nabla_w \Phi(\mu)) \circ X h \cdot h_* d\omega \\
+ \int_{\Omega^2} \nabla^2_{ww} \Phi(\mu)(X(\omega), X(\omega_*)) h(\omega) \cdot h_*(\omega_*) d\omega d\omega_*
\]

if \( \xi, \xi_* \in \text{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d) \) and \( h = \xi \circ X \) and \( h_* = \xi_* \circ X \).
Hilbert space regularity is too restrictive for the study of MFG

Let $\Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and let $\tilde{\Phi} \in C^2(\mathbb{H})$ be its lift. In this case, we have the special representation of the Hessian:

$$\nabla^2 \tilde{\Phi}(X)(h, h_*) = \int_\Omega D_x(\nabla_w \Phi(\mu)) \circ X h \cdot h_* d\omega$$

$$+ \int_{\Omega^2} \nabla^2_{ww} \Phi(\mu)(X(\omega), X(\omega_*)) h(\omega) \cdot h_*(\omega_*) d\omega d\omega_*$$

if $\xi, \xi_* \in \text{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ and $h = \xi \circ X$ and $h_* = \xi_* \circ X$.

Lemma (Gangbo-M., 2020)

Let $\alpha \in (0, 1]$ and assume $\tilde{\Phi} \in C^2_{\text{loc}}(\mathbb{H})$ is rearrangement invariant so that it is the lift of a function $\Phi$. If (2) holds for all $h, h_* \in \mathbb{H}$ then $D_x(\nabla_w \Phi(\mu)(\cdot))$ is a constant function on $\text{spt}(\mu)$. 
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Lemma (Gangbo-M., 2020)

Let $\alpha \in (0, 1]$ and assume $\tilde{\Phi} \in C^{2,\alpha}_{\text{loc}}(\mathbb{H})$ is rearrangement invariant so that it is the lift of a function $\Phi$. If (2) holds for all $h, h_* \in \mathbb{H}$ then $D_x(\nabla_w \Phi(\mu)(\cdot))$ is a constant function on $\text{spt}(\mu)$.

Corollary: if $\tilde{\Phi}_g^{(k)}(X) := \int_{\Omega^k} g(X(\omega_1), \cdots, X(\omega_k)) d\omega_1 \cdots d\omega_k \forall X \in \mathbb{H}$, then $\tilde{\Phi}_g^{(k)} \in C^{2,\alpha}_{\text{loc}}(\mathbb{H})$ if and only if $g^{(k)}$ is a polynomial of degree at most 2.
Further consequences

→ Actually, the previous corollary for locally representable functions \( \Phi_g^{(k)} \) holds even for the class \( C^2(\mathbb{H}) \) (instead of \( C^2_{\text{loc}}(\mathbb{H}) \)).
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→ We underline that such results would stand also for $\tilde{\mathcal{H}}$. Since this always has a local representation, imposing $C^2$ regularity on $\mathbb{H}$, would yield that $H$ is a polynomial of degree at most 2.
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→ Abandoning the Hilbert space technique, we worked out a method to obtain $U(t, \cdot) \in C^2_{\text{loc}}$ in an intrinsic way, working directly on $(\mathcal{P}_2(\mathbb{R}^d), W_2)$. 
Further consequences

→ Actually, the previous corollary for locally representable functions \( \Phi^{(k)}_g \) holds even for the class \( C^2(\mathbb{H}) \) (instead of \( (C^2_{\text{loc}}(\mathbb{H})) \)).

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→ Abandoning the Hilbert space technique, we worked out a method to obtain \( \mathcal{U}(t, \cdot) \in C^2_{\text{loc}, \alpha, w} \) in an intrinsic way, working directly on \( (\mathcal{P}_2(\mathbb{R}^d), W_2) \).

**Theorem (Gangbo-M., 2020)**

Let \( f, u_0 \) and \( \mathcal{F}, \mathcal{U}_0 \) are such that \( D_xf = \nabla_w F \) and \( D_xu_0 = \nabla_w U_0 \). Let moreover \( \mathcal{F}, \mathcal{U}_0 \) be of class \( C^{2,1,w}_{\text{loc}} \), \( \mathcal{U}_0 \) and \( \mathcal{L} + \mathcal{F} \) displacement convex and let \( L, H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be \( C^3 \). Then there exists a unique, global in time classical solution \( \mathcal{U} \) to the equation \( (\text{HJB-} \mathcal{P}_2) \) which is such that \( \mathcal{U}(t, \cdot) \in C^{2,1,w}_{\text{loc}} \). Moreover, there exists a unique global in time classical solution \( u \in C^{1,1}_{\text{loc}} ([0, +\infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \) to (Master) and \( D_xu(t, \cdot, \mu)(\cdot) = \nabla_w \mathcal{U}(t, \mu)(\cdot) \) on \( \text{spt}(\mu) \).
\( C^{2,1,w}_{\text{loc}} \) functions on \((\mathcal{P}_2(\mathbb{R}^d), W_2)\)

We propose the following (inspired by [Chow-Gangbo, JDE 2019])

**Definition**

Let \( \mathcal{B} \subseteq \mathcal{P}_2(\mathbb{R}^d) \) be open and convex. We say that \( \mathcal{U} \in C^{2,1,w}(\mathcal{B}) \), if \( \mathcal{U} \in C^1(\mathcal{B}) \), and if there exist \( \Lambda_0 : \mathbb{R}^d \times \mathcal{B} \rightarrow \mathbb{R}^{d \times d} \) and \( \Lambda_1 : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{B} \rightarrow \mathbb{R}^{d \times d} \) such that \( \Lambda_0 \in L^\infty(\mathbb{R}^d; \mu) \), \( \Lambda_1 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mu \otimes \mu) \) and there exists a constant \( C > 0 \) such that

\[
\begin{align*}
&\left| \nabla_w \mathcal{U}(\nu)(y) - \nabla_w \mathcal{U}(\mu)(x) - \Lambda_0(x, \mu)(y - x) - \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \Lambda_1(x, a, \mu)(b - a) \, d\gamma(a, b) \right| \\
&\quad \leq C \left( |x - y|^2 + W_2^2(\mu, \nu) \right),
\end{align*}
\]

for all \( \mu, \nu \in \mathcal{B} \), \( \gamma \in \Pi_o(\mu, \nu) \) and \( (x, y) \in \text{spt}(\mu) \times \text{spt}(\nu) \).

(2) \( \Lambda_0 \) and \( \Lambda_1 \) are **Lipschitz continuous**, i.e. there exists \( C > 0 \) such that

\[
|\Lambda_0(x, \mu) - \Lambda_0(y, \mu)|_{\infty} \leq C(|x - y| + W_2(\mu, \nu))
\]

and \( |\Lambda_1(x_1, x_2, \mu) - \Lambda_1(y_1, y_2, \nu)|_{\infty} \leq C(|x_1 - y_1| + |x_2 - y_2| + W_2(\mu, \nu)) \), for any \( \mu, \nu \in \mathcal{B} \) and \( (x, y), (x_1, y_1), (x_2, y_2) \in \text{spt}(\mu) \times \text{spt}(\nu) \).
In the previous definition $C^{2,1}_{\text{loc}}(\mathcal{P}_2(\mathbb{R}^d))$ means that it is satisfied on each $B_r$. Let $U: \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be defined as

$$U(\mu) = \int \nabla w \cdot \nabla \phi_0 \, d\mu + \frac{1}{2} \int \nabla w \cdot \nabla \phi_1 \, d\mu \otimes d\mu,$$

for $\phi_0, \phi_1 : \mathbb{R}^d \to \mathbb{R}$ of class $C^{2,1}_{\text{loc}}(\mathbb{R}^d)$ such that both of them have at most quadratic growth at infinity and bounded second order derivatives. Let moreover $\phi_1$ be even. Then, $U \in C^{2,1}_{\text{loc}}(\mathcal{P}_2(\mathbb{R}^d))$. Let us notice that we have

$$\nabla U(\mu)(x) = D\phi_0(x) + (D\phi_1 \ast \mu)(x)$$

and

$$\nabla^2 U(\mu)(x) = D^2\phi_0(x) + (D^2\phi_1 \ast \mu)(x); \quad \nabla^2 w U(\mu)(x,y) = D^2\phi_1(x-y).$$

Notice that in this case $\Lambda_0(x,\mu) = \nabla U(\mu)(x)$ and $\Lambda_1(x,y,\mu) = \nabla^2 w U(\mu)(x,y)$. 

Examples
→ In the previous definition $C^{2,1,w}_{\text{loc}}(\mathcal{P}_2(\mathbb{R}^d))$ means that it is satisfied on each $B_r$!

Let $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be defined as

$$
\mathcal{U}(\mu) = \int_{\mathbb{R}^d} \varphi_0(x) \, d\mu(x) + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_1(x - y) \, d\mu(x) \, d\mu(y),
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for $\varphi_0, \varphi_0 : \mathbb{R}^d \to \mathbb{R}$ of class $C^{2,1}_{\text{loc}}(\mathbb{R}^d)$ such that both of them have at most quadratic growth at infinity and bounded second order derivatives. Let moreover $\varphi_1$ be even.

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$$\nabla_w \mathcal{U}(\mu)(x) = D\varphi_0(x) + (D\varphi_1 * \mu)(x)$$

and

$$D_x \nabla_w \mathcal{U}(\mu)(x) = D^2\varphi_0(x) + (D^2\varphi_1 * \mu)(x); \quad D_{ww} \mathcal{U}(\mu)(x,y) = D^2\varphi_1(x-y).$$
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Notice that in this case $\Lambda_0(x, \mu) = D_x \nabla_w U(\mu)(x)$ and $\Lambda_1(x, y, \mu) = D_{ww}^2 U(\mu)(x, y)$. 
Strategy of the proof of our main theorem

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→ Let $m \in \mathbb{N}$, let $x = (x_1, \ldots, x_m) \in (\mathbb{R}^d)^m$ and let us define $\mu_x^{(m)} := \frac{1}{m} \sum_{i=1}^{m} \delta_{x_i}$. 
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→ Let $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and define $U^{(m)}(x_1, \ldots, x_m) := \mathcal{U}(\mu_x^{(m)})$. 
We work via a finite dimensional approximation. Let \( m \in \mathbb{N} \), let \( x = (x_1, \ldots, x_m) \in (\mathbb{R}^d)^m \) and let us define \( \mu_x^{(m)} := \frac{1}{m} \sum_{i=1}^{m} \delta_{x_i} \). Let \( \mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) and define \( U^{(m)}(x_1, \ldots, x_m) := \mathcal{U}(\mu_x^{(m)}) \). We notice that if \( \mathcal{U} \in C^{2,1,w}_{\text{loc}}(\mathcal{P}_2(\mathbb{R}^d)) \), then \( U^{(m)} \in C^{2,1}_{\text{loc}}((\mathbb{R}^d)^m) \) and we have the correspondences

\[
\nabla_w \mathcal{U}(\mu_x^{(m)})(x_i) = mD_{x_i} U^{(m)}(x),
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We work via a finite dimensional approximation.

Let $m \in \mathbb{N}$, let $x = (x_1, \ldots, x_m) \in (\mathbb{R}^d)^m$ and let us define $\mu^{(m)}_x := \frac{1}{m} \sum_{i=1}^m \delta_{x_i}$.

Let $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and define $U^{(m)}(x_1, \ldots, x_m) := \mathcal{U}(\mu^{(m)}_x)$.

We notice that if $\mathcal{U} \in C^{2,1,w}_{loc}(\mathcal{P}_2(\mathbb{R}^d))$, then $U^{(m)} \in C^{2,1}_{loc}((\mathbb{R}^d)^m)$ and we have the correspondences

$$\nabla_w \mathcal{U}(\mu^{(m)}_x)(x_i) = mD_{x_i} \mathcal{U}^{(m)}(x),$$

and

$$D^2_{x_i,x_j} \mathcal{U}^{(m)}(x) = \begin{cases} \frac{1}{m^2} D_{ww} \mathcal{U}(\mu^{(m)}_x)(x_i, x_j), & i \neq j, \\ \frac{1}{m} D_x \nabla_w \mathcal{U}(\mu^{(m)}_x)(x_i) + \frac{1}{m^2} D_{ww} \mathcal{U}(\mu^{(m)}_x)(x_i, x_i), & i = j. \end{cases}$$
From (HJB-$\mathcal{P}_2$) to (HJ-$(\mathbb{R}^d)^m$)

Lemma

Let the data be as in our theorem and let $U \in C^{1,1}_{\text{loc}}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ be a classical solution to (HJB-$\mathcal{P}_2$). Let $m \in \mathbb{N}$ and define $U^{(m)} : [0, T] \times (\mathbb{R}^d)^m \to \mathbb{R}$ be defined as $U^{(m)}(t, x) = \mathcal{U}(t, \mu^{(m)}_x)$. Then $U^{(m)}$ is of class $C^{1,1}_{\text{loc}}$ and the unique classical solution of

$$\begin{cases}
\partial_t U^{(m)}(t, x) + H^{(m)}(x, D_x U^{(m)}(t, x)) = F^{(m)}(x), & \text{in } (0, T) \times (\mathbb{R}^d)^m, \\
U^{(m)}(0, x) = U^{(m)}_0(x), & \text{in } (\mathbb{R}^d)^m,
\end{cases}$$

where $F^{(m)}(x) := \mathcal{F}(\mu^{(m)}_x)$, $U^{(m)}_0(x) := \mathcal{U}_0(\mu^{(m)}_x)$ and

$$H^{(m)}(x_1, \ldots, x_m, p_1, \ldots, p_m) := \frac{1}{m} \sum_{i=1}^m H(x_i, mp_i).$$
From (HJB-\(\mathcal{P}_2\)) to (HJ-(\(\mathbb{R}^d\))^m)

**Lemma**

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\begin{align*}
\partial_t U^{(m)}(t, x) + H^{(m)}(x, D_x U^{(m)}(t, x)) &= F^{(m)}(x), & \text{in } (0, T) \times (\mathbb{R}^d)^m, \\
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\end{align*}
\]

(\(\text{HJ-}(\mathbb{R}^d)^m\))

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\[H^{(m)}(x_1, \ldots, x_m, p_1, \ldots, p_m) := \frac{1}{m} \sum_{i=1}^{m} H(x_i, mp_i).\]

**Corollary**

As a consequence of the classical theory, one has also that \(U^{(m)} \in C^{2,1}_{\text{loc}}([0, T] \times (\mathbb{R}^d)^m)!\)
How to deduce the desired $C^{2,1}_\text{loc}^w$ properties of $U$?

We need fine quantitative estimates on derivatives of $U^{(m)}$!

**Theorem**

*Under the assumptions of our main theorem, we have that the solution $U^{(m)}$ of $(\text{HJ-}(\mathbb{R}^d)^m)$ satisfies the following. For all $t \in [0, T]$ and $r > 0$ there exists $C = C(t, r) > 0$ such that for all $x \in B^m_r$ we have*

$$
|D^2_{x_i x_j} U^{(m)}(t, x)|_\infty \leq \begin{cases} 
& \frac{C}{m}, \quad i = j; \\
& \frac{C}{m^2}, \quad i \neq j;
\end{cases} \quad (3)
$$

*and*

$$
|D^3_{x_i x_j x_k} U^{(m)}(t, x)|_\infty \leq \begin{cases} 
& \frac{C}{m}, \quad i = j = k; \\
& \frac{C}{m^2}, \quad i = j \neq k, i \neq j = k, i = k \neq j; \\
& \frac{C}{m^3}, \quad i \neq j \neq k,
\end{cases} \quad (4)
$$

*where $B^m_r := \{x \in (\mathbb{R}^d)^m : \frac{1}{m} \sum_{i=1}^m |x_i|^2 \leq r^2\}$.*
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→ We derive the regularity estimates on the associated finite dimensional Hamiltonian flow.

\[
\begin{align*}
\dot{Q}_i(s, x) &= D_pH(Q_i(s, x), mP_i(s, x)), \\
\dot{P}_i(s, x) &= -\frac{1}{m}D_xH(Q_i(s, x), mP_i(s, x)) + D_xF^{(m)}(Q_1(s, x), \ldots, Q_m(s, x)), \\
Q_i(0, x) &= x_i, \\
P_i(0, x) &= D_xU_0^{(m)}(x_1, \ldots, x_m),
\end{align*}
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How do we obtain such fine quantitative estimates?

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Q_i(0,x) &= x_i, \ P_i(0,x) = D_{x_i} U^{(m)}_0(x_1, \ldots, x_m),
\end{align*}
\]

Since $U^{(m)}$ is of class $C^{1,1}_{\text{loc}}$, we have

→ $P_i(s,x) = D_{x_i} U^{(m)}(s, Q_1(s,x), \ldots, Q_m(s,x))$. 
How do we obtain such fine quantitative estimates?

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Q_i(0, x) &= x_i, & P_i(0, x) = D_{x_i}U^{(m)}_0(x_1, \ldots, x_m),
\end{align*}
\]

Since \( U^{(m)} \) is of class \( C_{\text{loc}}^{1,1} \), we have

→ \( P_i(s, x) = D_{x_i}U^{(m)}(s, Q_1(s, x), \ldots, Q_m(s, x)) \).
→ Therefore, regularity estimates on derivatives of \( P(s, \cdot) \) and \( Q^{-1}(s, \cdot) \) will give the required estimates on \( U^{(m)}(t, \cdot) \).
From \((HJ-(\mathbb{R}^d)^m)) to \((HJB-\mathcal{P}_2)) and to \((\text{Master}))

**Theorem**

Suppose that the solution \(U^{(m)}\) to \((HJ-(\mathbb{R}^d)^m)) satisfies the fine quantitative derivative estimates up to order 3. Then \(U\), the solution to \((HJB-\mathcal{P}_2)\), is of such that \(U(t, \cdot) \in C^{2,1,w}_{\text{loc}}(\mathcal{P}_2(\mathbb{R}^d))\).
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This regularity is enough to deduce the existence of a **global classical solution to** the vectorial master equation.
Theorem

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From (HJ-(\(\mathbb{R}^d\))^m) to (HJB-P_2) and to (Master)

**Theorem**

*Suppose that the solution \(U^{(m)}\) to (HJ-(\(\mathbb{R}^d\))^m) satisfies the fine quantitative derivative estimates up to order 3. Then \(\mathcal{U}\), the solution to (HJB-P_2), is of such that \(\mathcal{U}(t, \cdot) \in C^{2,1,1}_{loc}(P_2(\mathbb{R}^d))\).*

\[\rightarrow\] This regularity is enough to deduce the existence of a **global classical solution** to the vectorial master equation.

\[\rightarrow\] The regularity of \(u(t, x, \cdot)\) does not follow immediately from these arguments! Need to have the regularity of \(\mu \mapsto \sigma_s (s \in (0, t))\), where \(\sigma_t = \mu\) and

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→ One needs to perform a ‘new’ discretization argument to show this remaining regularity!
From \((HJ-(\mathbb{R}^d)^m))\) to \((HJB-\mathcal{P}_2)\) and to \((Master)\)

**Theorem**

*Suppose that the solution \(U^{(m)}(m)\) to \((HJ-(\mathbb{R}^d)^m)\) satisfies the fine quantitative derivative estimates up to order 3. Then \(U\), the solution to \((HJB-\mathcal{P}_2)\), is of such that \(U(t, \cdot) \in C^{2,1}_{loc} (\mathcal{P}_2(\mathbb{R}^d)).\)*

→ This regularity is enough to deduce the existence of a **global classical solution** to the vectorial master equation.

→ The regularity of \(u(t, x, \cdot)\) does not follow immediately from these arguments! Need to have the regularity of \(\mu \mapsto \sigma_s (s \in (0, t))\), where \(\sigma_t = \mu\) and \(\partial_s \sigma_s + \nabla \cdot (\sigma_s \nabla w U(s, \sigma_s)(\cdot)) = 0.\)

→ One needs to perform a ‘new’ discretization argument to show this remaining regularity!

→ As a conclusion, we obtain \(u \in C^{1,1}_{loc} ([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))\) is a classical solution to \((Master)\). The uniqueness follows from the (strict) displacement convexity of the data!
Thank you for your attention!