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Homogenization of a stationary mean-field game via two-scale convergence

Rita Ferreira (in collaboration with Diogo Gomes & Xianjin Yang)

Workshop III: Mean Field Games and Applications (UCLA, May 2020)

www.kaust.edu.sa

Given:

$P \in \mathbb{R}^d$ — preferred direction of motion

$V : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ smooth, $\mathbb{Z}^d \times \mathbb{Z}^d$-periodic — potential (spacial preferences)

$H : \mathbb{R}^d \to \mathbb{R}$, $H(p) = \frac{1}{2}|p|^2$ — Hamiltonian (cost function)

$g : \mathbb{R}^+ \to \mathbb{R}$, $g(m) = \ln(m)$ — coupling (interactions)

$\epsilon > 0$ — length-scale of heterogeneities

Problem: asymptotic behavior as $\varepsilon \to 0$ of

Find $(u_\epsilon, m_\epsilon, \overline{H}_\epsilon) \in C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d) \times \mathbb{R}$, with $m_\epsilon > 0$, solving

$$\begin{cases} \dfrac{|P + \nabla u_\epsilon(x)|^2}{2} + V\left(x, \dfrac{x}{\epsilon}\right) = \ln(m_\epsilon(x)) + \overline{H}_\epsilon(P) & \text{in } \mathbb{T}^d \\ -\operatorname{div}\left(m_\epsilon(x)(P + \nabla u_\epsilon(x))\right) = 0 & \text{in } \mathbb{T}^d \\ \displaystyle\int_{\mathbb{T}^d} u_\epsilon(x)\,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^d} m_\epsilon(x)\,\mathrm{d}x = 1 \end{cases}$$

Key feature: spacial preferences of agents, given by V, depend on

- macroscopic variable, x
- microscopic or fast oscillating variable, $\frac{x}{\epsilon}$

Examples: Traffic-flow in a long road with (periodically) changing road conditions:

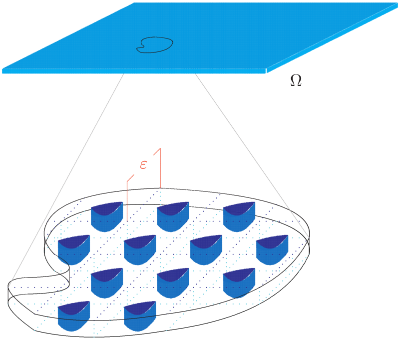
- x – position on the road
- $\frac{x}{\epsilon}$ – current road conditions

Agents moving through a forest or a minefield:

- x – position in the forest/minefield
- $\frac{x}{\epsilon}$ – current conditions: obstacle/no obstacle or mine/no mine

Underlying assumption:

heterogeneities (obstacles) are **evenly distributed** at a scale much smaller than that of the medium, allowing us to assume that the distribution is ε-periodic ($\varepsilon > 0$ small)



$Y^d := [0,1)^d$ reference cell

$\varepsilon Y^d = [0,\varepsilon)^d$ periodicity cell

Two scales characterize the problem:

- x:= macroscopic variable (position in Ω)
- $\frac{x}{\varepsilon}$:= microscopic variable (white or blue)

$$x \in \Omega \Rightarrow x \in \varepsilon(\kappa + Y^d) \Rightarrow \frac{x}{\varepsilon} = \kappa + y,\ \kappa \in \mathbb{Z}^d,\ y \in Y^d$$

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For $\epsilon \ll 1$, numerical methods for these problems are computationally very expensive, potentially unstable, and may breakdown

Questions: How to pass to the limit as $\varepsilon \to 0$?
Does the limit problem preserve the MFG structure?

Our main result in a nutshell

$u_\epsilon \rightharpoonup u_0$, $m_\epsilon \rightharpoonup m_0$, $\overline{H}_\epsilon \to \overline{H}$, with $m_0 > 0$ and

$$\begin{cases} \widetilde{H}(x, P + \nabla u_0(x)) = \ln(m_0(x)) + \overline{H}(P) & \text{in } \mathbb{T}^d \\ -\operatorname{div}\left(m_0(x) D_\Lambda \widetilde{H}(x, P + \nabla u_0(x))\right) = 0 & \text{in } \mathbb{T}^d \\ \int_{\mathbb{T}^d} u_0(x)\,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^d} m_0\,\mathrm{d}x = 1, & \end{cases}$$

where the *homogenized* Hamiltonian, $\widetilde{H} = \widetilde{H}(x, \Lambda)$, is given by an auxiliary problem on the reference cell, Y^d, called the *cell problem.*

Main tools: two-scale convergence, variational methods, PDE techniques

R. Ferreira, D. Gomes, X. Yang
Two-scale Homogenization of a stationary mean-field game.
ESAIM: Control, Optimisation and Calculus of Variations (2020)

On the literature within Homogenization of MFGs

Prior works:

- A. Cesaroni, N. Dirr, C. Marchi
  Homogenization of a mean field game system in the small noise limit.
  SIAM Journal on Mathematical Analysis (2016)

- S. Cacace, F. Camilli, A. Cesaroni, C. Marchi
  An ergodic problem for Mean Field Games: qualitative properties and numerical simulations.
  Minimax Theory and its Applications (2018)

Subsequent works:

- P.-L. Lions, P. E. Souganidis
  Homogenization of the backward-forward mean-field games systems in periodic environments.
  preprint arXiv:1909.01250

Two-scale convergence

Notion introduced by Nguetseng '89, further developed by Allaire '92

- theoretical improvement of the method of asymptotic expansions
- well suited for problems with a variational structure

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Idea of the of asymptotic expansions:

① Upon the observation that two scales characterize the problem, postulate that the solution of $L_\varepsilon w_\varepsilon = f$ admits an expansion of the form

$$w_\varepsilon(x) = w_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon w_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 w_2\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

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Two-scale convergence!

A brief idea on how the asymptotic expansions can provide the heuristics for the limiting behavior in our case:

- Postulate

$$\begin{cases} u_\epsilon(x) = \widetilde{u}_0(x) + \epsilon \widetilde{u}_1(x, \frac{x}{\epsilon}) \\ m_\epsilon(x) = \widetilde{m}_0(x)(\widetilde{m}_1(x, \frac{x}{\epsilon}) + \epsilon \widetilde{m}_2(x, \frac{x}{\epsilon})) \\ \overline{H}_\varepsilon = \overline{H} + \varepsilon \widetilde{H} \end{cases}$$

  with $\widetilde{m}_0$, $\widetilde{m}_1$, and $\widetilde{m}_2$ positive.

- Insert in

$$\begin{cases} \frac{|P+\nabla u_\epsilon(x)|^2}{2} + V(x, \frac{x}{\epsilon}) = \ln(m_\epsilon(x)) + \overline{H}_\epsilon(P) \\ -\operatorname{div}\left(m_\epsilon(x)(P + \nabla u_\epsilon(x))\right) = 0. \end{cases}$$

- Collect the terms containing different powers of ϵ to obtain a sequence of equations, which are of the form $E(x, x/\epsilon) = 0$.

- Separate the scales by denoting $y = x/\epsilon$ and using the formal assumption that $E(x, y) = 0$ holds for all $x \in \mathbb{T}^d$ and $y \in Y^d$

$$\begin{cases} u_\epsilon(x) = \widetilde{u}_0(x) + \epsilon \widetilde{u}_1(x, \frac{x}{\epsilon}) \\ m_\epsilon(x) = \widetilde{m}_0(x)(\widetilde{m}_1(x, \frac{x}{\epsilon}) + \epsilon \widetilde{m}_2(x, \frac{x}{\epsilon})) \\ \overline{H}_\varepsilon = \overline{H} + \varepsilon \widetilde{H} \end{cases}$$

- Workout the algebra to find that (formally)
  - $(\widetilde{u}_0, \widetilde{m}_0, \overline{H})$ solves the **homogenized problem**

$$\begin{cases} \widetilde{H}(x, P + \nabla \widetilde{u}_0(x)) = \ln \widetilde{m}_0(x) + \overline{H}, \\ -\operatorname{div}\left(\widetilde{m}_0(x) D_\Lambda \widetilde{H}(x, P + \nabla \widetilde{u}_0(x))\right) = 0, \end{cases}$$

  - where, for each $x \in \mathbb{T}^d$ and $\Lambda \in \mathbb{R}^d$, $(\widetilde{u}_1, \widetilde{m}_1, \widetilde{H})$ solves the **cell problem**

$$\begin{cases} \frac{|\Lambda + \nabla_y \widetilde{u}_1(x,y)|^2}{2} + V(x,y) = \ln \widetilde{m}_1(x,y) + \widetilde{H}(x, \Lambda), \\ -\operatorname{div}_y \left(\widetilde{m}_1(x,y)(\Lambda + \nabla_y \widetilde{u}_1(x,y))\right) = 0. \end{cases}$$

Definition of two-scale convergence

Let $q \in [1, +\infty)$, $(w_\epsilon)_\epsilon \subset L^q(\mathbb{T}^d)$ bounded, $w \in L^q(\mathbb{T}^d \times Y^d)$.

We say that $(w_\epsilon)_\epsilon$ **weakly two-scale converges** to w if for all $\psi \in C^\infty(\mathbb{T}^d; C^\infty_{\mathrm{per}}(Y^d)) \sim C^\infty(\mathbb{T}^d; C^\infty(\mathbb{T}^d))$, we have

$$\lim_{\epsilon \to 0} \int_{\mathbb{T}^d} w_\epsilon(x) \psi\left(x, \frac{x}{\epsilon}\right) \mathrm{d}x = \int_{\mathbb{T}^d} \int_{Y^d} w(x, y) \psi(x, y) \, \mathrm{d}y \mathrm{d}x.$$

Notation

$w_\epsilon \xrightarrow{2\text{-}sc} w$ in $L^q(\mathbb{T}^d \times Y^d)$

Compactness for $1 < q < \infty$

Let $(w_\epsilon)_\epsilon \subset L^q(\mathbb{T}^d)$ be bounded with $q \in (1, +\infty)$.

Then, there exist $w \in L^q(\mathbb{T}^d \times Y^d)$ and a subsequence $(w_{\epsilon'})_{\epsilon'}$ such that

$$w_{\epsilon'} \xrightarrow{2\text{-}sc} w \text{ in } L^q(\mathbb{T}^d \times Y^d).$$

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Compactness for $q = 1$

Let $(w_\epsilon)_\epsilon \subset L^1(\mathbb{T}^d)$ be bounded and **equi-integrable**.

Then, there exist $w \in L^1(\mathbb{T}^d \times Y^d)$ and a subsequence $(w_{\epsilon'})_{\epsilon'}$ such that

$$w_{\epsilon'} \xrightarrow{2\text{-}sc} w \text{ in } L^1(\mathbb{T}^d \times Y^d).$$

Relationship with the weak limit in L^q

Let $(w_\epsilon)_\epsilon \subset L^q(\mathbb{T}^d)$ with $q \in [1, +\infty)$. Then,

$$w_\epsilon \xrightarrow{2\text{-}sc} w \text{ in } L^q(\mathbb{T}^d \times Y^d) \Rightarrow \; w_\epsilon \rightharpoonup w_0 = \int_{Y^d} w(\cdot, y)\, \mathrm{d}y \text{ in } L^q(\mathbb{T}^d).$$

The **two-scale limit captures more information** on the oscillatory behavior of a bounded sequence in L^q than its weak limit in L^q.

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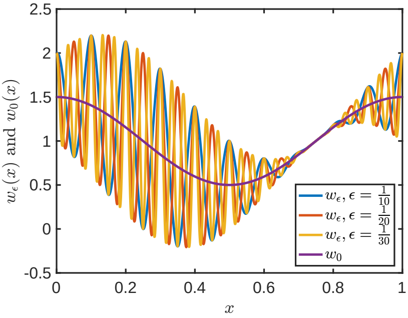
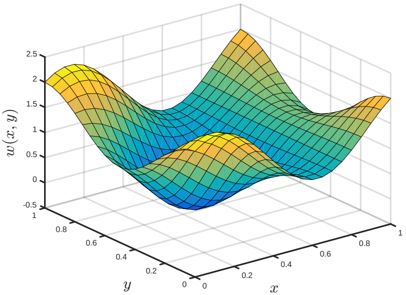
Possible weak two-scale limits:

Given $w \in L^q(\mathbb{T}^d \times Y^d)$, there exists a bounded sequence, $(w_\epsilon)_\epsilon \subset L^q(\mathbb{T}^d)$, such that

$$w_\epsilon \xrightarrow{2\text{-}sc} w \text{ in } L^q(\mathbb{T}^d \times Y^d).$$

Example





$$w_\epsilon(x) = \tfrac{1}{2}\cos(2\pi x) + 1 + \tfrac{1}{2}(\sin(2\pi x) + 1)\cos(2\pi \tfrac{x}{\epsilon})$$

$$w_\epsilon \xrightharpoonup{2\text{-}sc} w \text{ in } L^q(\mathbb{T} \times Y), \quad w(x, y) = \tfrac{1}{2}\cos(2\pi x) + 1 + \tfrac{1}{2}(\sin(2\pi x) + 1)\cos(2\pi y)$$

$$w_\epsilon \rightharpoonup w_0 \text{ in } L^q(\mathbb{T}), \quad w_0(x) = \tfrac{1}{2}\cos(2\pi x) + 1$$

More examples

- If $w_\varepsilon \to w$ in $L^q(\mathbb{T}^d)$, then

$$w_\epsilon \xrightarrow{2\text{-}sc} \tilde{w} \text{ in } L^q(\mathbb{T}^d \times Y^d) \text{ with } \tilde{w}(x,y) := w(x).$$

- Let $\psi \in L^q(\mathbb{T}^d; C_{\mathrm{per}}(Y^d))$, and set $\psi_\varepsilon(x) := \psi(x, \frac{x}{\varepsilon})$. Then,

$$\psi_\epsilon \xrightarrow{2\text{-}sc} \psi \text{ in } L^q(\mathbb{T}^d \times Y^d).$$

Relationship with Asymptotic Expansions

If $w_\varepsilon(x) = w_0\big(x, \frac{x}{\varepsilon}\big) + \varepsilon w_1\big(x, \frac{x}{\varepsilon}\big) + \varepsilon^2 w_2\big(x, \frac{x}{\varepsilon}\big) + \cdots$, w_i smooth, $w_i(x, \cdot)$ Y^d-periodic, **then** $w_\epsilon \xrightarrow{2\text{-}sc} w_0$, $w_0 = w_0(x, y)$

Consequently, existence of the first term, w_0, of the asymptotic expansion is justified

If $w_\epsilon \xrightarrow{2\text{-}sc} w$ in $L^q(\mathbb{T}^d \times Y^d)$, with $w \in L^q(\mathbb{T}^d; C_{\text{per}}(Y^d))$, and $\lim_{\varepsilon\to 0} \|w_\varepsilon\|_{L^q(\mathbb{T}^d)} = \|w\|_{L^q(\mathbb{T}^d \times Y^d)}$, **then**

$$\lim_{\varepsilon\to 0} \|w_\varepsilon - w(\cdot, \tfrac{\cdot}{\varepsilon})\|_{L^q(\mathbb{T}^d)} = 0.$$

Thus, convergence of the norms provides a sufficient condition for strong convergence of w_ε to the first term of its asymptotic expansion

In general, $\|\bar{w}\|_{L^q(\mathbb{T}^d)} \leqslant \|w\|_{L^q(\mathbb{T}^d \times Y^d)} \leqslant \liminf_{\varepsilon\to 0} \|w_\varepsilon\|_{L^q(\mathbb{T}^d)}$

Compactness in $W^{1,q}$ for $1 < q < \infty$

Let $(w_\epsilon)_\epsilon \subset W^{1,q}(\mathbb{T}^d)$ be bounded with $q \in (1, +\infty)$.

Then, there exist $w \in L^q(\mathbb{T}^d)$, $w_1 \in L^q(\mathbb{T}^d; W^{1,q}_{\text{per}}(Y^d)/\mathbb{R})$, and a subsequence $(w_{\epsilon'})_{\epsilon'}$ such that

$$
\begin{aligned}
& w_{\epsilon'} \rightharpoonup w \text{ in } W^{1,q}(\mathbb{T}^d), \\
& w_{\epsilon'} \xrightharpoonup{2\text{-}sc} w \text{ in } L^q(\mathbb{T}^d \times Y^d), \\
& \nabla w_{\epsilon'} \xrightharpoonup{2\text{-}sc} \nabla w + \nabla_y w_1 \text{ in } \left[L^q(\mathbb{T}^d \times Y^d)\right]^d.
\end{aligned}
$$

Remark: The term $\nabla_y w_1$ can be interpreted as the gradient limit at the microscale characterizing the problem.

Back to our problem:

Problem

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$$\begin{cases} \dfrac{|P + \nabla u_\epsilon(x)|^2}{2} + V\left(x, \dfrac{x}{\epsilon}\right) = \ln(m_\epsilon(x)) + \overline{H}_\epsilon(P) & \text{in } \mathbb{T}^d \\ -\operatorname{div}\big(m_\epsilon(x)(P + \nabla u_\epsilon(x))\big) = 0 & \text{in } \mathbb{T}^d \\ \displaystyle\int_{\mathbb{T}^d} u_\epsilon(x)\,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^d} m_\epsilon(x)\,\mathrm{d}x = 1 \end{cases}$$

As proved by Evans in

L. C. Evans
Some new PDE methods for weak KAM theory.
Calculus of Variations and Partial Differential Equations (2013)

this problem has a unique solution (when $\varepsilon^{-1} \in \mathbb{N}$), and is equivalent to

Variational Problem

Find $u_\epsilon \in C^\infty(\mathbb{T}^d)$ satisfying $\int_{\mathbb{T}^d} u_\epsilon(x)\,\mathrm{d}x = 0$ and

$$I_\epsilon[u_\epsilon] = \inf_{\substack{u \in C^1(\mathbb{T}^d) \\ \int_{\mathbb{T}^d} u(x)\,\mathrm{d}x = 0}} I_\epsilon[u],$$

where

$$\boxed{I_\epsilon[u] = \int_{\mathbb{T}^d} e^{\frac{|P+\nabla u(x)|^2}{2} + V(x,\frac{x}{\epsilon})}\,\mathrm{d}x} \quad \text{for } u \in C^1(\mathbb{T}^d).$$

through the identities

$$\overline{H}_\epsilon(P) = \ln I_\epsilon[u_\epsilon]$$

and

$$m_\epsilon = e^{\frac{|P+\nabla u_\epsilon(x)|^2}{2} + V(x,\frac{x}{\epsilon}) - \overline{H}_\epsilon(P)}.$$

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Note: Exponential growth makes this problem somewhat non-standard, and therefore with independent interest in the calculus of variations.

Exploiting both the PDE and the variational formulation, we establish

Uniform estimates in ε

Let $q \in [1, \infty)$. Then, there exist positive constants, $C = C(P)$, $C_q = C(q, P)$, and $C_\epsilon = C(\epsilon, P)$, such that

$$\inf_{\mathbb{T}^d \times Y^d} V \leqslant \overline{H}_\epsilon(P) \leqslant \frac{|P|^2}{2} + \sup_{\mathbb{T}^d \times Y^d} V$$

$$\sup_\epsilon \|u_\epsilon\|_{W^{1,q}(\mathbb{T}^d)} \leqslant C_q,$$

$$\frac{1}{C} \leqslant \inf_{\mathbb{T}^d} m_\epsilon \leqslant \sup_{\mathbb{T}^d} m_\epsilon \leqslant C_\varepsilon,$$

$$\sup_\epsilon \int_{\mathbb{T}^d} m_\epsilon(x) \ln(m_\epsilon(x))\, \mathrm{d}x \leqslant \frac{|P|^2}{2} + \sup_{\mathbb{T}^d \times Y^d} V - \inf_{\mathbb{T}^d \times Y^d} V.$$

Note: The last last estimate together with the de la Vallée Poussin criterion for equi-integrability allows us to use the compactness result for two-scale convergence in L^1 applied to $(m_\varepsilon)_\varepsilon$

Corollary

There exist $\alpha \in (0,1)$

$$u_0 \in C^{0,\alpha} \cap W^{1,q}(\mathbb{T}^d) \text{ with } \int_{\mathbb{T}^d} u_0 \,\mathrm{d}x = 0,$$
$$u_1 \in L^q(\mathbb{T}^d; W^{1,q}_{\mathrm{per}}(Y^d)/\mathbb{R}),$$
$$m \in L^1(\mathbb{T}^d \times Y^d) \text{ with } \int_{\mathbb{T}^d}\int_{Y^d} m(x,y)\,\mathrm{d}y\mathrm{d}x = 1,$$
$$\overline{H}(P) \in \mathbb{R}$$

such that, up to a subsequence,

$$u_\epsilon \to u_0 \text{ in } L^\infty(\mathbb{T}^d), \quad u_\epsilon \rightharpoonup u_0 \text{ in } W^{1,q}(\mathbb{T}^d),$$
$$\nabla u_\epsilon \xrightarrow{2\text{-}sc} \nabla u_0 + \nabla_y u_1 \text{ in } [L^q(\mathbb{T}^d \times Y^d)]^d,$$
$$m_\epsilon \xrightarrow{2\text{-}sc} m \text{ in } L^1(\mathbb{T}^d \times Y^d), \quad m_\epsilon \rightharpoonup m_0 = \int_{Y^d} m(\cdot,y)\,\mathrm{d}y \text{ in } L^1(\mathbb{T}^d),$$
$$\overline{H}_\epsilon(P) \to \overline{H}(P) \text{ in } \mathbb{R}.$$

$u_\epsilon \rightharpoonup u_0$ in $W^{1,q}(\mathbb{T}^d)$, $\nabla u_\epsilon \xrightharpoonup{2\text{-}sc} \nabla u_0 + \nabla_y u_1$ in $[L^q(\mathbb{T}^d \times Y^d)]^d$,
$m_\epsilon \xrightharpoonup{2\text{-}sc} m$ in $L^1(\mathbb{T}^d \times Y^d)$, $m_\epsilon \rightharpoonup m_0 = \int_{Y^d} m(\cdot, y)\,\mathrm{d}y$ in $L^1(\mathbb{T}^d)$,
$\overline{H}_\epsilon(P) \to \overline{H}(P)$ in $\mathbb{R}$

Question: What problem(s) do u_0, u_1, m, m_0, and $\overline{H}$ solve?

Additional assumption on the potential: V is separable in y; that is, there exist smooth functions, $V_i : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$, where $1 \leqslant i \leqslant d$, such that for all $x \in \mathbb{T}^d$ and $y \in \mathbb{R}^d$, $y = (y_1, \ldots, y_i, \ldots, y_d)$, we have

$$V(x,y) = \sum_{i=1}^{d} V_i(x, y_i).$$

Then, our **main theorem**, stated from the variational viewpoint is:

$u_\epsilon \rightharpoonup u_0$ in $W^{1,q}(\mathbb{T}^d)$, $\nabla u_\epsilon \xrightharpoonup{2\text{-}sc} \nabla u_0 + \nabla_y u_1$ in $[L^q(\mathbb{T}^d \times Y^d)]^d$,
$m_\epsilon \xrightharpoonup{2\text{-}sc} m$ in $L^1(\mathbb{T}^d \times Y^d)$, $m_\epsilon \rightharpoonup m_0 = \int_{Y^d} m(\cdot, y)\,\mathrm{d}y$ in $L^1(\mathbb{T}^d)$,
$\overline{H}_\epsilon(P) \to \overline{H}(P)$ in $\mathbb{R}$

① (u_0, u_1) is the unique solution to the

Variational two-scale homogenized problem

Find $u_0 \in C^\infty(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} u_0 \,\mathrm{d}x = 0$ and $u_1 \in C^\infty(\mathbb{T}^d; C^{2,\alpha}_{\mathrm{per}}(Y^d)/\mathbb{R})$ satisfying

$$I^{2\mathrm{sc}}_{\mathrm{hom}}[u_0, u_1] = \inf_{\substack{u \in W^{1,p}(\mathbb{T}^d),\ \int_{\mathbb{T}^d} u \,\mathrm{d}x = 0 \\ w \in L^p(\mathbb{T}^d; W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R})}} I^{2\mathrm{sc}}_{\mathrm{hom}}[u, w],$$

where

$$I^{2\mathrm{sc}}_{\mathrm{hom}}[u, w] := \int_{\mathbb{T}^d} \int_{Y^d} e^{\frac{|P + \nabla u(x) + \nabla_y w(x,y)|^2}{2} + V(x,y)} \,\mathrm{d}y\mathrm{d}x$$

for $(u, w) \in W^{1,p}(\mathbb{T}^d) \times L^p(\mathbb{T}^d; W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R})$

$u_\epsilon \rightharpoonup u_0$ in $W^{1,q}(\mathbb{T}^d)$, $\nabla u_\epsilon \xrightharpoonup{2\text{-}sc} \nabla u_0 + \nabla_y u_1$ in $[L^q(\mathbb{T}^d \times Y^d)]^d$,
$m_\epsilon \xrightharpoonup{2\text{-}sc} m$ in $L^1(\mathbb{T}^d \times Y^d)$, $m_\epsilon \rightharpoonup m_0 = \int_{Y^d} m(\cdot, y)\, \mathrm{d}y$ in $L^1(\mathbb{T}^d)$,
$\overline{H}_\epsilon(P) \to \overline{H}(P)$ in $\mathbb{R}$

② $\lim\limits_{\epsilon\to 0} I_\epsilon[u_\epsilon] = I^{2\mathrm{sc}}_{\mathrm{hom}}[u_0, u_1]$; that is,

$$\lim_{\epsilon\to 0} \int_{\mathbb{T}^d} e^{\frac{|P+\nabla u_\epsilon(x)|^2}{2} + V(x, \frac{x}{\epsilon})}\, \mathrm{d}x = \int_{\mathbb{T}^d} \int_{Y^d} e^{\frac{|P+\nabla u_0(x)+\nabla_y u_1(x,y)|^2}{2} + V(x,y)}\, \mathrm{d}y\mathrm{d}x$$

③ $\overline{H}(P) = \ln I^{2\mathrm{sc}}_{\mathrm{hom}}[u_0, u_1]$

④ $m(x,y) = e^{\frac{|P+\nabla u_0(x)+\nabla_y u_1(x,y)|}{2} + V(x,y) - \overline{H}(P)}$

⑤ u_0 is the unique solution of the

Variational homogenized problem

Find $u_0 \in C^\infty(\mathbb{T}^d)$ satisfying $\int_{\mathbb{T}^d} u_0 \,\mathrm{d}x = 0$ and

$$I_{\mathrm{hom}}[u_0] = \inf_{u \in W^{1,p}(\mathbb{T}^d),\, \int_{\mathbb{T}^d} u \,\mathrm{d}x = 0} I_{\mathrm{hom}}[u],$$

where

$$\boxed{I_{\mathrm{hom}}[u] := \int_{\mathbb{T}^d} e^{\widetilde{H}(x, P + \nabla u(x))} \,\mathrm{d}x} \quad \text{for } u \in W^{1,p}(\mathbb{T}^d).$$

Variational homogenized problem

Find $u_0 \in C^\infty(\mathbb{T}^d)$ satisfying $\int_{\mathbb{T}^d} u_0 \, \mathrm{d}x = 0$ and

$$I_{\mathrm{hom}}[u_0] = \inf_{u \in W^{1,p}(\mathbb{T}^d),\, \int_{\mathbb{T}^d} u \, \mathrm{d}x = 0} I_{\mathrm{hom}}[u],$$

where

$$\boxed{I_{\mathrm{hom}}[u] := \int_{\mathbb{T}^d} e^{\widetilde{H}(x, P + \nabla u(x))} \, \mathrm{d}x} \quad \text{for } u \in W^{1,p}(\mathbb{T}^d).$$

Here, $\widetilde{H} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is defined, for each $x \in \mathbb{T}^d$ and $\Lambda \in \mathbb{R}^d$, by

$$\boxed{\widetilde{H}(x, \Lambda) = \ln I_{\mathrm{cell}}[x, \Lambda; \widetilde{w}],}$$

where

$$\boxed{I_{\mathrm{cell}}[x, \Lambda; w] := \int_{Y^d} e^{\frac{|\Lambda + \nabla w(y)|^2}{2} + V(x,y)} \, \mathrm{d}y} \quad \text{for } w \in W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R}$$

and $\widetilde{w}$ is the unique solution of

Variational cell problem

For each $x \in \mathbb{T}^d$ and $\Lambda \in \mathbb{R}^d$, find $\widetilde{w} \in C^{2,\alpha}_{\mathrm{per}}(Y^d)/\mathbb{R}$, depending on x and Λ, satisfying

$$I_{\mathrm{cell}}[x,\Lambda;\widetilde{w}] = \inf_{w \in W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R}} I_{\mathrm{cell}}[x,\Lambda;w],$$

where

$$\boxed{I_{\mathrm{cell}}[x,\Lambda;w] := \int_{Y^d} e^{\frac{|\Lambda+\nabla w(y)|^2}{2}+V(x,y)}\,\mathrm{d}y} \quad \text{for } w \in W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R}$$

Adopting a PDE viewpoint, we revisit the slide "our main result in a nutshell" and prove the heuristics provided by the asymptotic expansion method:

$u_\epsilon \rightharpoonup u_0$ in $W^{1,q}(\mathbb{T}^d)$, $\nabla u_\epsilon \xrightharpoonup{2\text{-}sc} \nabla u_0 + \nabla_y u_1$ in $[L^q(\mathbb{T}^d \times Y^d)]^d$,
$m_\epsilon \xrightharpoonup{2\text{-}sc} m$ in $L^1(\mathbb{T}^d \times Y^d)$, $m_\epsilon \rightharpoonup m_0 = \int_{Y^d} m(\cdot, y)\,\mathrm{d}y$ in $L^1(\mathbb{T}^d)$,
$\overline{H}_\epsilon(P) \to \overline{H}(P)$ in $\mathbb{R}$

$$\overline{H}(P) = \ln I^{2\text{sc}}_{\text{hom}}[u_0, u_1], \qquad m(x,y) = e^{\frac{|P+\nabla u_0(x)+\nabla_y u_1(x,y)|}{2}+V(x,y)-\overline{H}(P)}$$

⑤' $(u_0, m_0, \overline{H})$ is the unique solution of

Homogenized problem

Find $u_0 \in C^\infty(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} u_0\,\mathrm{d}x = 0$, $m_0 \in C^\infty(\mathbb{T}^d)$ with $m_0 > 0$, and $\overline{H} \in \mathbb{R}$ satisfying

$$\begin{cases} \widetilde{H}(x, P + \nabla u_0(x)) = \ln(m_0(x)) + \overline{H}(P) & \text{in } \mathbb{T}^d \\ -\operatorname{div}\left(m_0(x) D_\Lambda \widetilde{H}(x, P + \nabla u_0(x))\right) = 0 & \text{in } \mathbb{T}^d \\ \displaystyle\int_{\mathbb{T}^d} m_0\,\mathrm{d}x = 1, \end{cases}$$

where $\widetilde{H}$ is determined by

Cell problem

For each $x \in \mathbb{T}^d$ and $\Lambda \in \mathbb{R}^d$, find $\widetilde{w} \in C^{2,\alpha}_{\text{per}}(Y^d)/\mathbb{R}$, $\widetilde{m} \in C^{1,\alpha}_{\#}(Y^d)$, and $\widetilde{H} \in \mathbb{R}$, depending on x and Λ, such that $(\widetilde{w}, \widetilde{m}, \widetilde{H})$ solves

$$\begin{cases} \dfrac{|\Lambda + \nabla_y \widetilde{w}(x,\Lambda,y)|^2}{2} + V(x,y) = \ln \widetilde{m}(x,\Lambda,y) + \widetilde{H}(x,\Lambda) & \text{in } Y^d \\ -\operatorname{div}_y \big(\widetilde{m}(x,\Lambda,y)(\Lambda + \nabla_y \widetilde{w}(x,\Lambda,y))\big) = 0 & \text{in } Y^d \\ \displaystyle\int_{Y^d} \widetilde{m}(x,\Lambda,y)\,\mathrm{d}y = 1. \end{cases}$$

Moreover, $(u_0, u_1, m, \overline{H})$ is the unique solution to

$u_\epsilon \rightharpoonup u_0$ in $W^{1,q}(\mathbb{T}^d)$, $\nabla u_\epsilon \xrightharpoonup{2\text{-}sc} \nabla u_0 + \nabla_y u_1$ in $[L^q(\mathbb{T}^d \times Y^d)]^d$,

$m_\epsilon \xrightharpoonup{2\text{-}sc} m$ in $L^1(\mathbb{T}^d \times Y^d)$, $m_\epsilon \rightharpoonup m_0 = \int_{Y^d} m(\cdot, y)\, \mathrm{d}y$ in $L^1(\mathbb{T}^d)$,

$\overline{H}_\epsilon(P) \to \overline{H}(P)$ in $\mathbb{R}$

Two-scale homogenized problem

Find $u_0 \in C^\infty(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} u_0\, \mathrm{d}x = 0$, $u_1 \in C^\infty(\mathbb{T}^d; C^{2,\alpha}_\#(Y^d)/\mathbb{R})$, $m \in C^\infty(\mathbb{T}^d; C^{1,\alpha}_\#(Y^d))$ with $\int_{\mathbb{T}^d} \int_{Y^d} m(x,y)\, \mathrm{d}y\mathrm{d}x = 1$, and $\overline{H} \in \mathbb{R}$ satisfying

$$\begin{cases} \dfrac{|P + \nabla u_0(x) + \nabla_y u_1(x,y)|^2}{2} + V(x,y) = \ln(m(x,y)) + \overline{H}(P) \\ -\operatorname{div}_x \left(\displaystyle\int_{Y^d} m(x,y)(P + \nabla u_0(x) + \nabla_y u_1(x,y)) dy \right) = 0 \\ -\operatorname{div}_y (m(x,y)(P + \nabla u_0(x) + \nabla_y u_1(x,y))) = 0 \end{cases}$$

in $\mathbb{T}^d \times Y^d$.

A lower semicontinuity result w.r.t. two-scale convergence

Assume that

- $f : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ is a Borel function such that $f(\cdot, p)$ is Y^d-periodic and $f(y, \cdot)$ is convex,
- $w_\epsilon \xrightarrow{2\text{-}sc} w$ in $[L^p(\mathbb{T}^d \times \mathbb{T}^d)]^d$.

Then, for all $\phi \in C^\infty(\mathbb{T}^d; C^\infty_{\text{per}}(Y^d))$ with $\phi \geqslant 0$, we have

$$\liminf_{\epsilon \to 0} \int_{\mathbb{T}^d} f\left(\frac{x}{\epsilon}, w_\epsilon(x)\right) \phi\left(x, \frac{x}{\epsilon}\right) \mathrm{d}x \geqslant \int_{\mathbb{T}^d} \int_{Y^d} f(y, w(x, y)) \phi(x, y) \, \mathrm{d}y \mathrm{d}x.$$

Taking

$$w_\varepsilon = \nabla u_\varepsilon \xrightarrow{2\text{-}sc} \nabla u_0 + \nabla_y u_1, \quad f(x, p) = \mathrm{e}^{\frac{|P+p|^2}{2}}, \quad \phi(x, y) = \mathrm{e}^{V(x,y)},$$

we prove a lower bound:

$$
\begin{aligned}
\liminf_{\epsilon\to 0} I_\epsilon[u_\epsilon] &= \liminf_{\epsilon\to 0} \int_{\mathbb{T}^d} e^{\frac{|P+\nabla u_\epsilon(x)|^2}{2}} e^{V(x,\frac{x}{\epsilon})}\,\mathrm{d}x \\
&\geqslant \int_{\mathbb{T}^d}\int_{Y^d} e^{\frac{|P+\nabla u_0(x)+\nabla_y u_1(x,y)|^2}{2}+V(x,y)}\,\mathrm{d}y\mathrm{d}x \\
&= I^{2\mathrm{sc}}_{\mathrm{hom}}[u_0,u_1] \\
&\geqslant \inf_{\substack{u\in W^{1,p}(\mathbb{T}^d),\ \int_{\mathbb{T}^d} u\,\mathrm{d}x=0 \\ w\in L^p(\mathbb{T}^d;W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R})}} I^{2\mathrm{sc}}_{\mathrm{hom}}[u,w]
\end{aligned}
$$

We prove a matching upper bound and uniqueness and regularity of minimizers by

- using a continuity argument with respect to *strong* two-scale convergence applied to convenient test functions:

$$
I_\epsilon[u_\epsilon] \leqslant I_\epsilon\left[\psi_0(\cdot) + \epsilon\psi_1\left(\cdot, \tfrac{\cdot}{\epsilon}\right)\right]
$$

- splitting the variational two-scale formulation into two subproblems as follows:

$$\inf_{\substack{u\in W^{1,p}(\mathbb{T}^d),\ \int_{\mathbb{T}^d} u\,\mathrm{d}x=0 \\ w\in L^p(\mathbb{T}^d;W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R})}} \int_{\mathbb{T}^d}\int_{Y^d} e^{\frac{|P+\nabla u(x)+\nabla_y w(x,y)|^2}{2}+V(x,y)}\,\mathrm{d}y\mathrm{d}x$$

$$= \underbrace{\inf_{\substack{u\in W^{1,p}(\mathbb{T}^d) \\ \int_{\mathbb{T}^d} u\,\mathrm{d}x=0}} \int_{\mathbb{T}^d} \underbrace{\inf_{w\in L^p(\mathbb{T}^d;W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R})} \int_{Y^d} e^{\frac{|P+\nabla u(x)+\nabla_y w(x,y)|^2}{2}+V(x,y)}\,\mathrm{d}y}\ \mathrm{d}x}$$

$$\inf_{\substack{u\in W^{1,p}(\mathbb{T}^d),\ \int_{\mathbb{T}^d} u\,\mathrm{d}x=0 \\ w\in L^p(\mathbb{T}^d;W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R})}} \int_{\mathbb{T}^d}\int_{Y^d} e^{\frac{|P+\nabla u(x)+\nabla_y w(x,y)|^2}{2}+V(x,y)}\,\mathrm{d}y\mathrm{d}x$$

$$= \underbrace{\inf_{\substack{u\in W^{1,p}(\mathbb{T}^d) \\ \int_{\mathbb{T}^d} u\,\mathrm{d}x=0}} \int_{\mathbb{T}^d} \underbrace{\inf_{w\in L^p(\mathbb{T}^d;W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R})} \int_{Y^d} e^{\frac{|P+\nabla u(x)+\nabla_y w(x,y)|^2}{2}+V(x,y)}\,\mathrm{d}y}_{e^{\widetilde{H}(x,\Lambda)} = \inf_{w\in W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R}} \int_{Y^d} e^{\frac{|\Lambda+\nabla_y w(x,y)|^2}{2}+V(x,y)}\,\mathrm{d}y} \quad \mathrm{d}x}$$

Cell problem: Existence given by the continuation method, where the implicit function theorem plays a role a provides regularity with respect to the parameters

$$\inf_{\substack{u\in W^{1,p}(\mathbb{T}^d),\ \int_{\mathbb{T}^d} u\,\mathrm{d}x=0\\ w\in L^p(\mathbb{T}^d;W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R})}} \int_{\mathbb{T}^d}\int_{Y^d} e^{\frac{|P+\nabla u(x)+\nabla_y w(x,y)|^2}{2}+V(x,y)}\,\mathrm{d}y\mathrm{d}x$$

$$= \underbrace{\inf_{\substack{u\in W^{1,p}(\mathbb{T}^d)\\ \int_{\mathbb{T}^d} u\,\mathrm{d}x=0}} \int_{\mathbb{T}^d} \underbrace{\inf_{w\in L^p(\mathbb{T}^d;W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R})}\int_{Y^d} e^{\frac{|P+\nabla u(x)+\nabla_y w(x,y)|^2}{2}+V(x,y)}\,\mathrm{d}y}_{e^{\widetilde{H}(x,\Lambda)} = \inf_{w\in W^{1,p}_{\mathrm{per}}(Y^d)/\mathbb{R}}\int_{Y^d} e^{\frac{|\Lambda+\nabla_y w(x,y)|^2}{2}+V(x,y)}\,\mathrm{d}y}\,\mathrm{d}x}_{\inf_{u\in W^{1,p}(\mathbb{T}^d),\ \int_{\mathbb{T}^d} u\,\mathrm{d}x=0}\int_{\mathbb{T}^d} e^{\widetilde{H}(x,P+\nabla u(x))}\,\mathrm{d}x}$$

Cell problem: Existence given by the continuation method, where the implicit function theorem plays a role a provides regularity with respect to the parameters

Homogenization problem: Existence, uniqueness, and regularity given by [Evans 2013] provided we prove that $\widetilde{H}$ satisfies the appropriate conditions - this is where the **separability** of V plays a role

More precisely, the homogenized problem

$$\inf_{u \in W^{1,p}\left(\mathbb{T}^d\right), \int_{\mathbb{T}^d} u \, \mathrm{d}x = 0} \int_{\mathbb{T}^d} e^{\widetilde{H}(x, P + \nabla u(x))} \, \mathrm{d}x$$

is considered in [Evans, 2003]. A unique smooth solution exists if $\widetilde{H}$ satisfies

- $\left| D_x \widetilde{H} \right| \leqslant C$, $\left| D_\Lambda \widetilde{H} \right| \leqslant C \left(1 + |\Lambda|\right)$,
- $\left| D_x^2 \widetilde{H} \right| \leqslant C$, $\left| D_\Lambda^2 \widetilde{H} \right| \leqslant C$, $\left| D_{x,\Lambda}^2 \widetilde{H} \right| \leqslant C$,
- $\xi^T D_\Lambda^2 \widetilde{H} \xi \geqslant C \left|\xi\right|^2$ for any $\xi \in \mathbb{R}^d$

If V is separable in y, $V(x, y) = \sum_{i=1}^d V_i(x, y_i)$, the solution $(\widetilde{m}, \widetilde{w}, \widetilde{H})$ of the cell problem is separable in y and can be written as

$$\widetilde{m}(x, y) = \prod_{i=1}^d \widetilde{m}_i(x, y_i), \ \widetilde{w}(x, y) = \sum_{i=1}^d \widetilde{w}_i(x, y_i), \ \widetilde{H}(x, \Lambda) = \sum_{i=1}^d \widetilde{H}_i(x, \Lambda_i),$$

Thus, the cell problem splits into one-dimensional systems:

$$\begin{cases} \dfrac{|\Lambda_i + (\widetilde{w}_i(x, y_i))_{y_i}|^2}{2} + V_i(x, y_i) = \ln\left(\widetilde{m}_i(x, y_i)\right) + \widetilde{H}_i(x, \Lambda_i) \\ \left(\widetilde{m}_i(x, y_i)(\Lambda_i + (\widetilde{w}_i(x, y_i))_{y_i})\right)_{y_i} = 0 \\ \displaystyle\int_0^1 \widetilde{m}_i(x, y_i)\,\mathrm{d}y = 1 \end{cases}$$

In the one-dimensional case, the current method gives strictly positive lower bounds on m_i that are **uniform** in Λ, and eventually allows us to prove that

$$\xi^T D^2_\Lambda \widetilde{H} \xi \geqslant C\,|\xi|^2 \text{ for any } \xi \in \mathbb{R}^d.$$

Back to the upper bound: Let

$$\psi_\epsilon(x) = \psi_0(x) + \epsilon\psi_1\left(x, \frac{x}{\epsilon}\right),$$

where $\psi_0 \in C^\infty(\mathbb{T}^d)$ and $\psi_1 \in C^\infty(\mathbb{T}^d; C^{2,\alpha}(\mathbb{T}^d)/\mathbb{R})$. The fact that u_ϵ minimizes $I_\epsilon[\cdot]$ and a continuity argument with respect to *strong* two-scale convergence yield

$$\limsup_{\epsilon\longrightarrow 0} I_\epsilon[u_\epsilon] \leqslant \limsup_{\epsilon\longrightarrow 0} I_\epsilon[\psi_\epsilon] = I_{\text{hom}}^{2\text{sc}}[\psi_0, \psi_1].$$

Thus, using the analysis on the iterated integrals for $I_{\text{hom}}^{2\text{sc}}$,

$$\limsup_{\epsilon\longrightarrow 0} I_\epsilon[u_\epsilon] \leqslant \inf_{\psi_0,\psi_1} I_{\text{hom}}^{2\text{sc}}[\psi_0, \psi_1] = \inf_{u,w} I_{\text{hom}}^{2\text{sc}}[u, w] = I_{\text{hom}}^{2\text{sc}}\overline{I}[\widehat{u}_0, \widehat{u}_1]$$

for some smooth $\widehat{u}_0$ and $\widehat{u}_1$. Therefore,

$$\lim_{\epsilon\longrightarrow 0} I_\epsilon[u_\epsilon] = I_{\text{hom}}^{2\text{sc}}[\widehat{u}_0, \widehat{u}_1] = I_{\text{hom}}^{2\text{sc}}[u_0, u_1].$$

Since the minimizer of $I_{\text{hom}}^{2\text{sc}}$ is unique, $\widehat{u}_0 = u_0$ and $\widehat{u}_1 = u_1$.

$$\begin{cases} \dfrac{|P+\nabla u_\epsilon(x)|^2}{2} + V\left(x, \dfrac{x}{\epsilon}\right) = \ln(m_\epsilon(x)) + \overline{H}_\epsilon(P) & \text{in } \mathbb{T}^d \\ -\operatorname{div}\left(m_\epsilon(x)(P+\nabla u_\epsilon(x))\right) = 0 & \text{in } \mathbb{T}^d \\ \displaystyle\int_{\mathbb{T}^d} u_\epsilon(x)\,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^d} m_\epsilon(x)\,\mathrm{d}x = 1 \end{cases}$$

$u_\epsilon \rightharpoonup u_0$ in $W^{1,q}(\mathbb{T}^d)$, $\nabla u_\epsilon \xrightarrow{2\text{-}sc} \nabla u_0 + \nabla_y u_1$ in $[L^q(\mathbb{T}^d \times Y^d)]^d$,

$m_\epsilon \xrightarrow{2\text{-}sc} m$ in $L^1(\mathbb{T}^d \times Y^d)$, $m_\epsilon \rightharpoonup m_0 = \int_{Y^d} m(\cdot, y)\,\mathrm{d}y$ in $L^1(\mathbb{T}^d)$,

$\overline{H}_\epsilon(P) \to \overline{H}(P)$ in $\mathbb{R}$

$$\begin{cases} \widetilde{H}(x, P+\nabla u_0(x)) = \ln(m_0(x)) + \overline{H}(P) & \text{in } \mathbb{T}^d \\ -\operatorname{div}\left(m_0(x) D_\Lambda \widetilde{H}(x, P+\nabla u_0(x))\right) = 0 & \text{in } \mathbb{T}^d \\ \displaystyle\int_{\mathbb{T}^d} u_0(x)\,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^d} m_0\,\mathrm{d}x = 1 \end{cases}$$

($\widetilde{H} = \widetilde{H}(x, \Lambda)$ is given by an auxiliary problem on the reference cell, Y^d)

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