Numerical methods for non-linear Fokker Planck equations and applications to Mean Field Games

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joint work with
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High Dimensional Hamilton-Jacobi PDEs
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Outline

1. Numerical approximation of FPK

2. Convergence Analysis

3. Mean Field Games
   - Non linear explicit case: a new Hughes type model

4. Lagrange Galerkin
A nonlinear Fokker-Planck equation

The nonlinear FP equation

\[
\begin{aligned}
\partial_t m - \frac{1}{2} \sum_{i,j} \partial_{x_i x_j} (a_{ij}[m](x, t)m) + \text{div}(b[m](x, t) m) &= 0 \quad \mathbb{R}^d \times \mathbb{R}^+ \\
\quad m(\cdot, 0) &= m_0(\cdot) \\
\end{aligned}
\]

- \(b\) is a given vector field depending on \(m\), non locally in space and possibly non locally in time
- \((a_{i,j}(m, x, t))\) is a given diffusion matrix (possible degenerate) depending on \(m\), non locally in space and possibly non locally in time; such that

\[
a_{i,j}(m, x, t) = \sum_{p=1}^{r} \sigma_{i,p} \cdot \sigma_{j,p} = (\sigma(\sigma^\top))_{ij},
\]

for all \(i, j = 1, \ldots d\), where \(r \in \mathbb{N} \setminus \{0\}\), and for all \(p = 1, \ldots, r\).
- the density of the initial law is given by \(m_0\):

\[
m_0 \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^d} m_0(x)dx = 1.
\]
A nonlinear Fokker-Planck equation

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- \( b \) is a given vector field depending on \( m \), non locally in space and possibly non locally in time.
- \( (a_{i,j}(m,x,t)) \) is a given diffusion matrix (possible degenerate) depending on \( m \), non locally in space and possibly non locally in time; such that \( a_{i,j}(m,x,t) = \sum_{p=1}^{r} \sigma_{i,p} \cdot \sigma_{j,p} = (\sigma(\sigma^\top))_{ij} \), for all \( i, j = 1, \ldots d \), where \( r \in \mathbb{N} \setminus \{0\} \), and for all \( p = 1, \ldots, r \).
- the density of the initial law is given by \( m_0 \):
  \( m_0 \geq 0 \) and \( \int_{\mathbb{R}^d} m_0(x)dx = 1 \).
Some applications

- Non local interactions due to collective phenomena (biophysics, social behavior)
- Hughes model $b[m](x, t) = -f^2(m(x, t))Dv[m](x, t)$ where $v[m]$ is the solution of a stationary HJB

$$|Dv| = \frac{1}{f(m(x, t))}$$

- Mean Filed Games: $b[m](x, t) = -DH(Dv[m](x, t))$ where $v[m]$ is the solution of a backward HJB

$$\begin{cases} 
-\partial_t v - \frac{\sigma^2}{2} \Delta v + H(Dv) = f(x, m(t)) \\
v(x, T) = g(x, m(T)).
\end{cases}$$
Probabilistic interpretation

\((FPK)\) describes the evolution of the law of the diffusion processes
\(X(t) \in \mathbb{R}^d\)

\[
\begin{cases}
    dX(t) = b(m, X(t), t)dt + \sigma(m, X(t), t)dW(t) & t \in [0, T], \\
    X(0) = X_0,
\end{cases}
\]

where the \(r\)-dimensional Brownian motion \(\{W\}\) independent of \(X_0\), the distribution of \(X_0\) is given by \(m_0\).

- \(b(m, x, t) = b(m(t), x, t)\) and \(\sigma_{i,j}(m, x, t) = \sigma_{i,j}(m(t), x, t)\), the FPK equation is called McKean-Vlasov equation well-posedness (T. Funaki '84, S. Méléard '96).
- existence in the general case: first order (V.I. Bogachev, M. Rockner, and S. V. Shaposhnikov 2009), second order case (O.A. Manita and S.V. Shaposhnikov 2013)
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- Kushner (1976): finite difference via probabilistic method
- Naess and Johnsen (1993): Path Integration Method
- Achdou, Camilli, Capuzzo Dolcetta (2012): implicit finite difference scheme s.t. preserves positivity, and mass of the distribution.
- M. Annunziato and A. Borzì (2013): implicit high order finite difference Chang Cooper scheme for optimal control FPK

Non Linear FP

- Drozdov, Morillo (1995): finite difference scheme s.t. preserves equilibrium states and mass of the distribution, (high order).
- Benamou, Carlier, Laborde (2015): semi implicit variant of JKO
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Representation formula for the Fokker Planck equation

\textbf{(Linear case)} \( b[m](x, t) = b(x, t), \quad \sigma[m](x, t) = \sigma(x, t) \)

\textbf{(Lip)} \( b \) and \( \sigma \) Lipschitz w.r. to \( x \), uniformly in \( t \in [0, T] \)

Let \( \Phi \) be the solution of

\[
\begin{align*}
\text{d}X(t) &= b(X(t), t)\text{d}t + \sigma(X(t), t)\text{d}W(t), \quad X(0) = X_0, \\
\Phi(\omega, x, 0, t) &= x + \int_0^t b(\Phi(\omega, x, 0, s), s)\text{d}s + \int_0^t \sigma(\Phi(\omega, x, 0, s), t)\text{d}W(s),
\end{align*}
\]

then

\[
m(t)(A) := \mathbb{E}(\Phi(\cdot, 0, t)\# m_0(A)) \quad \forall \ A \in \mathcal{B}(\mathbb{R}^d), \quad t \in [0, T].
\]

Analogously, we have that for each \( h > 0 \)

\[
m(t + h)(A) = \mathbb{E}(\Phi(\cdot, t, t + h)\# m(t)(A)) \quad \forall \ A \in \mathcal{B}(\mathbb{R}).
\]

or equivalently, for \( \phi \in C_0^0(\mathbb{R}^d), \)

\[
\int_{\mathbb{R}^d} \phi(x) d (m(t + h))(x) = \int_{\mathbb{R}^d} \mathbb{E}[\phi(\Phi(x, t, t + h))] d (m(t))(x).
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Let \( \Phi \) be the solution of

\[ dX(t) = b(X(t), t)dt + \sigma(X(t), t)dW(t), \quad X(0) = X_0, \]

\[ \Phi(\omega, x, 0, t) = x + \int_0^t b(\Phi(\omega, x, 0, s), s)ds + \int_0^t \sigma(\Phi(\omega, x, 0, s), t)dW(s), \]

then

\[ m(t)(A) := E(\Phi(\cdot, 0, t)\#\bar{m}_0(A)) \quad \forall \ A \in \mathcal{B}(\mathbb{R}^d), \quad t \in [0, T]. \]

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or equivalently, for \( \phi \in C^0_c(\mathbb{R}^d) \),

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\int_{\mathbb{R}^d} \phi(x)d(m(t + h))(x) = \int_{\mathbb{R}^d} \mathbb{E}[\phi(\Phi(x, t, t + h))]d(m(t))(x).
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Let $\Phi$ be the solution of

$$dX(t) = b(X(t), t)dt + \sigma(X(t), t)dW(t), \quad X(0) = X_0,$$

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Analogously, we have that for each $h > 0$

$$m(t + h)(A) = \mathbb{E}(\Phi(\cdot, t, t + h)\#m(t)(A)) \quad \forall \ A \in \mathcal{B}(\mathbb{R}).$$

or equivalently, for $\phi \in C^0_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \phi(x) d(m(t + h))(x) = \int_{\mathbb{R}^d} \mathbb{E}[\phi(\Phi(x, t, t + h))] d(m(t))(x).$$
Semi-discretization in time $d = 1$

Given $h > 0$, we set $t_k = kh$ for $k = 0, \ldots, N_T$.

A random walk discretization of the Brownian motion $W(\cdot)$: Weak Euler in dimension $d = 1$:

$$
\Phi^+_h(x, t_k) := x + hb(x, t_k) + \sigma(x, t_k)\sqrt{h},
$$
$$
\Phi^-_h(x, t_k) := x + hb(x, t_k) - \sigma(x, t_k)\sqrt{h}.
$$

$$
m(t_{k+1})(A) = \frac{1}{2} (\Phi^+(\cdot, t_k) \# m(t_k)(A)) + \frac{1}{2} (\Phi^-(\cdot, t_k) \# m(t_k)(A))
$$

or equivalently, for $\phi \in C^0_c(\mathbb{R})$,

$$
\int_{\mathbb{R}} \phi(x) d (m(t_{k+1}))(x) = \frac{1}{2} \int_{\mathbb{R}} [\phi (\Phi^+(x, t_k))] d (m(t_k))(x) + \frac{1}{2} \int_{\mathbb{R}} [\phi (\Phi^-(x, t_k))] d (m(t_k))(x).
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$$m(t_{k+1})(A) = \frac{1}{2} (\Phi^+ (\cdot, t_k) \diamond m(t_k)(A)) + \frac{1}{2} (\Phi^- (\cdot, t_k) \diamond m(t_k)(A))$$

or equivalently, for $\phi \in C^0_c(\mathbb{R})$,

$$\int_\mathbb{R} \phi(x) d (m(t_{k+1}))(x) = \frac{1}{2} \int_\mathbb{R} \left[ \phi (\Phi^+ (x, t_k)) \right] d (m(t_k))(x) +$$
$$+ \frac{1}{2} \int_\mathbb{R} \left[ \phi (\Phi^- (x, t_k)) \right] d (m(t_k))(x).$$
Fully-discrete scheme, \( d = 1 \)

Given \( \Delta x > 0 \), we set \( G_{\Delta x} := \{ x_i = i\Delta x, i \in \mathbb{Z} \} \) and \( S_{\Delta x,h} := \{ (\mu_{i,k})_{i \in \mathbb{Z}, k=0,\ldots,N} ; \mu_{i,k} \geq 0, \sum_{i \in \mathbb{Z}} \mu_{i,k} = 1 \} \),

**Discrete measure:**

\[
m_k = \sum_{j \in \mathbb{Z}} m_{j,k} \delta x_j \quad \forall \ k = 0, \ldots, N - 1.
\]

\[
\sum_j \phi(x_j)m_{j,k+1} = \frac{1}{2} \sum_j \phi(\Phi^+(x_j,t_k)) m_{j,k} + \frac{1}{2} \sum_j \phi(\Phi^-(x_j,t_k)) m_{j,k}.
\]

**P\(_1\)-projection:** \( \{ \beta_i \} \) are \( P_1 \)-basis function, \( \phi(x) = \beta_i(x) \).

\[
m_{i,k+1} = \frac{1}{2} \sum_j \beta_i(\Phi^+_{j,k}) m_{j,k} + \frac{1}{2} \sum_j \beta_i(\Phi^-_{j,k}) m_{j,k}.
\]

where

\[
\Phi^+_{j,k} := x_j + hb(x_j, t_k) + \sqrt{h}\sigma(x_j, t_k),
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\Phi^-_{j,k} := x_j + hb(x_j, t_k) - \sqrt{h}\sigma(x_j, t_k).
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Fully-discrete scheme, $d = 1$

Given $\Delta x > 0$, we set $G_{\Delta x} := \{x_i = i\Delta x, i \in \mathbb{Z}\}$ and $S_{\Delta x,h} := \{(\mu_{i,k})_{i \in \mathbb{Z}}, k=0,\ldots,N; \mu_{i,k} \geq 0, \sum_{i \in \mathbb{Z}} \mu_{i,k} = 1\}$.

Discrete measure:

$$m_k = \sum_{j \in \mathbb{Z}} m_{j,k} \delta_{x_j} \quad \forall \ k = 0, \ldots, N - 1.$$ 

$$\sum_{j} \phi(x_j)m_{j,k+1} = \frac{1}{2} \sum_{j} \phi \left( \Phi^+(x_j, t_k) \right) m_{j,k} + \frac{1}{2} \sum_{j} \phi \left( \Phi^-(x_j, t_k) \right) m_{j,k}.$$ 

$\mathbb{P}_1$-projection: $\{\beta_i\}$ are $\mathbb{P}_1$-basis function, $\phi(x) = \beta_i(x)$.

$$m_{i,k+1} = \frac{1}{2} \sum_{j} \beta_i \left( \Phi^+_{j,k} \right) m_{j,k} + \frac{1}{2} \sum_{j} \beta_i \left( \Phi^-_{j,k} \right) m_{j,k}.$$ 

where

$$\Phi^+_{j,k} := x_j + h b(x_j, t_k) + \sqrt{h} \sigma(x_j, t_k),$$

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Fully-discrete scheme for the non linear case

Given \( \mu \in \mathcal{S}^{\Delta x, h} \)

Non-linear discrete characteristics

\[
\Phi^+_{i,k}[\mu] := x_i + h b[\mu](x_i, t_k) + \sqrt{h} \sigma[\mu](x_i, t_k),
\]

\[
\Phi^-_{i,k}[\mu] := x_i + h b[\mu](x_i, t_k) - \sqrt{h} \sigma[\mu](x_i, t_k).
\]

The discretization of \((FPK)\) we propose is:

\[
\text{find } m \in \mathcal{S}^{\Delta x, h} \text{ such that }
\]

\[
\begin{cases}
  m_{i,0} = \bar{m}_0(E_i) \\
  m_{i,k+1} = \frac{1}{2} \sum_{j \in \mathbb{Z}} \left[ \beta_i(\Phi^+_{j,k}[m]) + \beta_i(\Phi^-_{j,k}[m]) \right] m_{j,k} \\
  \forall \ i \in \mathbb{Z}^d, \ k = 0, \ldots, N - 1.
\end{cases}
\]

where \( E_i = [x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}] \).
Fully-discrete scheme for the non linear case

Given $\mu \in \mathcal{S}^{\Delta x, h}$

Non linear discrete characteristics

$$\Phi_{i,k}^+[\mu] := x_i + h b[\mu](x_i, t_k) + \sqrt{h} \sigma[\mu](x_i, t_k),$$

$$\Phi_{i,k}^-[\mu] := x_i + h b[\mu](x_i, t_k) - \sqrt{h} \sigma[\mu](x_i, t_k).$$

The discretization of \((FPK)\) we propose is:

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\[
(S) \begin{cases}
  m_{i,0} = \bar{m}_0(E_i) \\
  m_{i,k+1} = \frac{1}{2} \sum_{j \in \mathbb{Z}} \left[ \beta_i(\Phi_{j,k}^+[m]) + \beta_i(\Phi_{j,k}^-[m]) \right] m_{j,k} \\
  \forall \ i \in \mathbb{Z}^d, \ k = 0, \ldots, N - 1.
\end{cases}
\]

where $E_i = [x_i - \frac{\Delta x}{2}, x_i + \frac{\Delta x}{2}]$. 
Main properties

- **Non-negative**: $m_{i,k} \geq 0$ for $k = 0, \ldots, N - 1, i \in \mathbb{Z}$
- Mass conservative: $\sum_i m_{i,k} = 1$ for $k = 0, \ldots, N - 1$
- Generalizable to any dimension
- Generalizable to handle Dirichlet and Neumann Boundary conditions
- Generalizable to handle degeneracy of the diffusion matrix
- Large time steps are allowed: inverse CFL type condition
  \[
  \frac{(\Delta x)^2}{h} \to 0
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- Duality property: this scheme is the DUAL of the classical Semi-Lagrangian scheme applied to Kolmogorov forward equation
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Dual Problem

Kolmogorov forward equation (FP)

\[
\begin{cases}
\partial_t m = \frac{1}{2} \sum_{i,j} \partial_{x_i x_j} (a_{ij}(x)m) - \text{div} (b(x)m) & \mathbb{R}^d \times (0, T] \\
m(\cdot, 0) = m_0
\end{cases}
\]

Kolmogorov backward equation (KB):

\[
\begin{cases}
-\partial_t u = \frac{1}{2} \sum_{i,j} a_{ij}(x) \partial_{x_i x_j} u + b(x)^\top Du & \mathbb{R}^d \times (0, T] \\
u(\cdot, T) = u_T
\end{cases}
\]  \hspace{1cm} (1)
Dual Problems

Kolmogorov forward equation (FP)

\[
\begin{align*}
\frac{\partial}{\partial t} m &= \frac{1}{2} \sum_{i,j} \partial_{x_i} \partial_{x_j} (a_{ij}(x)m) - \text{div} (b(x)m) = L^*(m) \\
\quad \text{in } \mathbb{R}^d \times (0, T] \\
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u(\cdot, T) = u_T
\end{cases}
\quad \mathbb{R}^d \times (0, T]
\]

$L^*$ is the dual of $L$ with respect to the $L_2$ inner product:

\[
\int L(f)g dx = \int L^*(g)f dx
\]
Dual Schemes $d = 1$

The **SL scheme for FP** can be written in vectorial form as

$$
\mu_{k+1} := B^* \mu_k
$$

where, $\mu_k = (\mu_{j,k})_k$ and $(B^*)_{i,j} = \frac{1}{2} (\beta_i (\Phi_{j,+}) + \beta_i (\Phi_{j,-}))$.

The **SL scheme for KB**

$$
u_{i,k} = \frac{1}{2} (I[v_{k+1}](\Phi_{i,+}) + I[v_{k+1}](\Phi_{i,-})) = \frac{1}{2} \sum_{j \in \mathbb{Z}} [\beta_j (\Phi_{i,+}) + \beta_j (\Phi_{i,-})] v_{j,k+1}
$$

can also be written in vectorial form as

$$
v_k := Bv_{k+1},
$$

where, $v_k = (v_{j,k})_k$ and $B^\top = B^*$, i.e.

$$(Bv_{k+1}, \mu_k) = (v_{k+1}, B^* \mu_k)$$
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The SL scheme for KB

$$v_{i,k} = \frac{1}{2} \left( I[v_{k+1}](\Phi_{i,+}) + I[v_{k+1}](\Phi_{i,-}) \right) = \frac{1}{2} \sum_{j \in \mathbb{Z}} [\beta_j (\Phi_{i,+}) + \beta_j (\Phi_{i,-})] v_{j,k+1}$$

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3. Mean Field Games
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4. Lagrange Galerkin
Main assumptions

\textbf{(H)}

- $\bar{m}_0 \in \mathcal{P}_2(\mathbb{R}^d)$.
- The maps $b$ and $\sigma$ are continuous.
- There exists $C > 0$ such that
  \[|b(m, x, t)| + |\sigma(m, x, t)| \leq C(1 + |x|) \quad \forall m, \quad x \in \mathbb{R}^{d_\ell}, \quad t \in [0, T].\]

\textbf{(Lip)}

- $b$ and $\sigma$ are Lipschitz w.r. to $x$, uniformly in $t \in [0, T]$
Well posedeness

**Proposition**

*Under assumption (H), there exists at least one solution \( m_{i,k} \in \mathcal{S}^{\Delta x}_h \) of \((S)\).*

Given \( m_{i,k} \in \mathcal{S}^{\Delta x,h} \), we define its extension \( m_{\Delta x}(t) \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d)) \)

\[
m_{\Delta x}(t) := \left( \frac{t - t_k}{h} \right) \sum_{i \in \mathbb{Z}^d} m_{i,k+1} \delta_{x_i} + \left( \frac{t_{k+1} - t}{h} \right) \sum_{i \in \mathbb{Z}^d} m_{i,k} \delta_{x_i}
\]

for \( t \in [t_k, t_{k+1}] \) and \( k = 0, \ldots, N - 1 \).
Let us denote the Wasserstein distance by

\[
d_1(\mu_1, \mu_2) = \sup \left\{ \int_{\mathbb{R}^d} f(x) d(\mu_1 - \mu_2)(x) ; f \in \text{Lip}_1(\mathbb{R}^d) \right\}.
\]
Convergence

Theorem

Under assumptions \((H)\) and \((\text{Lip})\), and \(\frac{(\Delta x)^2}{h} \to 0\), we have that as \((\Delta x) \to 0\)

\[(m_{\Delta x,h}) \to (m)\]

in \(C([0,T], \mathcal{P}_1)\), where \(m\) is solution of \((FPK)\) (there exists at least one) and \(m_{\Delta x,h}\), is solution of \((S)\).

Remark: The result holds in any dimension \(d \geq 1\)
Convergence non regular case

If (Lip) does not hold, and $b, \sigma$ verify only (H), it is necessary to regularize them, by using mollifiers.

$$b^\varepsilon[m](x, t) := \varphi \ast b[m](x, t), \quad \sigma^\varepsilon[m](x, t) := \varphi \ast \sigma[m](x, t)$$

We will apply this technique to approximate the solution of Mean Field Game Problem.

Find $m^\varepsilon \in S^{\Delta x, h}$ such that

$$\begin{align*}
(S^\varepsilon) \left\{ 
\begin{array}{l}
m_{i,0}^\varepsilon = \bar{m}_0(E_i) \\
m_{i,k+1}^\varepsilon = \frac{1}{2} \sum_{j \in \mathbb{Z}} \left[ \beta_i(\Phi_{j,k}^+, \varepsilon[m^\varepsilon]) + \beta_i(\Phi_{j,k}^-, \varepsilon[m^\varepsilon]) \right] m_{j,k}^\varepsilon \\
\forall i \in \mathbb{Z}^d, \quad k = 0, \ldots, N - 1.
\end{array}
\right.
\end{align*}$$
Convergence non regular case

Theorem

Under assumptions \((H)\), and \(\frac{(\Delta x)^2}{h} \to 0, \frac{h}{\varepsilon^2} \to 0\) we have that as 
\((\Delta x, h, \varepsilon) \to 0\)

\[ m^{\varepsilon}_{\Delta x, h} \to m \]

in \(C([0, T], \mathcal{P}_1)\), where \(m\) is solution of \((FPK)\) (there exists at least one) and \(m^{\varepsilon}_{\Delta x, h}\) are solution of \((S^{\varepsilon})\).

Remark

In the uniform elliptic case, the assumption \(h = o(\varepsilon)\)

Remark

In the degenerate elliptic case, we need to construct approximations which are absolutely continuous w.r. to the Lebesgue measure.

If \(d = 1\), uniform bound in \(L^\infty\) is shown for the approximated density and a convergence result is proved.
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**Theorem**

*Under assumptions (H), and \( \frac{(\Delta x)^2}{h} \to 0, \frac{h}{\varepsilon^2} \to 0 \) we have that as \((\Delta x, h, \varepsilon) \to 0\)\n
\[ m_{\Delta x, h}^\varepsilon \to m \]

in \( C([0, T], \mathcal{P}_1) \), where \( m \) is solution of \((FPK)\) (there exists at least one) and \( m_{\Delta x, h}^\varepsilon \) are solution of \((S_\varepsilon)\).*

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In this case the velocity field in the FP is

\[ b[m](x, t) = Dv[m](x, t) \]

where \( v[m] \) is the solution of the first equation in the following system:

\[
\begin{aligned}
-\partial_t v - \sigma \Delta v + \frac{1}{2} |Dv|^2 &= F(x, m(t)), & \text{in } \mathbb{R} \times (0, T), \\
\partial_t m - \sigma \Delta m - \text{div}(Dvm) &= 0, & \text{in } \mathbb{R} \times (0, T), \\
v(x, T) &= G(x, m(T)) & \text{in } \mathbb{R} \times \{T\} \\
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Model introduced independently by Huang-Malhamé-Caines and, independently, by Lasry-Lions in 2006.
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(H1) $F$ and $G$ are continuous.

(H2) There exists a constant $c_0 > 0$ such that for any $m \in \mathcal{P}_1$

$$\|F(\cdot, m)\|_{C^2} + \|G(\cdot, m)\|_{C^2} \leq c_0,$$

where $\|f(\cdot)\|_{C^2} := \sup_{x \in \mathbb{R}^d}\{ |f(x)| + |Df(x)| + |D^2f(x)| \}$.

(H3) The initial condition $m_0 \in \mathcal{P}_1$ is absolutely continuous w. r. to the Lebesgue measure, with density $m_0$ s.t. $\text{supp}(m_0) \subset B(0, c)$ and $\|m_0\|_{\infty} \leq c$, for $c > 0$.

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$$\int_{\mathbb{R}^d} [F(x, m_1) - F(x, m_2)] d[m_1 - m_2](x) \geq 0 \quad \text{for all } m_1, m_2 \in \mathcal{P}_1$$

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Some references of Numerical Approximation of MFG

- **Second order problem \((\sigma \neq 0)\)**
  - Y. Achdou, I. Capuzzo-Dolcetta ('10),
    Y. Achdou, F. Camilli, I. Capuzzo-Dolcetta ('12)
    (Semi-implicit Finite Difference scheme, Newton Iteration)
  - A. Lachapelle, M.-T. Wolfram (Steepest descent approach for the optimal control problem),
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- **First order problem \((\sigma = 0)\)**
  - F. Camilli, F. J. Silva ('12) (semi-discrete Semi-Lagrangian scheme)
  - S. Hadikhanloo, F. J. Silva ('19) (Semi-Lagrangian type with no interpolation)
  - Nuberkyan-Saude ('19) and Li-Jacobs-Li-Nuberkyan-Osher ('20) (Fourier methods)
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SL scheme for HJB

- We use a Semi-Lagrangian scheme to approximate $v$.
- We call $v_{\Delta x}$ the resulting interpolated discrete value functions.
- We regularize them by using space convolution

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- If $\sigma = 0$ then $Dv_{\Delta x_n}^\varepsilon[m_n] \to Dv[m]$ a.e., the convergence result has been proved only for the case $d = 1$
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If $\sigma = 0$ then $Dv_{\Delta x_n}^{\epsilon_n}[m_n] \rightarrow Dv[m]$ a.e., the convergence result has been proved only for the case $d = 1$.

If $\sigma \neq 0$ then $Dv_{\Delta x_n}^{\epsilon_n}[m_n] \rightarrow Dv[m]$ uniformly, the convergence is proved in general dimension.
We use a Semi-Lagrangian scheme to approximate $v$.

We call $v_{\Delta x}$ the resulting interpolated discrete value functions

We regularize them by using space convolution

$$v_{\Delta x,h}[m](\cdot, t) := \phi_{\varepsilon} \ast v_{\Delta x}[m](\cdot, t) \quad \forall \ t \in [0, T],$$

We approximate the drift by

$$b[m](x, t) := -Dv_{\Delta x}[m](x, t).$$

If $\sigma = 0$ then $Dv_{\Delta x_n}[m_n] \to Dv[m]$ a.e., the convergence result has been proved only for the case $d = 1$

If $\sigma \neq 0$ then $Dv_{\Delta x_n}[m_n] \to Dv[m]$ uniformly, the convergence is proved in general dimension.
Domain $\Omega \times (0, T) = (-3, 3) \times (0, 5)$.

Running cost

$$F(x, t, m(t)) = d(x, D)^2 V_\delta(x, m(t)),$$

$V_\delta(x, m) = (\phi_\delta * (\phi_\delta * m))(x)$, $\phi_\delta(x) := \frac{1}{\delta \sqrt{2\pi}} \exp\left(-\frac{x^2}{2\delta^2}\right)$, $\delta = 0.01$

$d(x, D)$ is the distance function from the set $D := [1, 1.5] \cup [-2, -2.5]$.

Final cost: $G(x, T, m(T)) = F(x, T, m(T))$

Initial mass distribution:

$$m_0(x) = \frac{\nu(x)}{\int_{\Omega} \nu(x) dx} \text{ with } \nu(x) = e^{-x^2/0.2}$$

Regularizing kernel $\phi_\varepsilon(x)$, with $\varepsilon = 0.15$.

Diffusion term $\sigma = 0$, first order MFG system

Discretization step $\Delta x = h = 0.02$.

Fix point: computed by a learning procedure as proposed by Cardaliaguet and Hadikhanloo.
Figure: Density evolution 3d and 2d view in the \((x, t)\) domain
Numerical test First order MFG

Figure: Density at time $t = 0, 0.6, T$ (black squares on the $x$ axis represents the ‘meeting areas’)
A new Hughes type model

In this model the velocity field in the FP is

\[ b[m](x, t) = Dv[m](x, t) \]

where \( v[m] \) is the solution of the first equation in the system.

\[
\begin{cases}
-\partial_s v(x, s) + \frac{1}{2}|Dv(x, s)|^2 = F(x, s, m(t)) & \text{in } \mathbb{R} \times (t, T), \\
\partial_t m - \text{div}(Dvm) = 0 & \text{in } \mathbb{R} \times (0, T), \\
v(x, T) = G(x, m(t)) & \text{for } x \in \mathbb{R}, \\
m(\cdot, 0) = m_0(\cdot)
\end{cases}
\]

(2)

where

\[ F(x, s, m(t)) = d(x, P)^2 V_\delta(x, m(t)). \]

Since \( b[m](x, t) \) depends on \( m(s) \) only at time past \( t \) we get an explicit scheme.
Numerical test new Hughes type model

Figure: Density evolution 3d and 2d view in the \((x, t)\) domain
Numerical test new Hughes type mode

Figure: Density at time $t = 0, 30h, T$ (black squares on the $x$ axis represents the ‘meeting areas’).
Numerical test new Hughes type mode

Figure: MFG (left) vs Hughes type model (right)
1. Numerical approximation of FPK

2. Convergence Analysis

3. Mean Field Games
   - Non linear explicit case: a new Hughes type model

4. Lagrange Galerkin
A Lagrange Galerkin scheme for the continuity equation

\[(CE) \begin{cases} \partial_t m + \text{div} \ (b(x, t)m) = 0 \quad (0, T) \times \mathbb{R}^d, \\
m(0, \cdot) = m_0(\cdot) \quad \mathbb{R}^d \end{cases} \]

where

- **(A1)** Let us suppose \( b(x, t) \in L^\infty(0, T; (W^{1, \infty}(\mathbb{R}^d))^d) \)
- **(A2)** \( m_0(\cdot) \in L^2(\mathbb{R}^d) \) with compact support

Representation formula: for any \( \phi \in C^\infty_c(\mathbb{R}^d) \),

\[ \int_{\mathbb{R}^d} \phi(x)m(x, t_{k+1})dx = \int_{\mathbb{R}^d} [\phi (\Phi(x, t_k))] \ m(x, t_k)dx \]

where \( \Phi(x, t_k) \) are the forward characteristics, solving

\[ \begin{cases} \dot{X}(s) = b(X(s), s), \quad s \in [0, h], \\
X(t_k) = x, \end{cases} \]
A Lagrange Galerkin scheme for the continuity equation

(CE) \[
\begin{aligned}
\partial_t m + \text{div} (b(x, t)m) &= 0 & (0, T) \times \mathbb{R}^d, \\
m(0, \cdot) &= m_0(\cdot) & \mathbb{R}^d
\end{aligned}
\]

where

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\[
\int_{\mathbb{R}^d} \phi(x)m(x, t_{k+1})dx = \int_{\mathbb{R}^d} [\phi(\Phi(x, t_k))] m(x, t_k)dx
\]

where \( \Phi(x, t_k) \) are the forward characteristics, solving

\[
\begin{aligned}
\dot{X}(s) &= b(X(s), s), & s \in [0, h], \\
X(t_k) &= x,
\end{aligned}
\]
A Lagrange Galerkin scheme for the continuity equation

Set

\[ \Phi_h(x, t_k) := x + hb(x, t_k) \]

Semi-discrete scheme

\[
\int_{\mathbb{R}^d} \phi(x) m(x, t_{k+1}) \, dx = \int_{\mathbb{R}^d} \left[ \phi(\Phi_h(x, t_k)) \right] m(x, t_k) \, dx
\]

Structured mesh \( G_{\Delta x} := \{ x_i = i\Delta x; i \in \mathbb{Z}^d \} \) with \( \Delta x > 0 \) a given space step,

Standard parallelepipedal finite elements basis \( \{ \beta_i \}_{i \in \mathbb{Z}^d} \), finite element space \( V_{\Delta x} = \{ v_{\Delta x} \in L^2(\mathbb{R}^d) \text{ such that } v_{\Delta x}(x) = \sum_{i \in \mathbb{Z}^d} v_i \beta_i(x) \} \)

We consider the following approximation of \( m(x, t) \)

\[ m_{\Delta x}(x, t_k) := \sum_{i \in \mathbb{Z}^d} m_{i,k} \beta_i(x) \quad \forall x \in \mathbb{R}^d, \]

for some weights \( \{ m_{i,k} \mid k = 0, \ldots, n, \ i \in \mathbb{Z}^d \} \subseteq \mathbb{R} \)
A Lagrange Galerkin scheme for the continuity equation

- Set

\[ \Phi_h(x, t_k) := x + h b(x, t_k) \]

**Semi-discrete scheme**

\[ \int_{\mathbb{R}^d} \phi(x) m(x, t_{k+1}) \, dx = \int_{\mathbb{R}^d} [\phi(\Phi_h(x, t_k))] \, m(x, t_k) \, dx \]

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\[ m_{\Delta x}(x, t_k) := \sum_{i \in \mathbb{Z}^d} m_i, k \beta_i(x) \quad \forall \ x \in \mathbb{R}^d, \]

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A Lagrange Galerkin scheme for the continuity equation

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\[ \Phi_h(x, t_k) := x + hb(x, t_k) \]

**Semi-discrete scheme**

\[ \int_{\mathbb{R}^d} \phi(x) m(x, t_{k+1}) dx = \int_{\mathbb{R}^d} [\phi(\Phi_h(x, t_k))] m(x, t_k) dx \]

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A Lagrange Galerkin scheme for the continuity equation

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for some weights \( \{ m_{i,k} \mid k = 0, \ldots, n, \ i \in \mathbb{Z}^d \} \subseteq \mathbb{R} \)
A Lagrange Galerkin scheme for the continuity equation

We project the semi discrete scheme

\[ \int_{\mathbb{R}^d} \phi(x)m(x, t_{k+1})dx = \int_{\mathbb{R}^d} [\phi(\Phi_h(x, t_k))] m(x, t_k)dx \]

in \( V_{\Delta x} \): find \( m_{i, k} \), with \( i \in \mathbb{Z}^d \) and \( k = 0, \ldots, N \)

(\( LG \))

\[ \begin{align*}
\sum_{i \in \mathbb{Z}^d} m_{i,k+1} \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)dx &= \sum_{i \in \mathbb{Z}^d} m_{i,k} \int_{\mathbb{R}^d} \beta_j(\Phi_h(t_k, x))\beta_i(x)dx \\
\sum_{i \in \mathbb{Z}^d} m_{i,0} \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)dx &= \int_{\mathbb{R}^d} m_0(x)\beta_j(x)dx.
\end{align*} \]

The \( (LG) \) scheme for \( (CE) \) can be written in vectorial form as

\[ Mm_{k+1} := Bm_k \]

where, \( m_k = (m_{j,k})_j \),

\( (M)_{i,j} = \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)dx \), \( (B)_{i,j} = \int_{\mathbb{R}^d} \beta_i(x)\beta_j(\Phi_h(t_k, x))dx \)

Ref. Morton, Priestley, Suli ('88)
A Lagrange Galerkin scheme for the continuity equation

We project the semi discrete scheme

\[ \int_{\mathbb{R}^d} \phi(x) m(x, t_{k+1}) dx = \int_{\mathbb{R}^d} \phi(\Phi_h(x, t_k)) m(x, t_k) dx \]

in \( V_{\Delta x} \): find \( m_{i,k} \), with \( i \in \mathbb{Z}^d \) and \( k = 0, ..., N \)

\[
\begin{cases}
\sum_{i \in \mathbb{Z}^d} m_{i,k+1} \int_{\mathbb{R}^d} \beta_i(x) \beta_j(x) dx = \sum_{i \in \mathbb{Z}^d} m_{i,k} \int_{\mathbb{R}^d} \beta_j(\Phi_h(t_k, x)) \beta_i(x) dx \\
\sum_{i \in \mathbb{Z}^d} m_{i,0} \int_{\mathbb{R}^d} \beta_i(x) \beta_j(x) dx = \int_{\mathbb{R}^d} m_0(x) \beta_j(x) dx.
\end{cases}
\]

The (LG) scheme for (CE) can be written in vectorial form as

\[
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\[
(M)_{i,j} = \int_{\mathbb{R}^d} \beta_i(x) \beta_j(x) dx, \quad (B)_{i,j} = \int_{\mathbb{R}^d} \beta_i(x) \beta_j(\Phi_h(t_k, x)) dx
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Ref. Morton, Priestley, Suli (’88)
A Lagrange Galerkin scheme for the continuity equation

**Proposition**

Under assumption (A1)-(A2), the following assertions hold true:

(i) **Well-posedness** There exists a unique solution to (LG).

(ii) **Non-negativity** If \( \{\beta_i\}_{i \in \mathbb{Z}^d} \) is s.t. \( \int_{\mathbb{R}^d} \beta_i(x) \beta_j(x) = \delta_{i,j} \) and \( \beta_i(x) \geq 0 \), we have \( m_{i,k} \geq 0 \)

(iii) **Mass conservation** \( \int_{\mathbb{R}^d} m_{\Delta x}(t_k, x) dx = 1 \)

(iv) **\( L^2 \)-stability** If \( h \) is sufficiently small, there exists \( C > 0 \), s.t.

\[
\|m_{\Delta x}(t_k, \cdot)\|_{L^2} \leq C \|m_0\|_{L^2}
\]

(v) **Equi-continuity** Suppose \( (\Delta x)^2 = O(h) \), for all \( t_1, t_2 \in [0, T] \), we have that

\[
d_1(m_{\Delta x}(t_1), m_{\Delta x}(t_2)) \leq C |t_1 - t_2|.
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A Lagrange Galerkin scheme for the continuity equation

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A Lagrange Galerkin scheme for the continuity equation

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A Lagrange Galerkin scheme for the continuity equation

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Under assumption (A1)-(A2), the following assertions hold true:

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\[
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\]

(v) **Equi-continuity** Suppose \((\Delta x)^2 = O(h)\), for all \( t_1, t_2 \in [0, T] \), we have that

\[
d_1(m_{\Delta x}(t_1), m_{\Delta x}(t_2)) \leq C|t_1 - t_2|.
\]
First order MFG

We consider the first order case $\sigma = 0$ and the particular case of a quadratic Hamiltonian:

$$
\begin{aligned}
(MFG) \quad &
\begin{cases}
-\partial_t v(x, t) + \frac{1}{2}|Dv(x, t)|^2 = F(x, m(t)) & \mathbb{R}^d \times (0, T) \\
v(x, T) = G(x, m(T)) & \mathbb{R}^d \\
\partial_t m(x, t) - \text{div}(Dv(x, t)m(x, t)) = 0 & \mathbb{R}^d \times (0, T) \\
m(0) = m_0 & \mathbb{R}^d
\end{cases}
\end{aligned}
$$
We use a Semi-Lagrangian scheme to approximate $v[m]$. We call $v_{\Delta x}[m]$ the resulting interpolated discrete value functions. We regularize them by using space convolution

$$v^\varepsilon_{\Delta x}[m](\cdot, t) := \phi \ast v_{\Delta x}[m](\cdot, t) \quad \forall t \in [0, T],$$

Lemma

For every $t \in [0, T]$, the following assertions hold true:

(i) Lipschitz property The function $v^\varepsilon_{\Delta x}[\mu](\cdot, t)$ is Lipschitz with constant $d_0$ independent of $(\Delta x, h, \mu, t)$.

(i) Semiconcavity There exists $d_1 > 0$ independent of $(\Delta x, h, \varepsilon, \mu, t)$, such that

$$\langle D^2 v^\varepsilon_{\Delta x}[\mu](x, t) y, y \rangle \leq d_1 \left(1 + \frac{\Delta x^2}{\varepsilon^4}\right) |y|^2 \quad \forall x, y \in \mathbb{R}^d. \quad (3)$$
SL scheme for HJB

- We use a Semi-Lagrangian scheme to approximate $v[m]$.
- We call $v_{\Delta x}[m]$ the resulting interpolated discrete value functions.
- We regularize them by using space convolution

$$v_{\Delta x}^\varepsilon[m](\cdot, t) := \phi_\varepsilon \ast v_{\Delta x}[m](\cdot, t) \quad \forall \ t \in [0, T],$$

**Lemma**

For every $t \in [0, T]$, the following assertions hold true:

(i) **Lipschitz property** The function $v_{\Delta x}^\varepsilon[\mu](\cdot, t)$ is Lipschitz with constant $d_0$ independent of $(\Delta x, h, \mu, t)$.

(i) **Semiconcavity** There exists $d_1 > 0$ independent of $(\Delta x, h, \varepsilon, \mu, t)$, such that

$$\langle D^2 v_{\Delta x}^\varepsilon[\mu](x, t) y, y \rangle \leq d_1 \left(1 + \frac{\Delta x^2}{\varepsilon^4}\right) |y|^2 \quad \forall \ x, y \in \mathbb{R}^d.$$  \hspace{1cm} (3)
We use a Semi-Lagrangian scheme to approximate $v[m]$. We call $v_{\Delta x}[m]$ the resulting interpolated discrete value functions. We regularize them by using space convolution:

$$v_{\Delta x}^\varepsilon[m](\cdot, t) := \phi_{\varepsilon} \ast v_{\Delta x}[m](\cdot, t) \quad \forall \ t \in [0, T],$$

**Lemma**

For every $t \in [0, T]$, the following assertions hold true:

(i) **Lipschitz property** The function $v_{\Delta x}^\varepsilon[\mu](\cdot, t)$ is Lipschitz with constant $d_0$ independent of $(\Delta x, h, \mu, t)$.

(ii) **Semiconcavity** There exists $d_1 > 0$ independent of $(\Delta x, h, \varepsilon, \mu, t)$, such that

$$\langle D^2 v_{\Delta x}^\varepsilon[\mu](x, t)y, y \rangle \leq d_1 \left(1 + \frac{\Delta x^2}{\varepsilon^4}\right) |y|^2 \quad \forall \ x, y \in \mathbb{R}^d. \quad (3)$$
We use a Semi-Lagrangian scheme to approximate $v[m]$.

We call $v_{\Delta x}[m]$ the resulting interpolated discrete value functions.

We regularize them by using space convolution:

$$v^{\varepsilon}_{\Delta x}[m](\cdot, t) := \phi_{\varepsilon} * v_{\Delta x}[m](\cdot, t) \quad \forall \ t \in [0, T],$$

**Lemma**

For every $t \in [0, T]$, the following assertions hold true:

(i) **Lipschitz property** The function $v^{\varepsilon}_{\Delta x}[\mu](\cdot, t)$ is Lipschitz with constant $d_0$ independent of $(\Delta x, h, \mu, t)$.

(ii) **Semiconcavity** There exists $d_1 > 0$ independent of $(\Delta x, h, \varepsilon, \mu, t)$, such that

$$\langle D^2 v^{\varepsilon}_{\Delta x}[\mu](x, t)y, y \rangle \leq d_1 \left(1 + \frac{\Delta x^2}{\varepsilon^4}\right) |y|^2 \quad \forall \ x, y \in \mathbb{R}^d.$$(3)
A Lagrange Galerkin scheme for deterministic MFG

Given \( \mu \in C([0, T]; \mathcal{P}_1) \) and \( \varepsilon > 0 \) let us define

\[
\Phi_h^\varepsilon[\mu](x, t_k) := x - hDv_\Delta x^\varepsilon[\mu](x, t_k)
\]

We propose the following scheme for (MFG):

Find \( \mu = (\mu^k_i) \) such that \( \mu_i, k = m^\varepsilon_{i, k}[\mu] \)

where \( m^\varepsilon_{i, k}[\mu] \) is defined as

\[
\begin{align*}
\sum_{i \in \mathbb{Z}^d} m^\varepsilon_{i, k+1} \int_{\mathbb{R}^d} \beta_i(x) \beta_j(x) dx &= \sum_{i \in \mathbb{Z}^d} m^\varepsilon_{i, k} \int_{\mathbb{R}^d} \beta_j(\Phi_h^\varepsilon[\mu](t_k, x)) \beta_i(x) dx \\
\sum_{i \in \mathbb{Z}^d} m^\varepsilon_{i, 0} \int_{\mathbb{R}^d} \beta_i(x) \beta_j(x) dx &= \int_{\mathbb{R}^d} m_0(x) \beta_j(x) dx
\end{align*}
\]

The (LG) scheme for (MFG) can be written in vectorial form as

\[
M m^\varepsilon_{k+1} := B^\varepsilon m^\varepsilon_k
\]

where, \( m^\varepsilon_k = (m^\varepsilon_{j, k})_j \), \( (B^\varepsilon)_{i, j} = \int_{\mathbb{R}^d} \beta_j(\Phi_h^\varepsilon[\mu](t_k, x)) \beta_i(x) dx \).
A Lagrange Galerkin scheme for deterministic MFG

Given $\mu \in C([0,T];\mathcal{P}_1)$ and $\varepsilon > 0$ let us define

$$\Phi^\varepsilon_h[\mu](x, t_k) := x - hDv^\varepsilon_{\Delta x}[\mu](x, t_k)$$

We propose the following scheme for (MFG):

Find $\mu = (\mu^k_i)$ such that $\mu_{i,k} = m_{i,k}^\varepsilon[\mu]$ where $m_{i,k}^\varepsilon[\mu]$ is defined as

$$\begin{cases}
\sum_{i \in \mathbb{Z}^d} m_{i,k+1}^\varepsilon \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)\,dx = \sum_{i \in \mathbb{Z}^d} m_{i,k}^\varepsilon \int_{\mathbb{R}^d} \beta_j(\Phi^\varepsilon_h[\mu](t_k, x))\beta_i(x)\,dx \\
\sum_{i \in \mathbb{Z}^d} m_{i,0}^\varepsilon \int_{\mathbb{R}^d} \beta_i(x)\beta_j(x)\,dx = \int_{\mathbb{R}^d} m_0(x)\beta_j(x)\,dx
\end{cases}$$

The (LG) scheme for (MFG) can be written in vectorial form as

$$Mm_{k+1}^\varepsilon := B^\varepsilon m_k^\varepsilon$$

where, $m_k^\varepsilon = (m_{j,k}^\varepsilon)_j$, $(B^\varepsilon)_{i,j} = \int_{\mathbb{R}^d} \beta_j(\Phi^\varepsilon_h[\mu](t_k, x))\beta_i(x)\,dx$. 
A Lagrange Galerkin scheme for deterministic MFG

Given $\mu \in C([0, T]; \mathcal{P}_1)$ and $\varepsilon > 0$ let us define

$$\Phi^{\varepsilon}_h[\mu](x, t_k) := x - hDv^{\varepsilon}_{\Delta x}[\mu](x, t_k)$$

We propose the following scheme for (MFG):

Find $\mu = (\mu^k_i)$ such that $\mu_i, k = m^{\varepsilon}_{i,k}[\mu]$ where $m^{\varepsilon}_{i,k}[\mu]$ is defined as

$$
\begin{cases}
\sum_{i \in \mathbb{Z}^d} m^{\varepsilon}_{i,k+1} \int_{\mathbb{R}^d} \beta_i(x) \beta_j(x) dx = \sum_{i \in \mathbb{Z}^d} m^{\varepsilon}_{i,k} \int_{\mathbb{R}^d} \beta_j(\Phi^{\varepsilon}_h[\mu](t_k, x)) \beta_i(x) dx \\
\sum_{i \in \mathbb{Z}^d} m^{\varepsilon}_{i,0} \int_{\mathbb{R}^d} \beta_i(x) \beta_j(x) dx = \int_{\mathbb{R}^d} m_0(x) \beta_j(x) dx
\end{cases}
$$

The (LG) scheme for (MFG) can be written in vectorial form as

$$M m^{\varepsilon}_{k+1} := B^{\varepsilon} m^{\varepsilon}_k$$

where, $m^{\varepsilon}_k = (m^{\varepsilon}_{j,k})_j$, $(B^{\varepsilon})_{i,j} = \int_{\mathbb{R}^d} \beta_j(\Phi^{\varepsilon}_h[\mu](t_k, x)) \beta_i(x) dx$. 
Convergence analysis LG for MFG

Key Property: Semiconcavity of $v^\varepsilon_{\Delta x}$

Proposition

Under assumptions (H1)-(H2)-(H3), the following assertions hold true:

(i) **$L^2$-stability** If $h$ is sufficiently small, there exists a constant $C > 0$, such that

$$
\| m^\varepsilon_{\Delta x}(t_k, \cdot) \|_{L^2} \leq c \| m_0 \|_{L^2}
$$

(ii) **Equicontinuity** Suppose $(\Delta x)^2 = O(h)$, for all $t_1, t_2 \in [0, T]$, we have that

$$
d_1(m^\varepsilon_{\Delta x}(t_1), m_{\Delta x}(t_2)) \leq C |t_1 - t_2|.
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Theorem

Under assumptions (H1)-(H2)-(H3), consider a sequence of positive numbers $\Delta x_n, h_n, \varepsilon_n$ satisfying that

$$
\Delta x_n/\varepsilon_n^2 \leq C \quad (\Delta x_n)^2/h_n \to 0, \quad h_n/\varepsilon_n \to 0
$$

as $\varepsilon_n \downarrow 0$. Let $\{m_{\Delta x_n}^{\varepsilon_n}\}_{n \in \mathbb{N}}$ be a sequence of solutions of (LG) for the corresponding parameters $\Delta x_n, h_n, \varepsilon_n$.

Then every limit point in $C([0, T]; \mathcal{P}_1)$ and in $L^2(\mathbb{R}^d \times [0, T])$-weak of $m_{\Delta x_n}^{\varepsilon_n}$ (there exists at least one) solves (MFG).

In particular, if (H4) holds we have that $m_{\Delta x_n}^{\varepsilon_n} \to m$ (the unique solution of (MFG)) in $C([0, T]; \mathcal{P}_1)$ and in $L^2(\mathbb{R}^d \times [0, T])$-weak.

Possible Setting of parameters

$$
h = \Delta x, \quad \varepsilon = \sqrt{\Delta x}
$$
LG + area-weighting

In general, the integral \( \int_{\mathbb{R}^d} \beta_j(\Phi^\varepsilon_h(t_k, x)) \beta_i(x) dx \) can not be exactly computed.

- inexact integration: quadrature formulae
- area-weighting: approximate the trajectories neglecting the deformation caused by advection and compute exact integration

Using area-weighting + basis \( \beta^0_i \in \mathbb{P}_0 \)

\[
(M)_{i,j} = \int_{\mathbb{R}^d} \beta^0_i(x) \beta^0_j(x) dx = \delta_{i,j}
\]

\[
(B^\varepsilon_{aw})_{i,j} = \int_{\mathbb{R}^d} \beta^0_j(x - x_i + \Phi^\varepsilon_h(t_k, x_i)) \beta^0_i(x) dx = \int_{\mathbb{R}^d} \beta^0_j(x - hDv^\varepsilon_{Ax}[\mu](t_k, x_i)) \beta^0_i(x) dx = \\
\beta^1_j(x_i - hDv^\varepsilon_{Ax}[\mu](t_k, x_i)) = (B^*)_i,j.
\]

Ref. Morton, Priestley, Suli ('88), Ferretti ('12)
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Conclusions and Future Works

Conclusions

- we have proposed a scheme for non-linear non-local FP
- it allows large time steps and is explicit
- it applies to get existence and numerical approximation of a new Hughes model
- it applies to approximate second order possibly degenerate MFG
- we have proposed a LG scheme for MFG first order getting convergence in arbitrary dimension

Future works

- Extension to non-linear non-local FP with general Neumann condition and application on MFG (with E. Calzola and F.J.Silva)
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