Second-order local minimal-time mean field games

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based on a joint work with Romain Ducasse and Filippo Santambrogio

High Dimensional Hamilton–Jacobi PDEs
Workshop III: Mean Field Games and Applications
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L2S, CentraleSupélec
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IPSA

Outline

Introduction

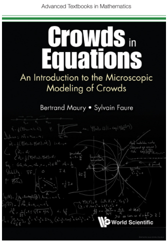
Macroscopic models for crowd motion

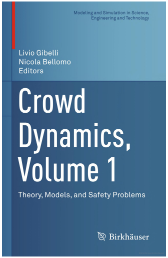
Crowd motion

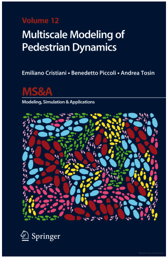
Shibuya Crossing, Tokyo, 2014

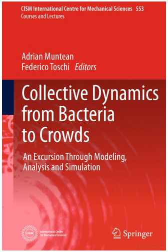
Introduction

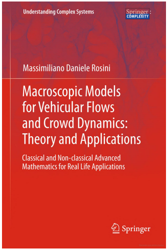
Macroscopic models for crowd motion

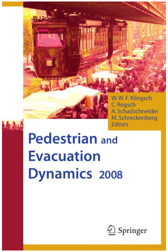












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Introduction

Macroscopic models for crowd motion

- Macroscopic models for crowd motion:

$$\partial_t \rho - \nu \Delta \rho + \operatorname{div}(\rho V) = 0 \qquad \text{in } \Omega$$

  - ⇝ $\rho(t, x)$: density of pedestrians at position $x \in \Omega$ in time t
  - ⇝ $V(t, x, \rho)$: velocity
  - ⇝ $\nu \geq 0$: viscosity
  - ⇝ Conservation law: $\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \rho = \int_{\partial\omega} (\nu \nabla \rho - \rho V) \cdot n, \qquad \omega \subset \Omega$
- How do pedestrians choose V?
- The MFG approach: pedestrians choose V by solving an optimal control problem, which depends on the average behavior of other pedestrians
- Goal: propose and study a MFG model inspired by crowd motion and taking into account some of its important features

Introduction

Macroscopic models for crowd motion

Other works on MFGs for (or related to) crowd motion: [Lachapelle, Wolfram; 2011], [Burger, Di Francesco, Markowich, Wolfram; 2013], [Cardaliaguet, Mészáros, Santambrogio; 2016], [Benamou, Carlier, Santambrogio; 2017].

Main features of our model:

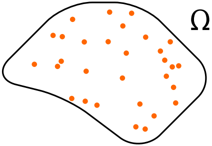
- Each agent solves an optimization criterion with free final time
  - ⇝ Pedestrians may stop at different times and the total travel time may be part of the optimization criterion of a pedestrian
- Congestion-dependent velocity constraint
  - ⇝ Maximal speed of a pedestrian depends on the density of pedestrians around them

Introduction

Previous results: the first-order case

Previous work on a first-order model considered in [M., Santambrogio; 2019] and [Dweik, M.; 2020]:

- Players of the game evolve on an open set $\Omega \subset \mathbb{R}^d$
- Goal of a player: reach the exit $\partial\Omega$ in minimal time
- Interaction through congestion: a player's maximal speed depends on the density of players around them



In this talk, $\partial\Omega$ is always C^2

Introduction

Previous results: the first-order case

Mathematically:

- Distribution of players at time t given by $\rho_t \in \mathcal{P}(\overline{\Omega})$
  - ⇝ ρ_0 is known, the goal is to determine ρ_t for $t > 0$
- Dynamics of a player given by the control system

$$\left.\begin{aligned} \dot{x}(t) &= k(\rho_t, x(t))u(t) \\ x(t) &\in \overline{\Omega} \text{ (state)} \\ \lvert u(t)\rvert &\leq 1 \text{ (control)} \end{aligned}\right\} \iff \lvert\dot{x}(t)\rvert \leq k(\rho_t, x(t))$$

- Choice of the control u: minimize the exit time

$$\inf\{T \geq 0 \mid \dot{x}(t) = k(\rho_t, x(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0,1),\ x(0) \in \overline{\Omega} \text{ fixed},\ x(T) \in \partial\Omega\}$$

- Characteristics of our model:
  - ⇝ Interaction between players through their dynamics
  - ⇝ Control constraint: $\lvert u(t)\rvert \leq 1$
  - ⇝ Optimization criterion with free final time

Introduction

Previous results: the first-order case

Formally:

- Introduce the value function φ: $\varphi(t, x)$ is the minimal time to reach $\partial\Omega$ for a pedestrian starting at (t, x)
  - ⇝ φ solves a Hamilton–Jacobi–Bellman equation
  - ⇝ Natural boundary condition: $\varphi\big|_{\partial\Omega} = 0$
- Optimal control for $\dot{x}(t) = k(\rho_t, x(t))u(t)$: $u(t) = -\frac{\nabla\varphi(t,x(t))}{|\nabla\varphi(t,x(t))|}$
  - ⇝ ρ solves a continuity equation
  - ⇝ No boundary condition: velocity field always points outwards
- MFG system:

$$\begin{cases} \partial_t\rho - \operatorname{div}\left(\rho k(\rho_t, \cdot)\dfrac{\nabla\varphi}{|\nabla\varphi|}\right) = 0 \\ -\partial_t\varphi + k(\rho_t, \cdot)|\nabla\varphi| - 1 = 0 \\ \rho(0, x) = \rho_0(x) \qquad \varphi\big|_{\partial\Omega} = 0 \end{cases}$$

Introduction

Previous results: the first-order case

- Ideally, k should be local: $k(\rho_t, x) = \kappa(\rho_t(x))$ for absolutely continuous ρ_t
  - ⇝ With κ non-increasing, e.g., $\kappa(\rho) = (1 - \rho)_+$
- Results available only in the non-local case, e.g.
$$k(\rho_t, x) = \kappa\left(\int_{\overline{\Omega}} \chi(x - y)\eta(y)\, \mathrm{d}\rho_t(y)\right)$$
with a uniform lower bounded $\kappa(\rho) \geq \kappa_{\min} > 0$
  - ⇝ Existence of solutions to the MFG system
    - — Weak (Lagrangian) notion of equilibrium: measure on the set of trajectories concentrated on optimal trajectories
    - — Regularity of optimal trajectories
    - — Semiconcavity of φ
    - — $\frac{\nabla\varphi}{|\nabla\varphi|}$ exists along optimal trajectories
    - — Techniques specific to the first-order case
  - ⇝ $\rho_0 \in L^p \implies \rho(t, \cdot) \in L^p \quad \forall t \geq 0$
  - ⇝ One may take some discontinuous η, e.g., $\eta = \mathbb{1}_\Omega$

[M., Santambrogio; 2019], [Dweik, M.; 2020]

Introduction

Previous results: the first-order case

$$\begin{cases} \partial_t \rho - \operatorname{div}\left(\rho k(\rho_t, \cdot)\frac{\nabla \varphi}{|\nabla \varphi|}\right) = 0 \\ -\partial_t \varphi + k(\rho_t, \cdot)|\nabla \varphi| - 1 = 0 \\ \rho(0, x) = \rho_0(x) \qquad \varphi\big|_{\partial\Omega} = 0 \end{cases}$$

Our MFG system above is related to Hughes model for crowd motion [Hughes; 2002]

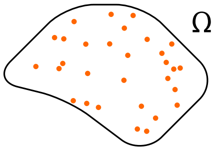
- Hughes model: At time t, a pedestrian solves an optimal control problem assuming others remain at the same position
- MFG model: At time t, a pedestrian solves an optimal control problem using rationality to determine future behavior

Introduction

The local second-order model

With respect to the first-order case:

- Random noise: players are submitted to additive independent Brownian motions



Mathematically:

- Dynamics of a player given by the stochastic control system

$$dX_t = k(\rho_t, X_t) U_t \, dt + \sqrt{2\nu} \, dW_t,$$

$$X_t: \text{ state}, \qquad U_t: \text{ control}, \quad |U_t| \leq 1, \qquad \nu > 0,$$

$$W_t: \text{ Brownian motion (mutually indep. for different players)}$$

- Exit time: $\tau = \inf\{t \geq 0 \mid X_t \notin \Omega\}$
  - ⇝ We assume that X_t stops after reaching $\partial\Omega$.
- Choice of the control U: minimize the expected exit time $\mathbb{E}[\tau]$

Introduction

The local second-order model

Motivation:

- Independent Brownian motions

  $\Rightarrow$ diffusion terms in the PDEs

  $\Rightarrow$ ρ and φ should be more regular

  $\Rightarrow$ possibility to treat the local case

  $$k(\rho_t, x) = \kappa(\rho(t, x))$$

  where $\kappa : \mathbb{R} \to (0, +\infty)$ is non-increasing.

We assume in the sequel that we are in the local case

Introduction

The local second-order model

Issue with free final time:

- Non-compact time interval $\implies$ difficulties when applying fixed-point techniques for existence of MFG equilibria
  - $\rightsquigarrow$ First-order case: $\exists T > 0$ s.t. all agents leave before T

Strategy:

- Prove existence for finite T, then let $T \to +\infty$

More precisely:

- $T \in (0, +\infty)$: time horizon
- $\psi : \overline{\Omega} \to \mathbb{R}_+$: penalization for players not leaving by T

$$\psi(x) = 0 \iff x \in \partial\Omega$$

- Optimization criterion:

$$\cancel{\min \mathbb{E}[\tau]} \quad \rightsquigarrow \quad \min \mathbb{E}[\min(\tau, T) + \psi(X_T)]$$

Value function:

$$\varphi(t_0, x_0) = \min_{U} \mathbb{E}_{(t_0, x_0)}[\min(\tau, T) + \psi(X_T)]$$

Introduction

The local second-order model

- Hamilton–Jacobi–Bellman equation for φ:

$$\begin{cases} -\partial_t \varphi + \kappa(\rho)\lvert\nabla\varphi\rvert - \nu\Delta\varphi - 1 = 0 \\ \varphi\big|_{\partial\Omega} = 0 \\ \varphi(T, \cdot) = \psi \end{cases}$$

- Optimal control:

$$U_t = -\frac{\nabla\varphi(t, X_t)}{\lvert\nabla\varphi(t, X_t)\rvert}$$

- Fokker–Planck equation for ρ:

$$\begin{cases} \partial_t \rho - \nu\Delta\rho - \operatorname{div}\left(\rho\kappa(\rho)\frac{\nabla\varphi}{\lvert\nabla\varphi\rvert}\right) = 0 \\ \rho\big|_{\partial\Omega} = 0 \\ \rho(0, \cdot) = \rho_0 \end{cases}$$

Introduction

The local second-order model

MFG system:

$$\begin{cases} \partial_t \rho - \nu \Delta \rho - \operatorname{div}\left(\rho \kappa(\rho) \frac{\nabla \varphi}{|\nabla \varphi|}\right) = 0 & \text{(FP)} \\ -\partial_t \varphi + \kappa(\rho) |\nabla \varphi| - \nu \Delta \varphi - 1 = 0 & \text{(HJB)} \\ \rho\big|_{\partial\Omega} = 0 \qquad \varphi\big|_{\partial\Omega} = 0 & \\ \rho(0, \cdot) = \rho_0 \qquad \varphi(T, \cdot) = \psi & \end{cases}$$

Sequel of the talk:

- Existence in finite time horizon
- The case of infinite time horizon by a limit procedure

MFG with finite time horizon
Existence of solutions

$$\begin{cases} \partial_t \rho - \nu \Delta \rho - \operatorname{div}\left(\rho \kappa(\rho) \frac{\nabla \varphi}{|\nabla \varphi|}\right) = 0 & \text{(FP)} \\ -\partial_t \varphi + \kappa(\rho)|\nabla \varphi| - \nu \Delta \varphi - 1 = 0 & \text{(HJB)} \\ \rho\big|_{\partial\Omega} = 0 \qquad \varphi\big|_{\partial\Omega} = 0 \\ \rho(0, \cdot) = \rho_0 \qquad \varphi(T, \cdot) = \psi \end{cases}$$

MFG with finite time horizon

Existence of solutions

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Definition

$(\rho, \varphi) \in \left[L^\infty_t L^2_x \cap L^2_t H^1_{0\,x}\right]^2$ is a solution if $\exists V \in L^\infty_{t,x}$ s.t.

- ρ is a weak solution with initial condition ρ_0 of
$$\partial_t \rho - \nu \Delta \rho + \operatorname{div}(\rho V) = 0$$
- φ is a weak solution with final condition ψ of
$$-\partial_t \varphi + \kappa(\rho)|\nabla \varphi| - \nu \Delta \varphi - 1 = 0$$
- V satisfies $|V| \leq \kappa(\rho)$ and $V \cdot \nabla \varphi = -\kappa(\rho)|\nabla \varphi|$

MFG with finite time horizon

Existence of solutions

Theorem

Let $T > 0$ *and assume* $\rho_0 \in L^2(\Omega)$, $\psi \in H_0^1(\Omega)$, $\kappa : \mathbb{R} \to (0, +\infty)$ *continuous and bounded.*

Then there exists a solution $(\rho, \varphi) \in \left[L_t^\infty L_x^2 \cap L_t^2 H_{0\,x}^1\right]^2$ *to the second-order local MFG system. Moreover*

$$\begin{aligned}
\rho &\in C_t L_x^2 \\
\partial_t \rho &\in L_t^2 H_x^{-1} \\
\varphi &\in C_t H_{0\,x}^1 \cap L_t^2 H_x^2 \\
\partial_t \varphi &\in L_{t,x}^2
\end{aligned}$$

MFG with finite time horizon

Ingredients of the proof

- Existence, uniqueness, and energy estimates for FP and HJB separately $\rightsquigarrow$ Classical results for parabolic PDEs
- Continuity of FP with respect to the velocity field
- Continuity of HJB with respect to $\kappa(\rho)$
- Fixed point argument to obtain a solution of the system

MFG with finite time horizon

The Fokker–Planck equation

$$\begin{cases} \partial_t \rho - \nu \Delta \rho + \operatorname{div}(\rho V) = 0 \\ \rho\big|_{\partial\Omega} = 0 \qquad \rho(0, \cdot) = \rho_0 \in L^2. \end{cases}$$

- Existence and uniqueness of weak solutions in $L_t^\infty L_x^2 \cap L_t^2 H_{0\,x}^1$ when $V \in L_{t,x}^\infty$
- Weak solutions also satisfy $\rho \in C_t L_x^2$, $\partial_t \rho \in L_t^2 H_x^{-1}$, and the energy estimate
$$\|\rho\|_{L_t^\infty L_x^2} + \|\rho\|_{L_t^2 H_{0\,x}^1} + \|\partial_t \rho\|_{L_t^2 H_x^{-1}} \leq C \|\rho_0\|_{L^2},$$
$C = C(d, \nu, T, \Omega, M)$, M upper bound on $\|V\|_{L_{t,x}^\infty}$
- Positivity: $\rho_0 \geq 0 \implies \rho(t, \cdot) \geq 0$

MFG with finite time horizon

The Fokker–Planck equation

$$\begin{cases} \partial_t \rho - \nu \Delta \rho + \operatorname{div}(\rho V) = 0 \\ \rho\big|_{\partial\Omega} = 0 \qquad \rho(0, \cdot) = \rho_0 \in L^2. \end{cases} \tag{FP}$$

Continuity: $V_n \overset{*}{\rightharpoonup} V$ in $L^\infty_{t,x} \implies \rho_n \to \rho$ in $L^2_{t,x}$

- Energy estimate $\implies (\rho_n)_n$ bounded in $L^2_t H^1_{0\,x}$
- Energy estimate $\implies (\partial_t \rho_n)_n$ bounded in $L^2_t H^{-1}_x$
- Aubin–Lions Lemma $\implies (\rho_n)_n$ compact in $L^2_{t,x}$
- Passing to the limit, any limit point ρ^* of $(\rho_n)_n$ must solve (FP).
- Uniqueness: $\rho^* = \rho$, then $\rho_n \to \rho$ in $L^2_{t,x}$

MFG with finite time horizon

The Hamilton–Jacobi–Bellman equation

$$\begin{cases} -\partial_t \varphi - \nu \Delta \varphi + K |\nabla \varphi| - 1 = 0 \\ \varphi|_{\partial\Omega} = 0, \qquad \varphi(T, \cdot) = \psi \in H_0^1 \end{cases}$$

- Existence and uniqueness of weak solutions in $L_t^\infty L_x^2 \cap L_t^2 {H_0^1}_x$ when $K \in L_{t,x}^\infty$
  - ⇝ Linear heat equation with source $1 - K|\nabla \varphi|$ & fixed point
- Weak solutions also satisfy $\varphi \in C_t {H_0^1}_x \cap L_t^2 H_x^2$, $\partial_t \varphi \in L_{t,x}^2$, and the energy estimates

$$\|\varphi\|_{L_t^\infty L_x^2} + \|\varphi\|_{L_t^2 {H_0^1}_x} + \|\partial_t \varphi\|_{L_t^2 H_x^{-1}} \leq C\left(\|\psi\|_{L^2} + 1\right),$$

$$\|\varphi\|_{L_t^\infty {H_0^1}_x} + \|\varphi\|_{L_t^2 H_x^2} + \|\partial_t \varphi\|_{L_{t,x}^2} \leq C\left(\|\psi\|_{H_0^1} + 1\right),$$

$C = C(d, \nu, T, \Omega, M)$, M upper bound on $\|K\|_{L_{t,x}^\infty}$

MFG with finite time horizon

The Hamilton–Jacobi–Bellman equation

$$\begin{cases} -\partial_t \varphi - \nu \Delta \varphi + K \lvert \nabla \varphi \rvert - 1 = 0 \\ \varphi\big|_{\partial\Omega} = 0, \qquad \varphi(T, \cdot) = \psi \in H_0^1 \end{cases} \tag{HJB}$$

Continuity: $K_n \overset{*}{\rightharpoonup} K$ in $L^\infty_{t,x} \implies \varphi_n \to \varphi$ in $L^2_t H^1_{0\,x}$

- Energy estimate $\implies$ $(\varphi_n)_n$ bounded in $L^2_t H^2_x$
- Energy estimate $\implies$ $(\partial_t \varphi_n)_n$ bounded in $L^2_{t,x}$
- Aubin–Lions Lemma $\implies$ $(\varphi_n)_n$ compact in $L^2_t H^1_{0\,x}$
- Passing to the limit, any limit point φ^* of $(\varphi_n)_n$ must solve (HJB).
- The stronger convergence $L^2_t H^1_{0\,x}$ is needed because of the non-linear term $K \lvert \nabla \varphi \rvert$
- Uniqueness: $\varphi^* = \varphi$, then $\varphi_n \to \varphi$ in $L^2_t H^1_{0\,x}$

MFG with finite time horizon

Kakutani fixed point

$$\begin{cases} \partial_t \rho - \nu \Delta \rho - \operatorname{div}\left(\rho \kappa(\rho) \frac{\nabla \varphi}{|\nabla \varphi|}\right) = 0 & \text{(FP)} \\ -\partial_t \varphi + \kappa(\rho)|\nabla \varphi| - \nu \Delta \varphi - 1 = 0 & \text{(HJB)} \\ \rho\big|_{\partial\Omega} = 0 \qquad \varphi\big|_{\partial\Omega} = 0 & \\ \rho(0, \cdot) = \rho_0 \qquad \varphi(T, \cdot) = \psi & \end{cases}$$

$$\begin{array}{ccccccccc} w^*\text{-}L^\infty_{t,x} & \to & L^2_{t,x} & \to & w^*\text{-}L^\infty_{t,x} & \to & L^2_t H^1_{0\,x} & \to & w^*\text{-}L^\infty_{t,x} \\ V & \mapsto & \rho & \mapsto & \kappa(\rho) & \mapsto & \varphi & \mapsto & \mathcal{V}(V) \end{array}$$

$$\mathcal{V}(V) = \left\{ \widetilde{V} \in L^\infty_{t,x} \;\middle|\; \begin{array}{c} |\widetilde{V}| \leq \kappa(\rho) \\ \widetilde{V} \cdot \nabla \varphi = -\kappa(\rho)|\nabla \varphi| \end{array} \right\}$$

MFG with finite time horizon

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MFG with finite time horizon

Kakutani fixed point

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MFG with finite time horizon

Kakutani fixed point

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$$\mathcal{V}(V) = \left\{ \widetilde{V} \in L^\infty_{t,x} \,\middle|\, \begin{array}{c} |\widetilde{V}| \leq \kappa(\rho) \\ \widetilde{V} \cdot \nabla \varphi = -\kappa(\rho) |\nabla \varphi| \end{array} \right\}$$

$$\text{Solution} \iff V \in \mathcal{V}(V)$$

MFG with finite time horizon

Kakutani fixed point

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Hypotheses for Kakutani fixed point theorem:

- $\mathcal{V}(V)$ non-empty, compact, convex
- $\mathcal{V}$ upper semi-continuous

MFG with finite time horizon

Kakutani fixed point

$$\begin{array}{ccccccccc} w^*\text{-}L^\infty_{t,x} & \to & L^2_{t,x} & \to & w^*\text{-}L^\infty_{t,x} & \to & L^2_t H^1_{0\,x} & \to & w^*\text{-}L^\infty_{t,x} \\ V & \mapsto & \rho & \mapsto & \kappa(\rho) & \mapsto & \varphi & \mapsto & \mathcal{V}(V) \end{array}$$

$$\mathcal{V}(V) = \left\{ \widetilde{V} \in L^\infty_{t,x} \;\middle|\; \begin{array}{c} |\widetilde{V}| \leq \kappa(\rho) \\ \widetilde{V} \cdot \nabla\varphi = -\kappa(\rho)|\nabla\varphi| \end{array} \right\}$$

Hypotheses for Kakutani fixed point theorem:

- $\mathcal{V}(V)$ non-empty, compact, convex
- $\mathcal{V}$ upper semi-continuous
  - ⇝ $V \mapsto \rho \mapsto \kappa(\rho) \mapsto \varphi$ all continuous
  - ⇝ $\mathcal{V}$ has a closed graph

$\Longrightarrow$ Existence of a fixed point

$\Longrightarrow$ Existence of a solution to the MFG system ■

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Existence of solutions

MFG system with infinite time horizon:

$$\begin{cases} \partial_t \rho - \nu \Delta \rho - \operatorname{div}\left(\rho \kappa(\rho) \frac{\nabla \varphi}{|\nabla \varphi|}\right) = 0 & t \geq 0,\ x \in \Omega \\ -\partial_t \varphi + \kappa(\rho)|\nabla \varphi| - \nu \Delta \varphi - 1 = 0 & t \geq 0,\ x \in \Omega \\ \rho\big|_{\partial\Omega} = 0 \qquad \varphi\big|_{\partial\Omega} = 0 \\ \rho(0, \cdot) = \rho_0 \end{cases}$$

Definition of solution: similar to the finite time horizon, but we also require φ to be globally bounded

- Boundedness of φ often required in optimal control to identify it as the value function of an optimal control problem
- We can prove existence of solutions with bounded φ
- Boundedness of φ is important to obtain results on the asymptotic behavior of ρ and φ as $t \to +\infty$.

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Existence of solutions

Theorem

Assume $\rho_0 \in L^2(\Omega)$, $\rho_0 \geq 0$, $\kappa : \mathbb{R} \to (0, +\infty)$ *continuous and bounded.*

Then there exists a solution $(\rho, \varphi) \in \left[L^\infty_t L^2_x \cap L^2_t H^1_{0\,x}\right]^2$ *to the second-order local MFG system in infinite horizon. Moreover*

$$\begin{aligned} \rho &\in C_t L^2_x \\ \partial_t \rho &\in L^2_t H^{-1}_x \\ \varphi &\in C_t H^1_{0\,x} \cap L^2_t H^2_x \\ \partial_t \varphi &\in L^2_{t,x} \end{aligned}$$

All summabilities in t are local

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Sketch of the proof

- Start from a sequence of solutions $(\rho_n, \varphi_n)_n$ in finite time horizon T_n, $T_n \to +\infty$
- We require $\varphi_n(T_n, \cdot) = \psi_n \geq 0$ to be bounded in $L^\infty \cap H_0^1$
- Goal: up to extracting a subsequence

$$\begin{aligned} \rho_n &\to \rho && \text{in } L^2_{\mathrm{loc},t} L^2_x \\ \varphi_n &\to \varphi && \text{in } L^2_{\mathrm{loc},t} {H^1_0}_x \end{aligned}$$

- If these convergences hold, it is easy to verify that (ρ, φ) is a solution in infinite horizon

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Sketch of the proof

- Fokker–Planck equation: given T, if $T_n \geq T$,
  $$\|\rho_n\|_{L^\infty_{(0,T)} L^2_x} + \|\rho_n\|_{L^2_{(0,T)} H^1_{0\,x}} + \|\partial_t \rho_n\|_{L^2_{(0,T)} H^{-1}_x} \leq C(T) \|\rho_0\|_{L^2_x}$$
  - ⇝ Boundedness of $(\rho_n)_n$ in $L^2_{(0,T)} H^1_{0\,x}$
  - ⇝ Boundedness of $(\partial_t \rho_n)_n$ in $L^2_{(0,T)} H^{-1}_x$
  - ⇝ Aubin–Lions Lemma $\implies$ compactness of $(\rho_n)_n$ in $L^2_{(0,T)} L^2_x$
  - ⇝ Convergence (up to subsequence) in $L^2_{\mathrm{loc},t} L^2_x$, as required

MFG with infinite time horizon

Sketch of the proof

- Fokker–Planck equation: given T, if $T_n \geq T$,

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  - ⇝ Convergence (up to subsequence) in $L^2_{\mathrm{loc},t} L^2_x$, as required
- Hamilton–Jacobi–Bellman equation: same strategy does not work immediately! Given T, if $T_n \geq T$,

$$\|\varphi_n\|_{L^\infty_{(0,T)} L^2_x} + \|\varphi_n\|_{L^2_{(0,T)} H^1_{0\,x}} + \|\partial_t \varphi_n\|_{L^2_{(0,T)} H^{-1}_x} \leq C(T) \left(\|\varphi_n(T)\|_{L^2_x} + 1\right)$$

$$\|\varphi_n\|_{L^\infty_{(0,T)} H^1_{0\,x}} + \|\varphi_n\|_{L^2_{(0,T)} H^2_x} + \|\partial_t \varphi_n\|_{L^2_{(0,T)} L^2_x} \leq C(T) \left(\|\varphi_n(T)\|_{H^1_{0\,x}} + 1\right)$$

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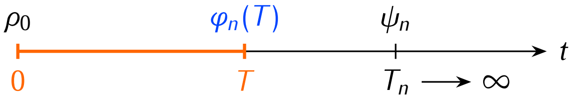
Sketch of the proof

$$\|\rho_n\|_{L^\infty_{(0,T)} L^2_x} + \|\rho_n\|_{L^2_{(0,T)} H^1_{0\,x}} + \|\partial_t \rho_n\|_{L^2_{(0,T)} H^{-1}_x} \leq C(T) \|\rho_0\|_{L^2_x}$$

$$\|\varphi_n\|_{L^\infty_{(0,T)} L^2_x} + \|\varphi_n\|_{L^2_{(0,T)} H^1_{0\,x}} + \|\partial_t \varphi_n\|_{L^2_{(0,T)} H^{-1}_x} \leq C(T) \left(\|\varphi_n(T)\|_{L^2_x} + 1\right)$$

$$\|\varphi_n\|_{L^\infty_{(0,T)} H^1_{0\,x}} + \|\varphi_n\|_{L^2_{(0,T)} H^2_x} + \|\partial_t \varphi_n\|_{L^2_{(0,T)} L^2_x} \leq C(T) \left(\|\varphi_n(T)\|_{H^1_{0\,x}} + 1\right)$$



To conclude, it suffices to show that $(\varphi_n)_n$ is bounded in $L^\infty_t L^2_x$ and $L^\infty_t H^1_{0\,x}$

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Sketch of the proof

$$\begin{cases} -\partial_t \varphi - \nu \Delta \varphi + K |\nabla \varphi| - 1 = 0 \\ \varphi\big|_{\partial\Omega} = 0, \qquad \varphi(T, \cdot) = \psi \end{cases} \tag{HJB}$$

Lemma

Assume that $T > 0$, $K \in L^\infty_{t,x}$, $\psi \in L^\infty \cap H^1_0$, $K \geq 0$, and $\psi \geq 0$ and let φ be the solution of (HJB).

Then $\exists C > 0$ depending only on ν, Ω, $\|\psi\|_{L^\infty}$, $\|\psi\|_{H^1_0}$ s.t.

$$0 \leq \varphi \leq C$$

$$\|\varphi\|_{L^\infty_t H^1_{0\,x}} \leq C$$

MFG with infinite time horizon
Sketch of the proof

Proof of the Lemma: To prove $0 \leq \varphi \leq C$:

- Comparison principle with
$$\begin{cases} -\nu\Delta\Phi - 1 = 0 \\ \Phi\big|_{\partial\Omega} = 0 \end{cases}$$
Since $0 \leq \psi \leq \Phi + \|\psi\|_{L^\infty}$, then $0 \leq \varphi \leq \Phi + \|\psi\|_{L^\infty}$
- Φ: expected time to leave Ω only with Brownian motion

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Sketch of the proof

Proof of the Lemma (cont.): To prove $\|\varphi\|_{L^\infty_t H^1_{0\,x}} \leq C$:

- Derivative of the L^2 norm:
  $$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega \varphi^2\right) = -\nu \int_\Omega |\nabla \varphi|^2 - \int_\Omega \kappa(\rho)|\nabla\varphi|\varphi + \int_\Omega \varphi$$
  Using that φ is bounded:
  $$\int_{t_1}^{t_2}\int_\Omega |\nabla\varphi|^2 \leq C(1 + |t_2 - t_1|)$$
- Derivative of the H^1_0 norm:
  $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega |\nabla\varphi|^2\right) = \nu\int_\Omega (\Delta\varphi)^2 - \int_\Omega \kappa(\rho)|\nabla\varphi|\Delta\varphi + \int_\Omega \Delta\varphi$$
  Young + Gronwall:
  $$\int_\Omega |\nabla\varphi(t)|^2 \leq C\int_t^{t+1}\int_\Omega |\nabla\varphi|^2 + C$$ ■

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Sketch of the proof

Finally,

$$\|\varphi_n\|_{L^\infty_{(0,T)} L^2_x} + \|\varphi_n\|_{L^2_{(0,T)} H^1_{0\,x}} + \|\partial_t \varphi_n\|_{L^2_{(0,T)} H^{-1}_x} \leq C(T) \left(\|\varphi_n(T)\|_{L^2_x} + 1\right)$$

$$\|\varphi_n\|_{L^\infty_{(0,T)} H^1_{0\,x}} + \|\varphi_n\|_{L^2_{(0,T)} H^2_x} + \|\partial_t \varphi_n\|_{L^2_{(0,T)} L^2_x} \leq C(T) \left(\|\varphi_n(T)\|_{H^1_{0\,x}} + 1\right)$$

MFG with infinite time horizon

Sketch of the proof

Finally,

$$\|\varphi_n\|_{L^\infty_{(0,T)} L^2_x} + \|\varphi_n\|_{L^2_{(0,T)} H^1_{0\,x}} + \|\partial_t \varphi_n\|_{L^2_{(0,T)} H^{-1}_x} \leq C(T)$$

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- Boundedness of $(\varphi_n)_n$ in $L^2_{(0,T)} H^2_x$
- Boundedness of $(\partial_t \varphi_n)_n$ in $L^2_{(0,T)} L^2_x$
- Aubin–Lions lemma $\implies$ compactness of $(\varphi_n)_n$ in $L^2_{(0,T)} H^1_{0\,x}$
- Convergence (up to subsequence) in $L^2_{\mathrm{loc},t} H^1_{0\,x}$, as required ■

MFG with infinite time horizon

Parabolic regularization and more general initial conditions

Proposition (Parabolic regularization)

Let V, F, f, g, u be C^∞, V, g be L^∞, $u \geq 0$, be s.t.

$$\partial_t u - \Delta u + \operatorname{div}(uV) + \nabla \cdot F + f + g \cdot \nabla u \leq 0$$

Then $\forall p > 1$, $\forall \delta > 0$, $\exists C > 0$ (indep. of F, f, and depending on V, g only through their L^∞ norms), $\forall t \geq 0$,

$$\|u(t+\delta)\|_{L^\infty} \leq C\left(\|u(t)\|_{L^p} + \|F\|_{L^\infty} + \|f\|_{L^\infty}\right)$$

- General formulation: can be applied both to Fokker–Planck and Hamilton–Jacobi–Bellman
- Can be adapted to non-smooth solutions by regularization
  - ⇝ Different regularization arguments for FP and HJB
- Allows one to consider initial condition $\rho_0 \in L^p$ for FP also for $1 < p < 2$: solution becomes L^∞ for $t > 0$
- With some extra work, one may get existence and uniqueness for FP also for $\rho_0 \in L^1$ with finite entropy

MFG with infinite time horizon

Asymptotic behavior of ρ

Proposition

Any solution (ρ, φ) of the MFG system in infinite time horizon with $\rho_0 \geq 0$ satisfies

$$\rho(t, \cdot) \xrightarrow[t \to +\infty]{} 0$$

exponentially in L^p for every $p \in [1, \infty]$

- First case: $p = 1$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho = \nu \int_{\partial\Omega} \frac{\partial \rho}{\partial n} \leq 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho\varphi = -\int_\Omega \rho$$

MFG with infinite time horizon

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- First case: $p = 1$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho = \nu \int_{\partial\Omega} \frac{\partial \rho}{\partial n} \leq 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho\varphi = -\int_\Omega \rho \leq -\frac{1}{\|\varphi\|_{L^\infty_{t,x}}} \int_\Omega \rho\varphi$$

$$\implies \int_\Omega \rho(t)\varphi(t) \leq e^{-\lambda t} \int_\Omega \rho(0)\varphi(0), \qquad \lambda \leq 1/\|\varphi\|_{L^\infty_{t,x}}$$

$$\implies \int_\Omega \rho(t) \leq C e^{-\lambda t}$$

MFG with infinite time horizon

Asymptotic behavior of ρ

- Second case: $p > 1$ close to 1

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho^2 \leq \frac{\kappa_{\max}^2}{\nu} \int_\Omega \rho^2$$

$$\implies \int_\Omega \rho(t)^2 \leq e^{\gamma t} \int_\Omega \rho(0)^2, \qquad \gamma = \frac{\kappa_{\max}^2}{\nu}$$

Exponential convergence in L^1 + exponential bound in L^2 + interpolation

$\implies$ exponential convergence in L^p for $p > 1$ close to 1

MFG with infinite time horizon

Asymptotic behavior of ρ

- Second case: $p > 1$ close to 1
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho^2 \leq \frac{\kappa_{\max}^2}{\nu} \int_\Omega \rho^2$$
$$\implies \int_\Omega \rho(t)^2 \leq e^{\gamma t} \int_\Omega \rho(0)^2, \qquad \gamma = \frac{\kappa_{\max}^2}{\nu}$$
Exponential convergence in L^1 + exponential bound in L^2 + interpolation
$\implies$ exponential convergence in L^p for $p > 1$ close to 1
- Third case: any p
  - ⇝ Parabolic regularization: $\forall p > 1$, $\forall \delta > 0$, $\exists C$, $\forall t \geq 0$,
$$\lVert \rho(t + \delta) \rVert_{L^\infty(\Omega)} \leq C \lVert \rho(t) \rVert_{L^p(\Omega)}$$
  - ⇝ Apply with p close to 1 from second case, $\delta = 1$: $\rho(t) \to 0$ in L^∞ exponentially as $t \to \infty$ ■

MFG with infinite time horizon

Asymptotic behavior of φ

What happens to φ as $t \to \infty$?

$$\begin{cases} -\partial_t \varphi + \kappa(\rho)|\nabla \varphi| - \nu \Delta \varphi - 1 = 0 \\ \varphi\big|_{\partial\Omega} = 0 \end{cases}$$

We know that $\rho \to 0$ exponentially fast in all L^p

Let Ψ be the solution of

$$\begin{cases} \kappa(0)|\nabla \Psi| - \nu \Delta \Psi - 1 = 0 \\ \Psi\big|_{\partial\Omega} = 0 \end{cases}$$

$\Psi(x)$: minimal time to reach $\partial\Omega$ when the density is 0

Proposition

Any solution (ρ, φ) of the MFG system in infinite time horizon with $\rho_0 \geq 0$ satisfies

$$\varphi(t, \cdot) \xrightarrow[t \to +\infty]{} \Psi$$

in L^∞ and in H^1_0

MFG with infinite time horizon

Asymptotic behavior of φ

Strategy of the proof: Let $t_n \to +\infty$

- $\varphi_n(t, x) = \varphi(t + t_n, x)$ converges in $L^2_{\mathrm{loc},t} H^1_0$ as $n \to \infty$ to some $\overline{\varphi}$ solution of

$$\begin{cases} -\partial_t \overline{\varphi} + \kappa(0)|\nabla \overline{\varphi}| - \nu \Delta \overline{\varphi} - 1 = 0 \\ \overline{\varphi}\big|_{\partial\Omega} = 0 \end{cases} \tag{HJB$_0$}$$

- Consider u_T, v_T solutions of (HJB$_0$) with $u_T(T, x) = 0$, $v_T(T, x) = \Phi + M$, where

$$\begin{cases} -\nu \Delta \Phi = 1 \\ \Phi\big|_{\partial\Omega} = 0 \end{cases}$$

  and $M \geq \overline{\varphi}$. Comparison principle $\implies$

$$0 \leq u_T(t, x) \leq \overline{\varphi}(t, x) \leq v_T(t, x) \leq \Phi(x) + M$$

MFG with infinite time horizon

Asymptotic behavior of φ

Strategy of the proof (cont.):

$$0 \leq u_T(t, x) \leq \overline{\varphi}(t, x) \leq v_T(t, x) \leq \Phi(x) + M$$

- Comparison principle between u_{T+h} and $u_T \implies (u_T)_{T>0}$ non-decreasing
- Comparison principle between v_{T+h} and $v_T \implies (v_T)_{T>0}$ non-increasing
- $\implies (u_T)_{T>0}$ and $(v_T)_{T>0}$ converge a.e. as $T \to +\infty$, and their limit must be Ψ
- $\implies \overline{\varphi} = \Psi$
- Parabolic regularization $\implies$ convergence in L^∞
- Bound of φ in $L^2_{(t_1,t_2)} H^2_x \implies$ convergence in H^1_0 ■