Periodic equilibria for a first order Mean Field Game

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The MFG system

- Potential MFG: the equilibrium condition is the optimality condition for an optimization problem in the class of density evolutions
- Non monotone case:
  Players do not avoid places with high concentration,
  no uniqueness, less regularity.
  Density constraints
- State space is the whole $\mathbb{R}^d$:
  no boundary or periodicity conditions.

Study long-time patterns, in particular periodic equilibria,
study instability properties of stationary states - related to existence of brake orbits and heteroclinics

Joint work with Marco Cirant (Padova)

Literature: Long time attractors for non monotone MFG

In general instability of steady states.

- Gomes, Sedjro 2017, traveling waves for 1 order non-monotone MFG with congestion- PDE methods.
- Cirant, 2019 family of solutions that exhibit an oscillatory behaviour in time, around stationary solutions (which are constants), for viscous MFG (also 2 populations) with aggregation- bifurcations methods
- Cirant and Nurbekyan 2019, existence of periodic solutions for viscous MFG systems - bifurcation methods around a constant solution.
- Masoero, 2019, an example of a MFG whose long time attractor does not contain stationary solutions- weak KAM methods.

Potential MFG

We will consider T **periodic (constrained) critical points** of the following energy

$$J_T(m,w) = \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{dw}{dt \otimes m(t,dx)} \right|^2 m(t,dx)dt + \int_0^T \mathcal{W}(m)dt,$$

- $m \in C([0,T], \mathcal{P}^r(\mathbb{R}^d))$, $m(t)$ **has a L^∞ density,** $\|m\|_\infty \leq \rho$,
- w is a Borel d-vector measure a.c. w.r.t. $dt \otimes m(t,dx)$, $-\partial_t m + \mathrm{div}(w) = 0$ in the sense of distributions,
- $f(x,m) = \frac{\delta}{\delta m}\mathcal{W}(m) \in C(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d))$ **is the interaction term in the MFG.** Non monotone case: No convexity assumption on $\mathcal{W}$

Optimality conditions

Minimizers $(\bar{m}, \bar{w})$ of J_T in some given set $\mathcal{K}$ give rise to mean field Nash equilibria: for any admissible competitor (m, w),

$$\int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{dw}{dt \otimes m(t,dx)} \right|^2 m(t,dx)dt + \int_0^T f(x,\bar{m}) m dt$$

$$\geq \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{d\bar{w}}{dt \otimes \bar{m}(t,dx)} \right|^2 \bar{m}(t,dx)dt + \int_0^T f(x,\bar{m}) \bar{m} dt$$

Ref: Briani, Cardaliaguet, 2018.
This in turn can be shown to provide solutions $(\bar{m}, \bar{u})$ in a suitable weak sense to MFG system (by duality methods)

$$\begin{cases} -\partial_t u + \frac{|\nabla u|^2}{2} = f(x,m) + \text{ pressure term} + \text{ ergodic constant} \\ \partial_t m - \mathrm{div}(\mathrm{m}\nabla \mathrm{u}) = 0 \\ 0 \leq m \leq \rho, \ \int_{\mathbb{R}^d} m(t,x)dx = 1. \end{cases}$$

Literature: MFG with density constraints

References
1 order Santambrogio 2012, Cardaliaguet, Meszaros, Santambrogio, 2016,
2 order, stationary Meszaros, Silva 2018,
regularity issues Lavenant, Santambrogio 2018, 2019

- the density constraint avoids formation of singularities in focusing MFG
- in the optimality conditions a pressure term appears: a scalar field, vanishing where the density does not saturate the constraint. Pressure is a price to pay to pass through saturated regions (issues about regularity)

Wasserstein space

$\mathcal{P}_2^r(\mathbb{R}^d)$ is the space of Borel positive measures - with density- and bounded 2-order moment

$$\int_{\mathbb{R}^d} |x|^2 m(x)dx < +\infty$$

endowed with the Wasserstein distance

$$d_2(m,m')^2 = \inf_{\gamma\in\Pi(m,m')} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^2 d\gamma(x,y)$$

where $\Pi(m,m')$ are Borel probability measures with marginals m, m'.

$$\mathcal{P}_{2,\rho}^r(\mathbb{R}^d) = \{m \in \mathcal{P}_2^r(\mathbb{R}^d), \|m\|_\infty \leq \rho\}.$$

Ref: books Ambrosio, Gigli, Savaré 2008, and Santambrogio, 2015

Assumptions on $\mathcal{W}$ (double well potential)

1. $\exists \min_{\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \mathcal{W}(=0)$ and $\exists \mathcal{M}^+, \mathcal{M}^- \subset\subset \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$ s.t. $d_2(\mathcal{M}^+, \mathcal{M}^-) := 2q_0 > 0$ and $\mathcal{W}(m) = 0 \Leftrightarrow m \in \mathcal{M}^\pm$,
2. lsc (in $\mathcal{P}_p$, $p < 2$):
   if $\lim_n d_p(m_n, m) = 0$ then $\liminf_n \mathcal{W}(m_n) \geq \mathcal{W}(m)$.
   Note that lsc of the kinetic part term in J_T is standard by convexity.
3. coercivity: $\exists C_{\mathcal{W}} > 0$, for all $m \in \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$

   $$-C_{\mathcal{W}} + C_{\mathcal{W}}^{-1} \int_{\mathbb{R}^d} |x|^2 m(x)dx \leq \mathcal{W}(m) \leq C_{\mathcal{W}} \left(1 + \int_{\mathbb{R}^d} |x|^2 m(x)dx\right).$$

   (so $\mathcal{W}$ has cpt sublevel sets in $\mathcal{P}_p(\mathbb{R}^d)$ for $p < 2$).
4. continuity: for any $\{m_n\} \subset \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$,
   if $\lim_n \mathcal{W}(m_n) = 0$, then $\lim_n d_2(m_n, \mathcal{M}^\pm) = 0$.
   If $\mathcal{W}$ is lsc and with cpt sublevel sets in $\mathcal{P}_2(\mathbb{R}^d)$, this follows from 1.
5. symmetry: $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ reflection s.t. $\mathcal{W}(\gamma_\# m) = \mathcal{W}(m)$.

A model case

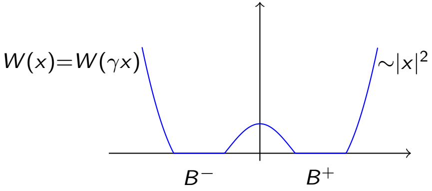
$$\mathcal{W}(m) = \int_{\mathbb{R}^d} W(x)m(dx) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(|x-y|)m(dx)m(dy).$$

In this case $f(x,m) = \delta_m \mathcal{W} = W(x) - 2\int_{\mathbb{R}^d} K(|x-y|)m(dy)$.

- interaction energy, nonlocal aggregative interaction
  kernel K is positive definite, radially symmetric, locally integrable and increasing at zero, e.g. the Riesz kernels

$$K(|x-y|) = \frac{1}{|x-y|^{d-\alpha}}, \quad \text{with } \alpha \in (0,d).$$

- confining potential W: spatial preference for aggregation,



The model case: stationary minimizers

$\min_{\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)} \int_{\mathbb{R}^d} m(x)W(x)dx - \int_{\mathbb{R}^{2d}} K(|x-y|)m(x)m(y)dxdy$:

- existence by standard direct methods.
- every minimizer has compact support by first variation

$$\exists c \in \mathbb{R} \ s.t. \quad -2K \star m(x) + W(x) \leq c \qquad x \in \text{supp } m$$

- each minimizer is a characteristic function of a compact set: $m = \rho\chi_E$, where $\rho|E| = 1$. Use the second variation:

$$\int_{\mathbb{R}^d} \xi(x)\xi(y)K(|x-y|)dxdy \leq 0.$$

If B^+, B^- are suff. large w.r.t. ρ

$$\mathcal{M}^+ = \{\chi_{B(x,r)}, B(x,r) \subseteq B^+\} \qquad \mathcal{M}^- = \gamma(\mathcal{M}^+).$$

Benamou-Brenier formulation

The energy

$$J_T(m,w) = \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \frac{|w(t,x)|^2}{m(t,x)}\, dxdt + \int_0^T \mathcal{W}(m(t))\, dt,$$

where $-\partial_t m + \mathrm{div}(w) = 0$, $\int_{\mathbb{R}^d} m(x)dx = 1$, $0 \le m \le \rho$ due to Benamou-Brenier formula can also be written as an energy on the space of absolutely continuous curves $t \mapsto m(t)$ with values in $\mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$

$$J_T(m) = \int_0^T \frac{1}{2}|m'_t|^2 + \mathcal{W}(m(t))\, dt,$$

where m'_t is the metric derivative of the curve and heuristically corresponds to the norm of the derivative in the Wasserstein space.

Minimal action problem for curves in functional spaces

Classical energetic approach to solution to the Newton equation $q''(t) = W'(q(t))$ (arising in mechanical systems with Hamiltonian structure) is through the minimization of the action functional

$$\int_0^T \frac{1}{2}|q'(t)|^2 + W(q(t))dt$$

A way to construct

1. heteroclinic solutions (solutions connecting 2 different stationary states)
2. periodic orbits, in particular
   brake orbits periodic orbits such that $q'(t_1), q'(t_2) = 0$ for some t_1, t_2.

Rabinowitz..

Literature: infinite dimensional setting

Extension to infinite dimensional setting:
curves $u : \mathbb{R} \to \mathcal{H}$, $\mathcal{H}$ functional space

- $\mathcal{H}$ Hilbert space, such as $H^1(\Omega)$, with appropriate boundary conditions and $\mathcal{W}(u) = \|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega W(x, u)dx$, existence of brake orbits and of heteroclinics
  - Alessio, 2016, Alessio Montecchiari 2017: minimize the action functional among curves with prescribed energy.
  - Fusco, Gronchi, Novaga, 2018: minimize the action functional among curves with fixed period.
- more generally $\mathcal{H}$ metric space,
  - Monteil, Santambrogio, 2019- reparametrize the action functional as a weighted length functional, heteroclinics as geodesics

Competitors for the energy J_T

Admissible competitors $(m, w) \in \mathcal{K}$ for the energy

$$\begin{cases} m \in C^0((-\infty,\infty), \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)), & \\ \int_s^t \int_{\mathbb{R}^d} \left|\frac{w(t,x)}{m(t,x)}\right|^2 m(t,x)\, dxdt < \infty & \text{for all } s < t \ , \\ -\partial_t m + \operatorname{div}(w) = 0 \text{ in } \mathbb{R}\times\mathbb{R}^d & \text{sense of distributions.} \end{cases}$$

Properties:

- $\int_{\mathbb{R}^d} m(t,x)dx = 1$ for a.e. t
- $w \in L^2([-L, L] \times \mathbb{R}^d)$ for all $L > 0$,

$$\int_s^t \int_{\mathbb{R}^d} |w|(1+|x|)dxdt < +\infty$$

- uniform continuity: for all $t, s \in (t_1, t_2)$,

$$d_2^2(m(t), m(s)) \le |t-s| \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left|\frac{w(\tau,x)}{m(\tau,x)}\right|^2 m(\tau,x)\, dxd\tau.$$

Periodic competitors

Finally we add perioditicity:

$$\mathcal{K}_T := \{(m, w) \in \mathcal{K} : m \ \ T\text{-periodic},$$

$$m(-t) = \gamma_\# m(t), \, \forall t \in \mathbb{R}$$

to rule out constant solution $m \equiv \bar{m}^\pm \in \mathcal{M}^\pm$

$$m\left(\frac{T}{4} + t\right) = m\left(\frac{T}{4} - t\right)\Bigg\}$$

look for trajectories oscillating (twice for period) around a trajectory with extremes at $m(\pm T/4)$.

Periodic orbits

Existence of periodic orbits oscillating around the steady states.

Let $0 < q \leq d_2(\mathcal{M}^+, \mathcal{M}^-)$.
Then there exists $\bar{T} = \bar{T}(q) > 4$ *such that, for any* $T \geq \bar{T}$,
there exists a minimizer $(m^T, w^T) \in \mathcal{K}_T$ *of* J_T *which satisfies*

$$\begin{cases} d_2(m^T(t), \mathcal{M}^+) < q & \forall t \in \left(s, \frac{T}{2} - s\right) \\ d_2(m^T(t), \mathcal{M}^-) < q & \forall t \in \left(-\frac{T}{2} + s, -s\right), \end{cases}$$

for some $s := s(q)$, *bdd in* q, *not depending on* T.
Moreover $\bar{T}(q) \to +\infty$ *as* $q \to 0$.

Let $(m^T, w^T) \in \mathcal{K}_T$ is a minimizer of J_T then

$$\int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{w^T(t,x)}{m^T(t,x)} \right|^2 m^T(t,x) + f(x, m^T(t)) m^T(t,x) dxdt$$

$$\leq \int_0^T \int_{\mathbb{R}^d} \frac{1}{2} \left| \frac{w(t,x)}{m(t,x)} \right|^2 m(t,x) + f(x, m^T(t)) m(t,x) dxdt$$

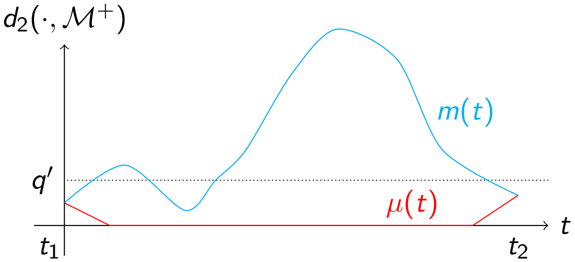
for all (also non-symmetric) $(m, w) \in \mathcal{K}$ (which are T-periodic).

Proof by symmetry of $\mathcal{W}$ (and f), and convexity of the kinetic part.

In the optimality conditions, no further multipliers related to $m(T/4 + t) = m(T/4 - t)$, $m(-t) = \gamma_\# m(t)$ appear.

Sktch of prooof: Cut lemma

if $(m, w) \in \mathcal{K}_T$ has bounded energy and $m(t)$ is sufficiently close to $\mathcal{M}^+$ (resp. $\mathcal{M}^-$) at some times t_1, t_2, then it is close to $\mathcal{M}^+$ (resp. $\mathcal{M}^-$) in the whole time interval; otherwise, it is indeed possible to modify it and decrease the energy.



Cut argument already used in the analysis of periodic orbits and heteroclinic connections for Hamiltonian systems, Coti Zelati, Rabinowitz 1991, Alikakos, Fusco 2008, Fusco, Gronchi, Novaga 2018.

Geodesics in $\mathcal{K}$

Exploiting displacement convexity McCann, 97

Lemma

Let $t_1 < t_2$, and $m_1, m_2 \in \mathcal{P}^r_{2,\rho}(\mathbb{R}^d)$. Then, there exists a couple $(m, w) \in \mathcal{K}$ which satisfies

$$m(t_i) = m_i \qquad \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| \frac{w(t,x)}{m(t,x)} \right|^2 m(t,x)\, dxdt = \frac{d_2^2(m_1, m_2)}{t_2 - t_1}.$$

m is constructed as the unique constant speed geodesic connecting m_1, m_2. The L^∞ constraint is satisfied since $m \mapsto \|m\|_{L^q(\mathbb{R}^d)}$ is *geodesically convex* in $\mathcal{P}_2(\mathbb{R}^d)$ for every $q > 1$.

Sketch of proof

- energy bounds: fix $m_0 = \gamma(m_0)$ with compact support, and $d = d_2(m_0, \mathcal{M}^{\pm})$ and construct a competitor $(\tilde{m}, \tilde{w})$ connecting m_0 to $\bar{m}^+ \in \mathcal{M}^+$ in finite time, ext. by symmetry and periodizing, $J_T(\tilde{m}, \tilde{w}) \leq C = C(d, C_{\mathcal{W}}, d_2(\mathcal{M}^{\pm}, \delta_0))$.
- cut lemma: minimizing sequences can be chosen to be close to $\mathcal{M}^{\pm}$: $2q' < d_2(\mathcal{M}^+, \mathcal{M}^-) \leq 2d_2(\mathcal{M}^+, m_n(0)) = 2d_2(\mathcal{M}^-, m_n(0))$, if $d_2(m_n(t), \mathcal{M}^{\pm}) > q'$, then this implies for $T \geq T(q')$ $J_T(m_n, w_n) > C$, so $\exists s \leq \bar{s}(q'), d_2(m_n(s), \mathcal{M}^+) = q' = d_2(m_n(T/2 - s), \mathcal{M}^+)$.
- direct methods: m_n minimizing sequence is uniformly continuous with values in$\mathcal{P}_2(\mathbb{R}^d)$, with uniformly bounded 2 moments: by Ascoli Arzelà pass to the limit (up to subseq.) in d_p, for $p < 2$. Limit also in $w^\star - (L^\infty)$ and $w_n \to w$ weakly in L^2..
  Conclusion by lsc of the energy.

Qualitative properties of the periodic orbits

Assume that the aggregating potential $\mathcal{W}$ is sufficiently strong, so that the stationary minimizers are compactly suppported and the density constraint is always saturated- as in the model case.

Question: under sufficiently strong aggregating potential (in particular as before), is it true that

minimal periodic orbits have **compact support**?

minimal periodic orbits are **characteristics for some evolving sets**? In this case in the support of m^T the constraint will be always saturated....

Qualitative properties of periodic equilibria- dimension 1

In dimension $d = 1$ for
$\mathcal{W}(m) = \int_{\mathbb{R}^d} W(x)m(dx) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(|x-y|)m(dx)m(dy)$ *under the additional assumptions*

$$\exists R_0 \geq 0 \ s.\ t. \ \begin{cases} \min_{|x|>R_0} \nabla W(x)\cdot x > 0 \\ \min_{|x|\leq 2R_0+2}(-K'(|x|)) \geq \max_{|x|\leq R_0+1} |\nabla W(x)| \end{cases}$$

there are (m^T, w^T) periodic minimizers of J_T such that

$$m^T(t) = \rho\chi_{(x(t),\ x(t)+1)} \qquad w^T(t) = -\rho\dot{x}(t)\chi_{(x(t),\ x(t)+1)}$$

where x minimizes

$$y_t \mapsto \int_0^T \frac{1}{2}(\dot{y}_t)^2 dt + \int_0^T \int_{y(t)}^{y(t)+1} W(s)dsdt$$

among T-periodic curves with $y_{t+T/4} = y_{T/4-t}$ and $y_t = -y_{-t} - 1$.

Discrete counterpart: the N-particle system:

We restrict J_T on curves with values on empirical measures $m^N(t) = \sum_{i=1}^N \delta_{x_i(t)}$,
where $t \to x_i(t)$ are T-periodic trajectories of N agents/ particles.
So $w^N(t) = \frac{1}{N} \sum_i \dot{x}_i(t) \delta_{x_i(t)}$

$$J_T^N(x) = \int_0^T \sum_{i=1}^N \frac{1}{2N} |\dot{x}_t^i|^2 dt + \int_0^T \mathcal{W}\left(\frac{1}{N} \sum_{i=1}^N \delta_{x_t^i}\right) dt$$

L^∞ constraint translates on a constraint on the minimal distance between particles (disks):
If $m^N = \frac{1}{N} \sum_i \delta_{x_i} \to \mu$ narrowly and $|x_t^i - x_t^j| \geq \frac{c}{N^{1/d}}$ for all $i \neq j$,
then μ has a density $m \in L^\infty(\mathbb{R}^d)$, and $\|m\|_\infty \leq 2^d c^{-d} \omega_d^{-1}$.

$d = 1$ $(\rho = 1)$, minimizing trajectories are uniformly bounded

$$x_t^1 < x_t^2 < \cdots < x_t^N, \qquad x_t^{i+1} - x_t^i \geq \frac{1}{N},$$
$$x_t^{N+1-i} = -x_{-t}^i, \qquad x_{\frac{T}{4}-t}^i = x_{\frac{T}{4}+t}^i,$$

Theorem

Assume that $W'(s) > 0$ for all $s > R_0$. Then, any minimizer J_T^N satisfies

$$|x_t^i| \leq R_0 + 1 \qquad \textit{for all } t \in [0, T] \textit{ and } i = 1, \ldots N.$$

Proof by using a truncation procedure.

$d = 1$, distance constraint is saturated

Theorem

Assume moreover that K is strong enough:

$$\min_{0<r\le 2R_0+2} |K'(r)| > \max_{|x|\le R_0+1} |W'(x)|,$$

x *minimizes* J_T^N. *Then,*

$$|x_t^{i+1} - x_t^i| = \frac{1}{N} \qquad \text{for all } t \text{ and } i.$$

Idea of the proof

- For any J, write down optimality conditions for the barycenters of trajectories $\{x^1, \dots, x^J\}$ and $\{x^{J+1}, \dots, x^N\}$ Due to the distance constraints, they hold as inequalities (and as equality in open sets where the distance constraint is not saturated),
- let $b_t^J := \frac{1}{N-J}\sum_{i=J+1}^N x_t^i - \frac{1}{J}\sum_{i=1}^J x_t^i$ and observe that if $x_t^{J+1} - x_t^J > \frac{1}{N}$ for some J and t,

$$\ddot{b}_t^J \geq \min_{0<r\leq 2R_0+2} |K'(r)| - \max_{|x|\leq R_0+1} |W'(x)| > 0.$$

- observe that b_t^J can be written as a combination of b_t^{J-1} and $x_t^J - x_t^{J-1}$ and look for maximizers of b_t^J.

$d = 1$, Γ convergence

$$J_T^N(x) \xrightarrow{\Gamma} J_T(m, mv) \qquad \text{as } N \to +\infty,$$

that is:

- $\Gamma - \liminf$: if $m_x^N \to m$ and $\frac{1}{N}\sum_{i=1}^N \dot{x}_t^i \delta_{x_t^i} \to v$, then
$$\liminf_{N\to\infty} J_T^N(x) \geq J_T(m, mv),$$
- $\Gamma - \limsup$: for every (m, v) there exists $x = x^N$ such that $m_x^N \to m$, $\frac{1}{N}\sum_{i=1}^N \dot{x}_t^i \delta_{x_t^i} \to v$ and
$$\limsup_{N\to\infty} J_T^N(x) \leq J_T(m, mv),$$

where

$$m_x^N \to m \ \text{ in } C([0, T], \mathcal{P}_2(\mathbb{R})), \qquad \frac{1}{N}\sum_{i=1}^N \dot{x}_t^i \delta_{x_t^i} \to v \text{ narrowly.}$$

As a consequence, every cluster point (m^T, w^T) of $(m_x^N, \frac{1}{N}\sum_{i=1}^N \dot{x}_t^i \delta_{x_t^i})$ minimizes J_T, is compactly supported and saturates the constraint.

Open questions - ongoing work

$d \geq 2$:

- Γ-convergence of discrete to continuous,
- $m(t)$ has compact support,
- $m(t) = \chi_{\Omega(t)}$.

(some) related literature:

- Di Francesco, Rosini, and Di Francesco, Fagioli, Radici :
  many-particle approximation of non-local transport PDE in dim 1
- convergence of the hard sphere system in Boltzmann problem