Some new regularity results for viscous Hamilton-Jacobi equations with unbounded right-hand side

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joint works with M. Cirant (Padova)

A glimpse on maximal L^q-regularity: linear equations...

For $q \in (1, \infty)$

- Poisson equation [Caldéron-Zygmund, '52]:

$$-\Delta u = f \in L^q \implies \|D^2 u\|_q \lesssim \|f\|_q \tag{P}$$

- Heat equation "Parabolic CZ" [Ladyzhenskaya-S.-U., '67]

$$\partial_t u - \Delta u = f \in L^q_{x,t} \implies \|u\|_{W^{2,1}_q} \lesssim \|f\|_q \tag{H}$$

provided $u(0) \in W^{2-2/q,q} = (L^q, W^{2,q})_{1-1/q,q}$. Actually true for more general evolution problems on a Banach space X

$$\begin{cases} u'(t) - \mathcal{A}u(t) = f(t) \in L^q(I; X) \\ u(0) = u_0 \end{cases}$$

Here, $u \in W^{1,q}(I; X) \cap L^q(I; D(\mathcal{A}))$ provided $u_0 \in (X, D(\mathcal{A}))_{1-1/q,q}$.
Cf [Lambèrton '87], see also [Da Prato-Grisvard, Denk-Hieber-Prüss, Prüss-Schnaubelt]

...quasi-linear equations

Question: What happens for quasi-linear PDEs like

$$-\operatorname{Tr}(A(x)D^2u) + H(x, Du) = f(x) \in L^q \tag{SHJ}$$

$$\partial_t u - \operatorname{Tr}(A(x,t)D^2u) + H(x, Du) = f(x,t) \in L^q_{x,t} \tag{PHJ}$$

with $H(x, Du) \sim |Du|^\gamma$, $\gamma > 1$ and $\lambda I \leq A \leq \Lambda I$?

Do they behave, respectively, as (P) and (H)? For which values of q?

Motivations- 1

Besides its own interest, this is motivated by the problem of smooth regularity in Mean-Field Game systems

$$\begin{cases} -\partial_t u - \sum_{i,j=1}^d a_{ij}\partial_{ij}u + H(x,Du) = f(m) & \text{in } Q_T \\ \partial_t m - \sum_{i,j=1}^d \partial_{ij}(a_{ij}m) - \mathrm{div}(mD_pH(x,Du)) = 0 & \text{in } Q_T \\ m(x,0) = m_0(x),\ u(x,T) = u_T(x) & \text{in } \mathbb{T}^d , \end{cases}$$

with local coupling (i.e. $f(m) \simeq \pm m^\alpha$ for some $\alpha > 0$, or $f(m) \simeq \log m$), that is still open in full generality (see e.g. [Cirant '16] for stationary problems, [Gomes et al '15-'16] for partial results on parabolic MFGs).

Motivation 2-Maximal regularity for quasi-linear problems

An important result in [P.-L. Lions, 85] for the stationary equation

$$-\Delta u(x) + |Du(x)|^{\gamma} = f(x)$$

states that Lipschitz regularity holds whenever

$$\|f\|_{L^q} < \infty, \qquad q > d,$$

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$$-\Delta u + |Du|^\gamma = f(x) \in L^q \quad \Rightarrow \quad D^2 u, |Du|^\gamma \in L^q .$$

if and only if

$$q > \frac{d}{\gamma'} .$$

⇝ Proof by [Cirant-G. ’20]

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Natural (open) question: What happens in the time-dependent case? Does HJ "behave" like (H)?

Related literature in the parabolic case

- When $f \in L^q$, $q > d+2$, and $\gamma \leq 2$, $W_q^{2,1}$ (and hence maximal) regularity dates back to [Ladyzhenskaya- Uralt'seva], see also [Weidemaier-Softova-Maugeri] for later results.
- For $\gamma > 2$, [Cardaliaguet-Silvestre, Stokols-Vasseur] Hölder regularity for the parabolic problem, even degenerate.
- [Gomes et al, '14-'15] Regularity for time-dependent MFG systems with $f(m) \sim m^\alpha$ and Lipschitz bounds via nonlinear adjoint method, up to $\gamma < 3$, and for smooth solutions.

What to expect? Scaling laws...

Looking at L^∞ gradient bounds, we perform a $W^{1,\infty}$ scaling $w(x,t) = \epsilon^{-1} u(\epsilon x, \epsilon^2 t)$ to see that

$$\partial_t w - \Delta w + \epsilon |Dw|^\gamma = \epsilon f(\epsilon x, \epsilon^2 t) =: g_\epsilon(x,t)\ .$$

Remark. $L^q_{x,t}$ norm of g_ϵ is invariant under the previous scaling precisely for the threshold $q = d + 2$

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For any $\tau \in (0, T)$, $x_0 \in \mathbb{T}^d$, consider the adjoint equation

$$\begin{cases} -\partial_t \rho - \Delta \rho - \mathrm{div}(D_p H(Du)\rho) = 0 & \text{in } \mathbb{T}^d \times (0, \tau) , \\ \rho(\tau) = \delta_{x_0} & \text{on } \mathbb{T}^d. \end{cases}$$

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and get the representation formula

$$\partial_\xi u(x_0, \tau) = \iint_{\mathbb{T}^d \times (0,\tau)} \partial_\xi f \, \rho + \int_{\mathbb{T}^d} v\rho(0)$$

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and get the representation formula

$$\partial_\xi u(x_0, \tau) = \underbrace{-\iint_{\mathbb{T}^d \times (0,\tau)} f\, \partial_\xi \rho}_{\leq \|f\|_q \|D\rho\|_{q'}} + \int_{\mathbb{T}^d} \partial_\xi u\, \rho(0).$$

Remark. $\|\partial_\xi u\|_\infty$ depends on $\|\partial_\xi \rho\|_{q'}$.

On the adjoint problem

Lemma

Suppose that ρ is a solution to the adjoint equation with drift b, knowing $|b| \in L^{\gamma'}(\rho\, dxdt)$. Then

$$\|D\rho\|_{L^{q'}} \leq C\left(\iint |b(x,t)|^{\gamma'}\rho\, dxdt + \|\rho_\tau\|_{L^1(\mathbb{T}^d)}\right)$$

where

$$q' < \begin{cases} \frac{d+2}{d+1} & \gamma \leq 2 \\ \frac{d+2}{d+3-\gamma'} & \gamma > 2\,. \end{cases}$$

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Proof inspired by [Metafune-Pallara-Rhandi], via parabolic Caldéron-Zygmund regularity and duality. Similar results appeared in [Porretta] in more general contexts but for $\gamma = 2$.

Remark. For $\gamma \leq 2$, same regularity of the heat equation. When $\gamma \to \infty$, the regularity deteriorates ($\gamma' \to 1$)

Theorem

- $a_{ij} \in C(0,T;W^{2,\infty}(\mathbb{T}^d))$ and are uniformly parabolic,
- $H(p) \approx |p|^\gamma$,
- $f \in L^q(\mathbb{T}^d \times (0,T))$, for some

$$q > d+2 \qquad \text{and} \qquad q \geq \frac{d+2}{\gamma'-1},$$

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Let u be a continuous *weak* solution to (PHJ) such that

$$D_p H(\cdot, Du) \in L^{\mathcal{P}}_{loc}(\mathbb{T}^d \times (0,T)) \qquad \text{for some} \quad \mathcal{P} \geq d+2.$$

Then, $u(\cdot,\tau) \in W^{1,\infty}(\mathbb{T}^d)$ for all $\tau \in (0,T]$. In particular, for all $t_1 \in (0,T)$ there exists a positive constant C_1 depending on t_1 and the data such that

$$\|u(\cdot,\tau)\|_{W^{1,\infty}(\mathbb{T}^d)} \leq C_1 \qquad \text{for all} \quad \tau \in [t_1, T].$$

Some remarks

- We assume the integrability condition $b = D_pH(Du) \in L^{\mathcal{P}}$ for some $\mathcal{P} \geq d+2$ (or better $b = D_pH(Du) \in L^{\mathcal{Q}}(L^{\mathcal{P}})$ for $\frac{d}{2\mathcal{P}} + \frac{1}{\mathcal{Q}} \leq \frac{1}{2}$). This is the so-called Aronson-Serrin condition, which ensures the well-posedness of the adjoint problem.

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that are NOT Lipschitz.
- We basically prove the maximal regularity for (PHJ) via

$$Du \in L^{(d+2)(\gamma-1)} \implies Du \in L^{\infty} \implies u \in W_p^{2,1}$$

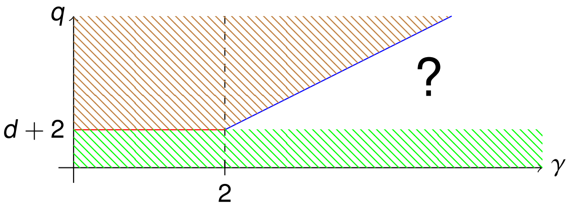
for every $\gamma > 1$. First attempt to generalize in the whole superlinear regime [Lions, '85] to the parabolic setting.
- False if $q \leq d+2$, expect to be true for $f \in L^{d+2,1}(Q_T)$ (by duality $D\rho \in L^{\frac{d+2}{d+1},\infty}(Q_T)$), [G., in progress]. See results from nonlinear potential theory...

Summary on Lipschitz regularity

$f \in L^q(\mathbb{T} \times (0, T))$, for some

$$q > d + 2 \qquad \text{and} \qquad q \geq \frac{d+2}{\gamma' - 1}.$$

$q \leq d + 2$: Lipschitz regularity fails.



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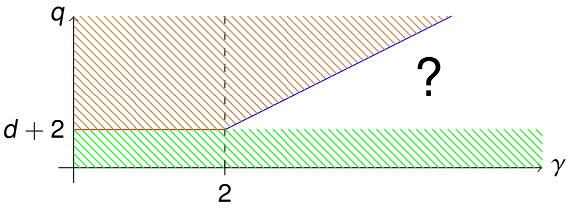

For $\gamma \leq 2$, “viscous HJ behaves like the heat equation”.

Some improved a priori estimates

We prove that $\|Du\|_\infty$ can be bounded by $\|f\|_q$ if

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We prove that $||Du||_\infty$ can be bounded by $||f||_q$ if

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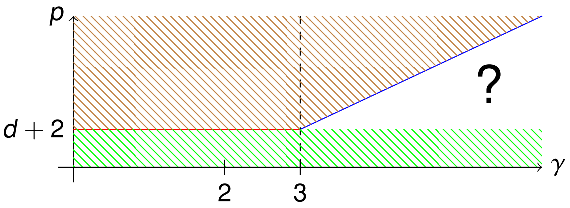
For $\gamma \leq 3$, "viscous HJ behaves like the heat equation":



The supercritical range $q \leq d + 2$: parabolic Lions' conjecture

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if and only if $q > \frac{d+2}{\gamma'}$.

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Remark. Typically, integrability exponents for parabolic problems can be deduced from the corresponding elliptic ones via the substitution $d \rightsquigarrow d+2$.

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Therefore, "adding 1 dimension" $\rightsquigarrow$ "add 2 to the exponents", heuristically because of the correspondence

$$\text{1 time derivative} \rightarrow \partial_t u = \Delta u \leftarrow \text{2 space derivatives}$$

Some observations

- Again by scaling, setting $v(x,t) = \lambda^{\frac{2-\gamma}{\gamma-1}} u(\lambda x, \lambda^2 t)$ we get

$$\partial_t v - \Delta v + |Dv|^\gamma = \lambda^{\gamma'} f(\lambda x, \lambda^2 t) = g_\lambda(x,t)$$

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  and the L^q norm of g_λ is invariant when $q = (d+2)/\gamma'$.
- Regard the equation as a perturbation of the heat equation, i.e. $\partial_t u - \Delta u = -|Du|^\gamma + f$, apply maximal L^q-regularity and Sobolev embeddings to deduce

$$|||Du|||_{L^{\frac{(d+2)q}{d+2-q}}} \underbrace{\lesssim}_{\text{Sobolev's inequality}} \|u\|_{W_q^{2,1}} \underbrace{\lesssim}_{CZ} |||Du|||_{\gamma q}^{\gamma} + \|f\|_q \,.$$

  Gain in regularity whenever

$$\frac{1}{q} - \frac{1}{d+2} < \frac{1}{q\gamma} \implies q > \frac{d+2}{\gamma'} \,.$$

Main ideas

1. Write HJ as $\partial_t u - \Delta u = -H(x, Du) + f(x, t)$ and apply CZ regularity for heat equations to get

$$(\|D^2 u\|_q \lesssim)\|u\|_{W^{2,1}_q(Q_T)} \leq C(\|f\|_{L^q(Q_T)} + \||Du|\|^{\gamma}_{L^{q\gamma}(Q_T)} + \|u(0)\|_{W^{2-2/q,q}(\mathbb{T}^d)})$$

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2. Use Gagliardo-Nirenberg inequalities.
   - When $\gamma < 2$ we use

$$\|Du\|^{\gamma}_{L^{\gamma q}} \lesssim \|D^2 u\|^{\frac{\gamma}{2}}_{L^q_{x,t}} \|u\|^{\frac{\gamma}{2}}_{L^\infty(L^p)}$$

with $p = \infty$ for $q > \frac{d+2}{2}$, $p < \infty$ for $q \in \left(\frac{d+2}{\gamma'}, \frac{d+2}{2}\right]$.

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   - When $\gamma = 2$ we apply [Nirenberg-Miranda]

$$\|Du\|^{2}_{L^{2q}} \lesssim \|D^2 u\|^{2\sigma}_{L^q_{x,t}} \|u\|^{2(1-\sigma)}_{L^\infty(C^\alpha)}$$

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     with $2\sigma < 1$, $\alpha \in (0,1)$.
   - For $\alpha > \frac{\gamma-2}{\gamma-1}$, $q > \frac{(d+2)(\gamma-1)}{2}$ we can apply

$$\|Du\|^{\gamma}_{L^{\gamma q}} \lesssim \|D^2 u\|^{\gamma\sigma}_{L^q_{x,t}} \|u\|^{\gamma(1-\sigma)}_{L^\infty(C^\alpha)} \quad \gamma\sigma < 1, \gamma > 2\ .$$

The next is a crucial regularity results to deduce integral estimates for solutions to HJ via a refinement of the Evans' adjoint method

Lemma

Suppose that ρ is a solution to the adjoint equation with drift b and $\rho_\tau \in L^{p'}(\mathbb{T}^d)$, p arbitrarily large, knowing $|b| \in L^{\gamma'}(\rho\, dxdt)$, $\gamma \in \left[\frac{d+2}{d+1}, 2\right)$. Then

$$\|D\rho\|_{L^{q'}} \leq C\left(\iint |b(x,t)|^{\gamma'} \rho\, dxdt + \|\rho_\tau\|_{L^{p'}(\mathbb{T}^d)}\right)$$

where

$$\frac{d+2}{d+1} \leq q' \leq \frac{d+2}{d+3-\gamma'}\ .$$

Remark. When $q' = \frac{d+2}{d+1}$, $\rho_\tau \in L\log L$ and $\gamma = 2$ are proved by [Porretta].

Main regularity estimates: integral estimates

- There exists a constant $C > 0$, depending on $\|f\|_{L^r(Q_\tau)}$, $r > \frac{d+2}{2}$, $\|u_0\|_{C(\mathbb{T}^d)}$, d, q, T, γ, $\gamma > 1$, such that any strong solution to HJ satisfies
$$\|u(\cdot,\tau)\|_{C(\mathbb{T}^d)} \leq C \text{ for all } \tau \in [0,T] .$$
- Let $\gamma > 1$ and $\frac{d+2}{\gamma'} < r \leq \frac{d+2}{2}$. There exists a constant $C > 0$, depending in particular on $\|f\|_{L^r(Q_\tau)}, \|u_0\|_{L^p(\mathbb{T}^d)}$, such that any strong solution to HJ satisfies
$$\|u\|_{L^\infty(0,\tau;L^p(\mathbb{T}^d))} \leq C$$
for $1 \leq p < \infty$.

Remark 1. Integral and sup-norm estimates have been obtained by [Gomes et al], but they are not sharp. In the degenerate diffusion case see also [Cardaliaguet-Graber-Porretta-Tonon].

Main a priori estimates

- Let u be a strong solution. Then, then there exists a positive constant C depending in particular on $\|u_0\|_{C^\alpha(\mathbb{T}^d)}$, $\|f\|_{L^r(Q_T)}$, such that

$$\|u\|_{L^\infty(0,T;C^\alpha(\mathbb{T}^d))} \leq C \ .$$

with $\alpha < 2 - \frac{d+2}{r}$ for $r > \frac{d+2}{2}$ and $\gamma = 2$, while $\alpha \leq \gamma' - \frac{d+2}{r}$ for $r > \frac{d+2}{\gamma'}$ and $\gamma > 2$.

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Remark 2. First Hölder continuity estimates for parabolic viscous problems having explicit Hölder exponents, cf [Cardaliaguet-Silvestre, Stokols-Vasseur].

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Remark 2. First Hölder continuity estimates for parabolic viscous problems having explicit Hölder exponents, cf [Cardaliaguet-Silvestre, Stokols-Vasseur]. Moreover, Hölder bounds are obtained via the Evans' duality method and such application of the nonlinear adjoint method is new.

Main a priori estimates

- Let u be a strong solution. Then, then there exists a positive constant C depending in particular on $\|u_0\|_{C^\alpha(\mathbb{T}^d)}$, $\|f\|_{L^r(Q_T)}$, such that

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Remark 2. First Hölder continuity estimates for parabolic viscous problems having explicit Hölder exponents, cf [Cardaliaguet-Silvestre, Stokols-Vasseur]. Moreover, Hölder bounds are obtained via the Evans' duality method and such application of the nonlinear adjoint method is new.

Remark 3. A slight modification of the proof allows to show a regularization effect for weak energy solutions to HJ starting from a continuous initial data.

Parabolic Lions' conjecture: final results

- **Sub-quadratic case** $\gamma < 2$. Let u be a strong solution to HJ with $u_0 \in W^{2-\frac{2}{q},q}(\mathbb{T}^d)$ and $\gamma \in \left(\frac{d+2}{d+1}, 2\right)$, and assume that $f \in L^q(Q_T)$, $q > \frac{d+2}{\gamma'}$. Then

$$\|\partial_t u\|_{L^q(Q_T)} + \|u\|_{L^q(0,T;W^{2,q}(\mathbb{T}^d))} + \||Du|^\gamma\|_{L^q(Q_T)} \lesssim \|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}$$

Remark. Known for $q > \frac{d+2}{2}$, cf [Gomes et al]. We go beyond since $\frac{d+2}{\gamma'} < \frac{d+2}{2}$ for $\gamma < 2$.

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- **Quadratic case** $\gamma = 2$. Let u be a strong solution to HJ with $u_0 \in W^{2-\frac{2}{q},q}(\mathbb{T}^d)$, and assume that $f \in L^q(Q_T)$, $\frac{d}{2} + 1 < q < d + 2$. Then

$$\|\partial_t u\|_{L^q(Q_T)} + \|u\|_{L^q(0,T;W^{2,q}(\mathbb{T}^d))} + \||Du|^2\|_{L^q(Q_T)} \lesssim \|f\|_{L^q(Q_T)} + \|u_0\|_{W^{2-\frac{2}{q},q}(\mathbb{T}^d)}$$

  Remark. Known for $q \geq d + 1$, cf [Softova-Weidemaier].

- **Super-quadratic case** $\gamma > 2$. Let u be a strong solution to HJ with $u_0 \in W^{2-\frac{2}{q},q}(\mathbb{T}^d)$ and $\gamma > 2$, and assume that $f \in L^q(Q_T)$, $\frac{(d+2)(\gamma-1)}{2} < q < \frac{d+2}{\gamma'-1}$. Then

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Remark. $\frac{(d+2)(\gamma-1)}{2} > \frac{d+2}{\gamma'}$. New in the super-quadratic regime.

Applications to MFGs

Consider the system

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) = \pm m^\alpha & \text{in } Q_T \\ \partial_t m - \Delta m - \operatorname{div}(m D_p H(x, Du)) = 0 & \text{in } Q_T \\ m(x, 0) = m_0(x),\ u(x, T) = u_T(x) & \text{in } \mathbb{T}^d\ , \end{cases} \tag{1}$$

with $H(x, Du) \sim |Du|^\gamma$.

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- **Defocusing case** $\rightsquigarrow f(m) \sim m^\alpha$. From second order estimates

$$\iint |D^2 u|^2 m + \underbrace{m^{\alpha-1}|Dm|^2}_{|D(m^{\frac{\alpha+1}{2}})|^2} \leq C \quad \forall \alpha > 0$$

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- **Focusing case** $\rightsquigarrow f(m) \sim -m^\alpha$. In this case, one has

$$\iint m^{\alpha+1} < \infty \text{ for } \alpha < \min\left\{\frac{\gamma'}{d+2-\gamma'}, \frac{\gamma'}{d}\right\}$$

Theorem (Focusing → $f(m) = -m^\alpha$)

There exists a smooth solution to (1) *in the following cases*

(i) *When* $\frac{d+2}{d+1} \le \gamma \le 2$ *and* $\alpha < \frac{\gamma'}{d}$

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Remark. When $\frac{d+2}{d+1} < \gamma < 2$, known under the additional restriction $\alpha < \min\left\{\frac{\gamma'}{d}, \frac{\gamma'-2}{d+2-\gamma'}\right\}$, cf [Cirant-Tonon]. New in the super-quadratic regime.

Consider $f(m) = -m^\alpha$ and $\gamma < 2$. Knowing that $\|m^\alpha\|_{L^{(1+\frac{1}{\alpha})}} < \infty$, by maximal L^q-regularity we have

$$\||Du|^\gamma\|_{L^q_{x,t}} \lesssim \|m^\alpha\|_{L^q_{x,t}}, q > \frac{d+2}{\gamma'} .$$

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Take $q = 1 + \frac{1}{\alpha}$ to see that $\||Du|^\gamma\|_{L^q_{x,t}} < \infty$ provided that

$$\left(1 + \frac{1}{\alpha}\right) > \frac{d+2}{\gamma'} \implies \alpha < \frac{\gamma'}{d+2-\gamma'} .$$

and recall that $\frac{\gamma'}{d} < \frac{\gamma'}{d+2-\gamma'}$ for $\gamma < 2$. Then, $\alpha < \gamma'/d$.

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Furthermore, the drift of the Fokker-Planck equation $|b| = |Du|^{\gamma-1}$ verifies the integrability condition

$$\||b|\|_{L^r_{x,t}} < \infty\,, r > d+2 \implies m \in C^{\beta,\beta/2} \text{ for some } \beta \in (0,1)$$

by [Aronson-Serrin].

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by [Aronson-Serrin]. Then, Schauder estimates give that solutions of the MFG system are smooth.

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Thanks for the attention!