Correlated equilibria and mean field games
– Joint work with Markus Fischer (Padova Uni) –

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1 Introduction

2 Correlated $N$-player and mean field games

3 Convergence of correlated equilibria in restricted strategies

4 Construction of approximate correlated equilibria

5 Conclusions
Mean field games (MFGs), introduced by [Huang et al., 2006] and [Lasry & Lions, 2007], arise as limit systems for certain symmetric stochastic differential $N$-player games with mean field interaction as the number of players $N \to \infty$. Mean field interaction: each player interacts with her competitors only via the empirical distribution of their positions (closely related to anonymous games). When $N \to \infty$, one expects the empirical distribution to converge to the law of the "representative player" (Law of Large Numbers aka Propagation of Chaos). In MFGs the "representative player" reacts optimally to the behaviour of the population, which in turn should arise (at equilibrium) by aggregation of all (identical) players best responses.
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Connection between MFG and $N$-player games

Rigorous connection between MFGs and underlying $N$-player games can be established in two directions:

1. **Construction of approximate Nash equilibria** for $N$-player games starting from a solution to the MFG
   (for instance, [Huang et al., 2006], [Carmona & Delarue, 2013], [Gomes et al., 2013], . . .).

2. **Convergence to solutions of the MFG** of (approximate) $N$-player Nash equilibria, as $N \to \infty$.

Crucial, especially in second direction, is the choice of admissible strategies in definition of $N$-player Nash eq. Standard choices:

- stochastic open-loop;
- feedback or closed-loop Markov over
  - full system state,
  - only individual players’ states (restricted or decentralized strategies, “Markov open-loop”).
Convergence to the MFG limit

Difficult, especially for non-stationary finite horizon problems: Convergence of full state Markov feedback Nash equilibria.

- Breakthrough in [Cardaliaguet et al., 2015]: convergence of Nash equilibria through master equation if well-posed, thus under uniqueness of MFG solutions.

- In this situation, also CLT and LDP from the MFG limit: [Cecchin & Pelino, 2017] and [Bayraktar & Cohen, 2017] for finite state games, [Delarue et al., 2018a] in the diffusion setting.
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- Recent case studies under non-uniqueness: [Nutz et al., 2018] for optimal stopping problems, [Delarue & Foguen Tchuendom, 2018] example of restoration of uniqueness through common noise, [Cecchin et al., 2018], [Bayraktar & Zhang, 2019] for two-state models.
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- In [Lacker, 2018], general convergence result in the non-degenerate diffusion setting, but to weak solutions of the MFG; in particular, stochastic flow of measures (even without common noise).
Correlated equilibria: a two-player textbook example

Generalization of Nash eq that allows for correlation between players’ strategies due to R. Aumann [Aumann, 1974, Aumann, 1987].

For extension to discrete time stochastic games see, e.g., [Solan, 2001].

Huge literature on correlated equilibria and its generalisations (e.g. communication equilibria): F. Forges, S. Hart, E. Lehrer, R. Myerson ...

Game of “chicken” (Rebel without a cause): D=Dare, S=Swerve

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>D</th>
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<tbody>
<tr>
<td>S</td>
<td>(6,6)</td>
<td>(2,7)</td>
</tr>
<tr>
<td>D</td>
<td>(7,2)</td>
<td>(0,0)</td>
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Two pure Nash: (S,D) and (D,S), with resp. payoffs (2, 7) and (7, 2).

One mixed Nash with payoff \((4\frac{2}{3}, 4\frac{2}{3})\): the players independently of each other select S with prob 2/3, D with prob 1/3.
Aumann’s idea: a mediator or correlation device randomly selects a strategy profile according to some publicly known distribution, then recommends each player in private a strategy according to the profile.

A probability distribution on the space of strategy profiles is a correlated equilibrium (CE) if no player has an incentive to unilaterally deviate from the mediator’s recommendation.
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Nash equilibria, pure or mixed, are correlated equilibria. In the example:

\[
\begin{array}{c|cc}
\text{Nash (S,D)} & S & D \\
\hline
S & 0 & 1 \\
D & 0 & 0 \\
\end{array}
\quad
\begin{array}{c|cc}
\text{Nash (D,S)} & S & D \\
\hline
S & 0 & 0 \\
D & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|cc}
\text{Mixed Nash} & S & D \\
\hline
S & \frac{4}{9} & \frac{2}{9} \\
D & \frac{2}{9} & \frac{1}{9} \\
\end{array}
\]

But also new equilibria:

Convex combinations of NE

Outside convex hull of NE
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<table>
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<tr>
<th>Nash (S,D)</th>
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<tr>
<td>S</td>
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S   | D   |
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<tr>
<td>0</td>
<td>α</td>
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<td>D(1−α)</td>
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Outside convex hull of NE

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Aim and scope

Consider correlated equilibria (CE) for a simple class of symmetric finite horizon $N$-player games. Find definition of CE for the limiting MFG ($N \to \infty$).

Justify definition in two ways:

- by showing convergence of $N$-player CE to the MFG limit
- by constructing approximate CE starting from a solution of the MFG.

Simplifying assumptions:

- dynamics in discrete time over finite horizon
- finite state space, finite set of control actions
- equilibria in restricted strategies.
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Set-up

Let $\mathcal{X}$, $\Gamma$ be finite sets, the set of individual states and control actions, respectively.

Let $T \in \mathbb{N}$ be the time horizon. The dynamics are determined by a meas. system function

$$\Psi : \{0, \ldots, T-1\} \times \mathcal{X} \times \mathcal{P}(\mathcal{X}) \times \Gamma \times \mathcal{Z} \rightarrow \mathcal{X},$$

where $\mathcal{Z} = [0, 1]$ is the space of noise states, equipped with $\nu \overset{\text{d}}{=} \text{Uniform}([0, 1])$.

Let $\mathcal{R}$ denote the set of admissible individual (restricted) strategies:

$$\mathcal{R} \overset{\text{d}}{=} \{ \varphi : \{0, \ldots, T-1\} \times \mathcal{X} \rightarrow \Gamma \}.$$

Strategies only depend on time and players’ own positions: restricted strategies (decentralized Markov strategies).
Let $m_0 \in \mathcal{P}(\mathcal{X})$, $\gamma \in \mathcal{P}(\mathcal{R}^N)$, and let $u : \mathcal{R} \to \mathcal{R}$; $m_0$ is called an initial distribution, $\gamma$ a correlated profile, and $u$ a strategy modification.
The $N$-player game: dynamics I

Let $m_0 \in \mathcal{P}(\mathcal{X})$, $\gamma \in \mathcal{P}(\mathcal{R}^N)$, and let $u : \mathcal{R} \to \mathcal{R}$; $m_0$ is called an initial distribution, $\gamma$ a correlated profile, and $u$ a strategy modification.

Let

$$(\Omega, \mathcal{F}, \mathcal{P}), \Phi_1^N, \ldots, \Phi_N^N, X_0^{N,1}, \ldots, X_T^{N,N}, \xi_1^{N,1}, \ldots, \xi_T^{N,N})$$

be a realization of the triple $(m_0, \gamma, u)$ for player $i$: for all $j \in \{1, \ldots, N\}$ the r.v.s

$$\Phi_j^N, \quad X_0^{N,j}, \ldots, X_T^{N,j}, \quad \xi_{1,j}, \ldots, \xi_{T,j}$$

takes values in $\mathcal{R}$, $\mathcal{X}$, and $\mathcal{Z}$, respectively, s.t.

- $X_0^{N,1}, \ldots, X_0^{N,N}$ are i.i.d. $\sim m_0$;
- $\mathcal{P} \circ (\Phi_1^N, \ldots, \Phi_N^N)^{-1} = \gamma$;
- $\xi_t^{N,j}, j = 1, \ldots, N, t = 1, \ldots, T$, are i.i.d. $\sim \nu$;

Moreover ... (see next slide)
... we also assume:

- \((\xi^{N,j})_{j=1}^N, (X^N_0)_{j=1}^N, and (\Phi^N_j)_{j=1}^N\) are independent;

- Dynamics: P-a.s., for every \(t \in \{0, \ldots, T - 1\}\),

\[
\begin{align*}
X^{N,i}_{t+1} &= \psi \left( t, X^N_t, \mu^N_t, u \circ \Phi^N_i(t, X^N_t), \xi^N_{t+1} \right), \\
X^{N,j}_{t+1} &= \psi \left( t, X^N_t, \mu^N_t, \Phi^N_j(t, X^N_t), \xi^N_{t+1} \right), \quad j \neq i,
\end{align*}
\]

where \(\mu^N_t = \frac{1}{N-1} \sum_{l \neq k} \delta_{X^N_{l,t}}\).
The $N$-player game: costs

The costs for player $i$ associated with $(m_0, \gamma, u)$ are

$$J_i^N(m_0; \gamma, u) = \mathbb{E} \left[ \sum_{t=0}^{T-1} f(t, X_t^{N,i}, \mu_t^{N,i}, u \circ \Phi_i^N(t, X_t^{N,i})) + F(X_T^{N,i}, \mu_T^{N,i}) \right],$$

where

- $f: \{0, \ldots, T-1\} \times \mathcal{X} \times \mathcal{P}(\mathcal{X}) \times \Gamma \to \mathbb{R}$ is the running cost,
- $F: \mathcal{X} \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ is the terminal costs,
- and the expected value $\mathbb{E}$ is determined according to a realization of $(m_0, \gamma, u)$ for player $i$.

*Interpretation:* In player $i$ costs, all other players follow the mediator recommendation (values of $\Phi_j^N$, $j \neq i$), while player $i$ applies modified strategy ($u \circ \Phi_i^N$ instead of $\Phi_i^N$).

Player $i$ accepts the mediator’s suggestion if $u = \text{Id}$. 
The $N$-player game: correlated equilibria (CE)

**Definition 1.**

Let $m_0 \in \mathcal{P}(X)$, $\varepsilon \geq 0$. A correlated profile $\gamma \in \mathcal{P}(\mathcal{R}^N)$ is called an $\varepsilon$-CE if for every $i \in \{1, \ldots, N\}$, every strategy modification $u$ on $\mathcal{R}$,

$$J_i^N(m_0; \gamma, \text{Id}) \leq J_i^N(m_0; \gamma, u) + \varepsilon.$$ 

When $\varepsilon = 0$, we say that $\gamma$ is a CE.

**Observations:**

1. Here, CE is defined with respect to restricted strategies; analogous definition for full feedback strategies.

2. When $\gamma$ is a Dirac distribution ($\Phi_1^N, \ldots, \Phi_N^N$ constant), then Def 1 reduces to that of a Nash eq in pure (restricted) strategies.

3. When $\gamma$ has product form ($\Phi_1^N, \ldots, \Phi_N^N$ independent), then Def 1 corresponds to Nash in mixed (restricted) strategies.
The $N$-player game: correlated equilibria (CE)

**Proposition.**

Let $m^N \in \mathcal{P}(\mathcal{X}^N)$ be exchangeable. Then there exists a symmetric CE with initial distribution $m^N$.

Sketch of the proof (based on Hart & Schmeidler (1989)):

- **Auxiliary 2-player zero-sum game** with player I min over symmetric $\gamma$, while player II max over modifications $u$.
- Player II pays to player I the amount
  \[
  \sum_{\varphi \in \mathcal{R}^N} \gamma(\varphi) \sum_{u \in \mathcal{U}} \theta(u) \left( J^N_1(m^N, \delta\varphi, u) - J^N_1(m^N, \delta\varphi, \text{Id}) \right).
  \]
- A symmetric CE is a symmetric strategy for player I that gives a payoff $\geq 0$ for any player II strategy.
- Use **Minimax Theorem** to find it.
The mean field game: dynamics

Any $\rho \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1})$ is called correlated flow. Let $u$ be a strategy modification on $\mathcal{R}$.

Let $((\Omega, \mathcal{F}, \mathbb{P}), \Phi, \mathcal{X}, \mu, \xi)$ be a realization of the triple $(m_0, \rho, u)$:

$$\Phi, \quad X_0, \ldots, X_T, \quad \mu_0, \ldots, \mu_T, \quad \xi_1, \ldots, \xi_T$$

are r.v.s with values in $\mathcal{R}$, $\mathcal{X}$, $\mathcal{P}(\mathcal{X})$, and $\mathcal{Z}$, respectively, such that

- $\mathbb{P} \circ (X_0)^{-1} = m_0$
- $\mathbb{P} \circ (\Phi, \mu_0, \ldots, \mu_T)^{-1} = \rho$
- $\xi_t, \ t = 1, \ldots, T$, are i.i.d. $\sim \nu$
- $\xi, X_0$, and $(\Phi, \mu)$ are independent
- $\mathbb{P}$-a.s., for every $t = 0, \ldots, T - 1$,

$$(1) \quad X_{t+1} = \Psi (t, X_t, \mu_t, u \circ \Phi (t, X_t), \xi_{t+1}).$$
The mean field game: costs

The costs associated with \((m_0, \rho, u)\) are given by

\[
J(m_0; \rho, u) = \mathbb{E} \left[ \sum_{t=0}^{T-1} f(t, X_t, \mu_t, u \circ \Phi(t, X_t)) + F(X_T, \mu_T) \right],
\]

where the expected value is determined according to a realization of \((m_0, \rho, u)\).

The cost functional is well defined, since any two realizations of \((m_0, \rho, u)\) generate the same expected value in the definition of \(J(m_0; \rho, u)\).
Definition 2.

Let $m_0 \in \mathcal{P}(\mathcal{X})$. A correlated flow $\rho \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1})$ is called a correlated solution of the MFG (in restricted strategies) if:

i. Optimality: For every strategy modification $u$ on $\mathcal{R}$,

$$J(m_0; \rho, \text{Id}) \leq J(m_0; \rho, u).$$

ii. Consistency: If $((\Omega, \mathcal{F}, \mathbb{P}), \Phi, X, \mu, \xi)$ is a realization of the triple $(m_0, \rho, \text{Id})$, then for every $t$,

$$\mu_t(\cdot) = \mathbb{P} [X_t \in \cdot \mid \mathcal{F}_{\tau}^\mu],$$

where $\mathcal{F}_{\tau}^\mu \equiv \sigma(\mu_s : s = 0, \ldots, T)$. 
Correlated solutions and weak solutions

Definition from [Lacker, 2018]; compatible with open-loop formulation from [Lacker, 2016] and work by R. Carmona and F. Delarue:

**Definition 2.5.** A weak semi-Markov mean field equilibrium (or simply a weak MFE) is a tuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \alpha^*, X^*, \mu)$, where $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space and:

1. $\mu$ is a continuous $\mathbb{F}$-adapted $\mathcal{P}(\mathbb{R}^d)$-valued process, $W$ is a $\mathbb{F}$-Brownian motion, and $X^*$ is a continuous $\mathbb{R}^d$-valued $\mathbb{F}$-adapted process with $\mathbb{P} \circ (X_0^*)^{-1} = \lambda$.
2. $\alpha^*: [0, T] \times \mathbb{R}^d \times C([0, T]; \mathcal{P}(\mathbb{R}^d)) \to A$ is semi-Markov.
3. $X_0^*, \mu$, and $W$ are independent.
4. The state equation holds:
   \[ dX_t^* = b(t, X_t^*, \mu_t, \alpha^*(t, X_t^*, \mu))dt + dW_t. \]
5. For every alternative semi-Markov $\alpha: [0, T] \times \mathbb{R}^d \times C([0, T]; \mathcal{P}(\mathbb{R}^d)) \to A$ we have
   \[ \mathbb{E} \left[ \int_0^T f(t, X_t^*, \mu_t, \alpha^*(t, X_t^*, \mu))dt + g(X_T^*, \mu_T) \right] \]
   \[ \geq \mathbb{E} \left[ \int_0^T f(t, X_t, \mu_t, \alpha(t, X_t, \mu))dt + g(X_T, \mu_T) \right], \]
   where $X$ is the solution (see Remark 2.6 below) of
   \[ dX_t = b(t, X_t, \mu_t, \alpha(t, X_t, \mu))dt + dW_t, \quad X_0 = X_0^*. \]  
   (2.2)
6. The consistency condition holds: $\mu_t = \mathbb{P}(X_t^* \in \cdot | \mathcal{F}_t^\mu)$ a.s. for each $t \in [0, T]$, where $\mathcal{F}_t^\mu = \sigma(\mu_s : s \leq t)$.
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Assumptions

For $N \in \mathbb{N}$, let $m_0 \in \mathcal{P}(\mathcal{X})$, $\gamma^N \in \mathcal{P}(\mathcal{R}^N)$, $\varepsilon_N \geq 0$, and $m_0 \in \mathcal{P}(\mathcal{X})$.

Assume:

A1 Continuity of $\Psi$: $\exists w : [0, \infty) \rightarrow [0, 1]$ measurable s.t. $w(s) \xrightarrow{s \to 0^+} 0$ and, for every $(t, x, a) \in \{0, \ldots, T - 1\} \times \mathcal{X} \times \Gamma$

$$\int_{\mathcal{Z}} 1_{\psi(t, x, m, a, z) \neq \psi(t, x, \tilde{m}, a, z)} \nu(dz) \leq w(\text{dist}(m, \tilde{m}))$$

for all $m, \tilde{m} \in \mathcal{P}(\mathcal{X})$.

Moreover, for every $t \in \{0, \ldots, T - 1\}$, every $\tau \in \mathcal{P}(\mathcal{X} \times \Gamma \times \mathcal{P}(\mathcal{X}))$, $\Psi(t, \cdot)$ is $\tau \otimes \nu$-a.e. continuous.

A2 The costs $f$, $F$ are continuous.

A3 $\gamma^N$ is a symmetric probability measures, for all $N \in \mathbb{N}$.

A4 Each $\gamma^N$ is an $\varepsilon_N$-CE in restricted strategies with $\varepsilon_N \to 0$ as $N \to \infty$. 
For $N \in \mathbb{N} \setminus \{1\}$, let

$$
((\Omega_N, \mathcal{F}_N, P_N), \Phi^N_1, \ldots, \Phi^N_N, X^{N,1}, \ldots, X^{N,N}, \xi^{N,1}, \ldots, \xi^{N,N})
$$

be a realization of the triple $(m_0, \gamma^N, \text{Id})$, and set

$$
\rho^N = P_N \circ \left( \Phi^N_1, \mu^N_0, \ldots, \mu^N_T \right)^{-1},
$$

where $\mu^N_t = \frac{1}{N-1} \sum_{j=2}^N \delta_x^{N,j}$. Then:

**Theorem 1.**

Grant (A1)-(A4). Then $(\rho^N)_{N \in \mathbb{N}}$ is relatively compact in $\mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1})$, and any limit point is a correlated solution of the MFG with initial law $m_0$. 

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Construction of approximate correlated equilibria

Idea: disintegrate a correlated MFG solution \( \rho \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}) \) as

\[
\rho(d\varphi, dm) = \rho_1(\varphi|m)\rho_2(dm).
\]

The mediator generates a flow \( m \sim \rho_2 \), which she uses as correlation device to recommend i.i.d. strategies \( \varphi_1, \ldots, \varphi_N \) from \( \rho_1(\cdot|m) \).

**Theorem 2.**

Let \( m_0 \in \mathcal{P}(\mathcal{X}) \). Grant (A1) and (A2). Suppose that \( \rho \in \mathcal{P}(\mathcal{R} \times \mathcal{P}(\mathcal{X})^{T+1}) \) is a correlated solution of the MFG. For \( N \in \mathbb{N} \), define \( \gamma_N \in \mathcal{P}(\mathcal{R}^N) \) as

\[
\gamma_N(C_1 \times \ldots \times C_N) \doteq \int_{\mathcal{P}(\mathcal{X})^{T+1}} \prod_{i=1}^{N} \rho_1(C_i|m) \rho_2(dm).
\]

Then there exists \( \varepsilon_N \geq 0 \) such that \( \gamma_N \) is an \( \varepsilon_N \)-CE with \( \varepsilon_N \to 0 \) as \( N \to \infty \).
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Conclusions

Definition of correlated equilibrium for a simple class of MFGs; extends in some sense concept of weak solution.

Justification of definition
- through convergence of $N$-player correlated equilibria to the MFG limit;
- through construction of approximate $N$-player correlated equilibria starting from the MFG.

Limitation: equilibria only in restricted strategies.

Some open questions:

1. Extension to more general classes, in particular, with infinite set of strategy modifications.

2. Examples.
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Mean field games.

Convergence to the mean field game limit: a case study.

E. Solan.
Characterization of correlated equilibria in stochastic games.