Weak Solutions of Second Order Master Equations for Mean Field Games with Common Noise

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Mean Field Games and Applications

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Outline

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1. Mean field game with common noise
   - Master equations with common noise
   - Mean field game system with common noise
   - Forward backward McKean-Vlasov SDEs
2. Well-posedness results
   - Well-posedness for FBSDEs
   - Well-posedness for master solutions
3. Convergence results
   - Nash systems
   - Convergence of solutions to Nash systems
   - Propagation of chaos

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Master equations with common noise

- The master equation with common noise:

$$\begin{cases} \partial_t V + \frac{1+\beta^2}{2}\mathrm{tr}(\partial_{xx} V) + H(x, \partial_x V) + F + \mathcal{M}V = 0, \\ V(T, x, \mu) = G(x, \mu), \end{cases} \tag{1}$$

where $F, G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and

$$\mathcal{M}V(t,x,\mu) := \mathrm{tr}\Big(\mathbb{E}\Big[\frac{1+\beta^2}{2}\partial_{\tilde{x}\mu} V(t,x,\mu,\xi) + \partial_\mu V(t,x,\mu,\xi)\partial_p H(\xi, \partial_x V(t,\xi,\mu)) + \beta^2 \partial_{x\mu} V(t,x,\mu,\xi) + \frac{\beta^2}{2}\tilde{\mathbb{E}}[\partial_{\mu\mu} V(t,x,\mu,\tilde{\xi},\xi)]\Big]\Big).$$

- Purpose: To find a Nash equilibrium for a mean field game.

Mean field game

- Let r.v. ξ be such that $\mathcal{L}_\xi = \mu$ and let $X^{\xi,\alpha}$ be

$$X_t^{\xi,\alpha} = \xi + \int_0^t \alpha_s ds + B_t + \beta B_t^0,$$

- Let $(Y^{\xi;\alpha',\alpha}, Z^{\xi;\alpha',\alpha}, Z^{0,\xi;\alpha',\alpha})$ solve

$$\begin{aligned} Y_t^{\xi;\alpha',\alpha} &= G(X_T^{\xi,\alpha'}, \mathcal{L}_{X_T^{\xi,\alpha}|B^0}) \\ &\quad + \int_t^T [F(X_s^{\xi,\alpha'}, \mathcal{L}_{X_s^{\xi,\alpha}|B^0})] - L(X_s^{\xi,\alpha'}, \alpha_s')ds \\ &\quad - \int_t^T Z_s^{\xi;\alpha',\alpha} dB_s - \int_t^T Z_s^{0,\xi;\alpha',\alpha} dB_s^0. \end{aligned}$$

Mean field game

- The cost functional

$$J(t,\mu;\alpha',\alpha)=\mathbb{E}[Y_0^{\xi;\alpha',\alpha}].$$

- The maximization problem:

$$V(t,\mu;\alpha)=\sup_{\alpha'} J(t,\mu;\alpha',\alpha)$$

Definition (Nash equilibrium)

We say that (α^*,μ^*) is a Nash equilibrium for the above mean field game problem if

$$V(t,\mu;\alpha^*)=J(t,\mu;\alpha^*,\alpha^*) \quad \text{and} \quad \mu_t^*=\mathcal{L}_{X_t^{\xi,\alpha^*}|B^0}.$$

Derivation of the master equation

- By comparison principle for BSDEs and the definition of the Nash equilibrium, we have

$$
\begin{aligned}
X_t^{\alpha^*} &= \xi + B_t^{\alpha^*} + \beta B_t^0, \\
Y_t^{\alpha^*} &= G(X_T^{\alpha^*}, \mathcal{L}_{X_T^{\alpha^*}|B^0}), \\
&\quad + \int_t^T F(X_s^{\alpha^*}, \mathcal{L}_{X_s^{\alpha^*}|B^0}) + H(X_s^{\alpha^*}, Z_s^{\alpha^*}) ds \\
&\quad - \int_t^T Z_s^{\alpha^*} dB_s^{\alpha^*} - \int_t^T Z_s^{0,\alpha^*} dB_s^0.
\end{aligned} \tag{2}
$$

where $\alpha_t^* = \partial_p H(X_t^{\alpha^*}, Z_t^{\alpha^*})$ and $dB_t^{\alpha^*} = \alpha_t^* dt + dB_t$.

Derivation of the master equation

- (2) is equivalent to the following FBSDE system

$$
\begin{aligned}
X_t^\xi &= \xi + \int_0^t \partial_p H(X_s^\xi, Z_s^\xi) ds + B_t + \beta B_t^0, \\
Y_t^\xi &= G(X_T^\xi, \mu_T), \\
&\quad + \int_t^T F(X_s^\xi, \mu_s) - L(X_s^\xi, \partial_p H(X_s^\xi, Z_s^\xi)) ds \\
&\quad - \int_t^T Z_s^\xi dB_s - \int_t^T Z_s^{0,\xi} dB_s^0.
\end{aligned}
$$

  where $\mu_t = \mathcal{L}_{X_t^\xi | B^0}$.

- Define $Y_t^\xi = V(t, X_t^\xi, \mu_t)$ and it can be shown that V satisfies the master equation (1) in $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ with $V(T, x, \mu) = G(x, \mu)$.

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Mean field game system

- Define $u(t,x) := V(t,x,\mu_t)$ and it can be shown that (μ_t, u_t) solves the following SPDE system

$$\begin{cases} d\mu = [\frac{1+\beta^2}{2}\mathrm{tr}(\partial_{xx}\mu) - \mathrm{div}(\mu\partial_p H(x,\partial_x u(t,x)))]dt - \beta\partial_x\mu dB_t^0; \\ du = -[\mathrm{tr}(\frac{1+\beta^2}{2}\partial_{xx}\mathrm{u} + \beta\partial_x\gamma(t,x)) + H(x,\partial_x u(t,x)) + F(x,\mu_t)]dt \\ \qquad +\gamma(t,x)dB_t^0 \\ u(T,x) = G(x,\mu(T)), \quad \mu(0) = \mu. \end{cases} \tag{3}$$

Remark:

- The vector function γ is part of the solution of the stochastic HJ equation.
- If the decoupling field V is smooth, it can be shown that

$$\gamma(t,x) = \beta\int_{\mathbb{R}^d}\partial_\mu V(t,x,\mu_t,y)\mu_t(dy).$$

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FBSDEs

- The mean field game equation (3) is equivalent to the following forward backward McKean-Vlasov SDEs

$$
\begin{aligned}
X_t^{\xi} &= \xi + \int_0^t \partial_p H(X_s^{\xi}, Z_s^{\xi})ds + B_t + \beta B_t^0, \\
Y_t^{\xi} &= G(X_T^{\xi}, \mu_T) + \int_t^T F(X_s^{\xi}, \mu_s) - L(X_s^{\xi}, \partial_p H(X_s^{\xi}, Z_s^{\xi}))ds \\
&\quad - \int_t^T Z_s^{\xi} dB_s - \int_t^T Z_s^{0,\xi} dB_s^0; \\
X_t^{x,\xi} &= x + \int_0^t \partial_p H(X_s^{x,\xi}, Z_s^{x,\xi})ds + B_t + \beta B_t^0, \\
Y_t^{x,\xi} &= G(X_T^{x,\xi}, \mu_T) + \int_t^T F(X_s^{x,\xi}, \mu_s) - L(X_s^{x,\xi}, \partial_p H(X_s^{x,\xi}, Z_s^{x,\xi}))ds \\
&\quad - \int_t^T Z_s^{x,\xi} dB_s - \int_t^T Z_s^{0,x,\xi} dB_s^0;
\end{aligned} \tag{4}
$$

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Known results

- Potential Mean Field Game:
  1. Degenerate: Bensoussan-Yam, Gangbo-Meszaros, Gangbo-Swiech, Mayorga,...
  2. Individual noise: Bensoussan-Graber-Yam, Pham-Wei, Wu-Zhang,...
  3. Common noise: Gangbo-Mayorga-Swiech,...
- General Mean Field Game:
  1. Degenerate: ???
  2. Individual noise: Chassagneux-Crisan-Delarue...
  3. Common noise: Cardaliaguet-Delarue-Lasry-Lions, Carmona-Delarue...

Known results

- Assume that F, G are smooth and satisfy some convexity assumption, the master equation (1) admits a unique classical solution for arbitrary long time T.
- Assume that F, G are smooth, the master equation (1) admits a unique classical solution for short time T.

An important open problem

- Can we define a notion of "weak" solution to the master equation and show its well-posedness if the above assumptions are not satisfied?

Assumptions

- $H, \partial_x H, \partial_p H$ are Lipschitz in each D_R and $\exists \epsilon_R > 0$

$$\partial_{pp} H \geq \epsilon_R I_d \text{ in } D_R = \{(x, p) \in \mathbb{R}^{2d} : |p| \leq R\}.$$

- $F, G, \partial_x F, \partial_x G$ are Lipschitz in x and μ (under $\mathcal{W}_1$)
- F, G are monotone.

Monotonicity

Definition (Monotonicity)

We say that $F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is monotone if $\forall \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \left(F(x,\mu_1) - F(x,\mu_2)\right)\left(\mu_1(x) - \mu_2(x)\right) dx \geq 0.$$

Well-posedness for FBSDEs

Theorem (Mou-Zhang, 2020)

The forward backward McKean-Vlasov SDEs (4) is well posed.

Remark:

- If F, G, H are smooth, the representation formulas for $\partial_\mu V, \partial_{x\mu} V, \partial_{\tilde{x}\mu} V$ and $\partial_{\mu\mu} V$ are given and V solves the master equation classically.
- The monotonicity is needed for keeping the Lipschitz constant of V in μ.

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good solution

Definition (good solution)

We say that V is a good solution if $\forall t_0 \in [0, T)$ and smooth V_n such that $\mathcal{L}V_n(t, x, \rho) \to 0$ in $L^\infty([t_0 - \delta, t_0] \times \mathbb{R}^d \times \mathcal{P}_2^M(\mathbb{R}^d))$ and $V_n(t_0, \cdot, \cdot) \to V(t_0, \cdot, \cdot)$ in $L^\infty(\mathbb{R}^d \times \mathcal{P}_2^M(\mathbb{R}^d))$ for some δ and any $M > 0$. Then $V_n \to V$ in $L^\infty([t_0 - \delta', t_0] \times \mathbb{R}^d \times \mathcal{P}_2^M(\mathbb{R}^d))$ for some δ'.

Well-posedness for good solutions

Theorem (Mou-Zhang, 2020)

The decoupling field V of the FBSDE (4) is the unique good solution to the master equation (1) in $[0, T]$ for any $T > 0$.

Remark:

- The monotonicity assumption is only needed to construct the decoupling field.

A main ingredient

We construct smooth mollifiers for functions on Wasserstein space.

Theorem

Let $U \in C^0(\mathcal{P}_1(\mathbb{R}^d))$. Then $\exists U_n \in C^\infty(\mathcal{P}_2) \cap C^0(\mathcal{P}_1(\mathbb{R}^d))$.

- $\lim_{n\to\infty} \|U_n - U\|_{L^\infty(\mathcal{M})} = 0$ *for any* $\mathcal{M} \subset\subset \mathcal{P}_1(\mathbb{R}^d)$.
- *If* $U \in \mathrm{Lip}(\mathcal{P}_1(\mathbb{R}^d))$ *with Lipschitz constant* L, *then* $U_n \in \mathrm{Lip}(\mathcal{P}_1(\mathbb{R}^d))$ *with Lipschitz constant* CL, *where* C *is independent of* n.
- *If* $U \in C^1(\mathcal{P}_2)$, *and* $\partial_\mu U$ *is uniformly continuous in* $(\mathcal{M} \cap \mathcal{P}_2(\mathbb{R}^d)) \times K$ *under* $\mathcal{W}_1$, *where* $\mathcal{M} \subset \mathcal{P}_1(\mathbb{R}^d)$ *and* $K \subset\subset \mathbb{R}^d$, *then*

$$\lim_{n\to\infty} \sup_{\mu \in \mathcal{M}\cap\mathcal{P}_2(\mathbb{R}^d)} \int_K |\partial_\mu U_n(\mu,x) - \partial_\mu U(\mu,x)| dx = 0.$$

Remark: Our smooth mollifier does not keep monotonicity.

Weak solution

Definition (Weak solution)

We say that $V \in C^{0,1,0}([0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ is a weak solution if for any $0 \leq t_0 \leq t_1 \leq T$ and any initial condition μ_0, the SPDE on $[t_0, t_1]$

$$\begin{cases} d\mu = [\frac{1+\beta^2}{2}\mathrm{tr}(\partial_{\mathrm{xx}}\mu) - \mathrm{div}(\mu\partial_p H(x, \partial_x V(t,x,\mu_t)))]dt - \beta\partial_x\mu dB_t^0; \\ \mu(t_0) = \mu_0. \end{cases}$$

has a weak solution μ; moreover, for such μ, $u(t,x) = V(t,x,\mu_t)$ is a weak solution to the BSPDE on $[t_0, t_1]$

$$\begin{cases} du = -[\mathrm{tr}(\frac{1+\beta^2}{2}\partial_{\mathrm{xx}}\mathrm{u} + \beta\partial_x\gamma(t,x)) + H(x, \partial_x u(t,x)) + F(x,\mu_t)]dt \\ \qquad + \gamma(t,x)dB_t^0 \\ u(T,x) = G(x, \mu(T)). \end{cases}$$

Well-posedness for weak solutions

Theorem (Mou-Zhang, 2020)

- *Any function is a weak solution of master equation (1) if and only if it is a good solution.*
- *Consequently, the decoupling field V of the FBSDE (4) is also the unique weak solution of the master equation.*

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Nash systems

- The Nash system with common noise

$$\begin{cases} \partial_t v^{N,i} + \frac{1}{2}\mathrm{tr}(\partial_{\vec{x}\vec{x}} v^{N,i}) + \frac{\beta^2}{2}\sum_{j,k}\mathrm{tr}(\partial_{x_j x_k} v^{N,i}) + H(x_i, \partial_{x_i} v^{N,i}) \\ +F(x_i, m_{\vec{x}}^{N,i}) + \sum_{j\neq i}\partial_p H(x_j, \partial_{x_j} v^{N,j})\partial_{x_j} V^{N,i} = 0, \\ V^{N,i}(T, \vec{x}) = G(x_i, m_{\vec{x}}^{N,i}), \end{cases} \tag{5}$$

where

$$m_{\vec{x}}^{N,i} = \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j} \quad \text{for any } \vec{x} = (x_1, \dots, x_N) \in \mathbb{R}^{dN}.$$

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Known results

- Cardaliaguet-Delarue-Lasry-Lions, Carmona-Delarue, Delarue-Lacker-Ramanan, Lacker...

Convergence of solutions to Nash systems

Theorem

For any $i \in \{1, ..., N\}$ *and* $(t, \vec{x}) \in [0, T] \times \mathbb{R}^{dN}$

$$|V(t, x_i, m_{\vec{x}}^{N,i}) - v^{N,i}(t, \vec{x})| \leq \frac{C}{N}(1 + |x_i|^2 + \frac{1}{N}\sum_j |x^j|^2)^{\frac{1}{2}}.$$

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Propagation of chaos

- Let ξ_i are i.i.d. such that $\mathcal{L}_{\xi_i} = \mu$.
- $X_{i,t} = \xi_i + \int_0^t \partial_p H(X_{i,s}, \partial_x V(s, X_{i,s}, \mathcal{L}_{X_{i,s}|B^0})) + B_t^i + \beta B_t^0$.
- $X_{i,t}^N = \xi_i + \int_0^t \partial_p H(X_{i,s}^N, \partial_{x_i} v^{N,i}(s, X_s^N) + B_t^i + \beta B_t^0$.

Theorem

For any $\eta > 0$, there exists a constant $C_\eta > 0$, independent of N, such that for any $i\{1, ..., N\}$

$$\mathbb{E}[\sup_{t\in[0,T]} |X_{i,t} - X_{i,t}^N|] \leq \frac{C_\eta}{N^{1/\max\{d,2+\eta\}}}.$$

Thank you for your attention!