Weak Solutions of Second Order Master Equations for Mean Field Games with Common Noise

Chenchen Mou (UCLA) with Jianfeng Zhang (USC)

Mean Field Games and Applications

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Outline

1. Mean field game with common noise
   - Master equations with common noise
   - Mean field game system with common noise
   - Forward backward McKean-Vlasov SDEs

2. Well-posedness results
   - Well-posedness for FBSDEs
   - Well-posedness for master solutions

3. Convergence results
   - Nash systems
   - Convergence of solutions to Nash systems
   - Propagation of chaos
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Master equations with common noise

- The master equation with common noise:

\[
\begin{aligned}
\partial_t V + \frac{1+\beta^2}{2} \text{tr}(\partial_{xx} V) + H(x, \partial_x V) + F + \mathcal{M} V &= 0, \\
V(T, x, \mu) &= G(x, \mu),
\end{aligned}
\]  

where \( F, G : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) and

\[
\mathcal{M} V(t, x, \mu) :=
\text{tr} \left( \mathbb{E} \left[ \frac{1 + \beta^2}{2} \partial_{x\mu} V(t, x, \mu, \xi) + \partial_\mu V(t, x, \mu, \xi) \partial_\rho H(\xi, \partial_x V(t, \xi, \mu)) \right] 
+ \beta^2 \partial_{x\mu} V(t, x, \mu, \xi) + \frac{\beta^2}{2} \mathbb{E} [\partial_{\mu\mu} V(t, x, \mu, \tilde{\xi}, \xi)] \right).
\]

- Purpose: To find a Nash equilibrium for a mean field game.
Let r.v. $\xi$ be such that $\mathcal{L}_\xi = \mu$ and let $X^{\xi,\alpha}_t$ be

$$X^{\xi,\alpha}_t = \xi + \int_0^t \alpha_s ds + B_t + \beta B^0_t,$$

Let $(Y^{\xi;\alpha',\alpha}, Z^{\xi;\alpha',\alpha}, Z^{0,\xi;\alpha',\alpha})$ solve

$$Y^{\xi;\alpha',\alpha}_t = G(X^{\xi,\alpha'}_T, \mathcal{L}X^{\xi,\alpha}_T|B^0)$$

$$+ \int_t^T [F(X^{\xi,\alpha'}_s, \mathcal{L}X^{\xi,\alpha}_s|B^0)] - L(X^{\xi,\alpha'}_s, \alpha'_s) ds$$

$$- \int_t^T Z^{\xi;\alpha',\alpha}_s dB_s - \int_t^T Z^{0,\xi;\alpha',\alpha}_s dB^0_s.$$
Mean field game

- The cost functional

\[ J(t, \mu; \alpha', \alpha) = \mathbb{E}[Y_{0}^{\xi; \alpha', \alpha}]. \]

- The maximization problem:

\[ V(t, \mu; \alpha) = \sup_{\alpha'} J(t, \mu; \alpha', \alpha) \]

**Definition (Nash equilibrium)**

We say that \((\alpha^*, \mu^*)\) is a Nash equilibrium for the above mean field game problem if

\[ V(t, \mu; \alpha^*) = J(t, \mu; \alpha^*, \alpha^*) \quad \text{and} \quad \mu_t^* = \mathcal{L}_{X_t^{\xi; \alpha^*}} |_{B^0}. \]
By comparison principle for BSDEs and the definition of the Nash equilibrium, we have

\[ \begin{align*}
X_t^{\alpha^*} &= \xi + B_t^{\alpha^*} + \beta B_t^0, \\
Y_t^{\alpha^*} &= G(X_T^{\alpha^*}, \mathcal{L}X_T^{\alpha^*} | B^0), \\
&\quad + \int_t^T F(X_s^{\alpha^*}, \mathcal{L}X_s^{\alpha^*} | B^0) + H(X_s^{\alpha^*}, Z_s^{\alpha^*}) \, ds \\
&\quad - \int_t^T Z_s^{\alpha^*} \, dB_s^{\alpha^*} - \int_t^T Z_s^{0, \alpha^*} \, dB_s^0.
\end{align*} \] (2)

where \( \alpha_t^* = \partial_{\alpha} H(X_t^{\alpha^*}, Z_t^{\alpha^*}) \) and \( dB_t^{\alpha^*} = \alpha_t^* \, dt + dB_t. \)
Derivation of the master equation

(2) is equivalent to the following FBSDE system

\[
X^\xi_t = \xi + \int_0^t \partial_p H(X^\xi_s, Z^\xi_s) \, ds + B_t + \beta B^0_t,
\]

\[
Y^\xi_t = G(X^\xi_T, \mu_T),
\]

\[
+ \int_t^T F(X^\xi_s, \mu_s) - L(X^\xi_s, \partial_p H(X^\xi_s, Z^\xi_s)) \, ds
\]

\[
- \int_t^T Z^\xi_s \, dB_s - \int_t^T Z^0,^\xi_s \, dB^0_s.
\]

where \( \mu_t = \mathcal{L}_{X^\xi_t | B^0} \).

Define \( Y^\xi_t = V(t, X^\xi_t, \mu_t) \) and it can be shown that \( V \) satisfies the master equation (1) in \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\) with \( V(T, x, \mu) = G(x, \mu) \).
Mean field game system with common noise

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Mean field game system

- Define $u(t, x) := V(t, x, \mu_t)$ and it can be shown that $(\mu_t, u_t)$ solves the following SPDE system

$$
\begin{cases}
    d\mu = \left[ \frac{1+\beta^2}{2} \text{tr}(\partial_{xx}\mu) - \text{div}(\mu \partial_p H(x, \partial_x u(t, x))) \right] dt - \beta \partial_x \mu dB^0_t; \\
    du = -\left[ \text{tr}\left( \frac{1+\beta^2}{2} \partial_{xx} u + \beta \partial_x \gamma(t, x) \right) + H(x, \partial_x u(t, x)) + F(x, \mu_t) \right] dt \\
    + \gamma(t, x) dB^0_t \\
    u(T, x) = G(x, \mu(T)), \quad \mu(0) = \mu.
\end{cases}
$$

(3)

Remark:
- The vector function $\gamma$ is part of the solution of the stochastic HJ equation.
- If the decoupling field $V$ is smooth, it can be shown that

$$
\gamma(t, x) = \beta \int_{\mathbb{R}^d} \partial_{\mu} V(t, x, \mu_t, y) \mu_t(dy).
$$
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The mean field game equation (3) is equivalent to the following forward backward McKean-Vlasov SDEs

\[ X_t^\xi = \xi + \int_0^t \partial_p H(X_s^\xi, Z_s^\xi) ds + B_t + \beta B_0^t, \]

\[ Y_t^\xi = G(X_T^\xi, \mu_T) + \int_t^T F(X_s^\xi, \mu_s) - L(X_s^\xi, \partial_p H(X_s^\xi, Z_s^\xi)) ds \]

\[ - \int_t^T Z_s^\xi dB_s - \int_t^T Z_s^0,\xi dB_0^s; \]

\[ X_t^{x,\xi} = x + \int_0^t \partial_p H(X_s^{x,\xi}, Z_s^{x,\xi}) ds + B_t + \beta B_0^t, \]

\[ Y_t^{x,\xi} = G(X_T^{x,\xi}, \mu_T) + \int_t^T F(X_s^{x,\xi}, \mu_s) - L(X_s^{x,\xi}, \partial_p H(X_s^{x,\xi}, Z_s^{x,\xi})) ds \]

\[ - \int_t^T Z_s^{x,\xi} dB_s - \int_t^T Z_s^{0,x,\xi} dB_0^s; \]
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Known results

- Potential Mean Field Game:
  1. Degenerate: Bensoussan-Yam, Gangbo-Meszaros, Gangbo-Swiech, Mayorga,…
  2. Individual noise: Bensoussan-Graber-Yam, Pham-Wei, Wu-Zhang,…
  3. Common noise: Gangbo-Mayorga-Swiech,…

- General Mean Field Game:
  1. Degenerate: ???
  2. Individual noise: Chassagneux-Crisan-Delarue…
  3. Common noise: Cardaliaguet-Delarue-Lasry-Lions, Carmona-Delarue…
Well-posedness for FBSDEs

**Known results**

- Assume that $F, G$ are smooth and satisfy some convexity assumption, the master equation (1) admits a unique classical solution for arbitrary long time $T$.
- Assume that $F, G$ are smooth, the master equation (1) admits a unique classical solution for short time $T$. 
An important open problem

- Can we define a notion of "weak" solution to the master equation and show its well-posedness if the above assumptions are not satisfied?
Well-posedness for FBSDEs

Assumptions

- $H, \partial_x H, \partial_p H$ are Lipschitz in each $D_R$ and $\exists \epsilon_R > 0$

  $$\partial_{pp} H \geq \epsilon_R I_d \text{ in } D_R = \{(x, p) \in \mathbb{R}^{2d} : |p| \leq R\}.$$  

- $F, G, \partial_x F, \partial_x G$ are Lipschitz in $x$ and $\mu$ (under $\mathcal{W}_1$)

- $F, G$ are monotone.
Well-posedness for FBSDEs

Monotonicity

**Definition (Monotonicity)**

We say that $F : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is monotone if

$$\forall \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$$

$$\int_{\mathbb{R}^d} (F(x, \mu_1) - F(x, \mu_2)) (\mu_1(x) - \mu_2(x)) \, dx \geq 0.$$
Theorem (Mou-Zhang, 2020)

*The forward backward McKean-Vlasov SDEs (4) is well posed.*

**Remark:**

- If $F, G, H$ are smooth, the representation formulas for $\partial_\mu V, \partial_{x\mu} V, \partial_{\tilde{x}\mu} V$ and $\partial_{\mu\mu} V$ are given and $V$ solves the master equation classically.

- The monotonicity is needed for keeping the Lipschitz constant of $V$ in $\mu$. 
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Definition (good solution)

We say that $V$ is a good solution if $\forall t_0 \in [0, T)$ and smooth $V_n$ such that $\mathcal{L}V_n(t, x, \rho) \to 0$ in $L^\infty([t_0 - \delta, t_0] \times \mathbb{R}^d \times \mathcal{P}_2^M(\mathbb{R}^d))$ and $V_n(t_0, \cdot, \cdot) \to V(t_0, \cdot, \cdot)$ in $L^\infty(\mathbb{R}^d \times \mathcal{P}_2^M(\mathbb{R}^d))$ for some $\delta$ and any $M > 0$. Then $V_n \to V$ in $L^\infty([t_0 - \delta', t_0] \times \mathbb{R}^d \times \mathcal{P}_2^M(\mathbb{R}^d))$ for some $\delta'$. 
Well-posedness for master solutions

Well-posedness for good solutions

**Theorem (Mou-Zhang, 2020)**

The decoupling field $V$ of the FBSDE (4) is the unique good solution to the master equation (1) in $[0, T]$ for any $T > 0$.

**Remark:**

- The monotonicity assumption is only needed to construct the decoupling field.
A main ingredient

We construct smooth mollifiers for functions on Wasserstein space.

**Theorem**

Let $U \in C^0(\mathcal{P}_1(\mathbb{R}^d))$. Then $\exists U_n \in C^\infty(\mathcal{P}_2) \cap C^0(\mathcal{P}_1(\mathbb{R}^d))$.

- $\lim_{n \to \infty} \|U_n - U\|_{L^\infty(\mathcal{M})} = 0$ for any $\mathcal{M} \subset \subset \mathcal{P}_1(\mathbb{R}^d)$.

- If $U \in \text{Lip}(\mathcal{P}_1(\mathbb{R}^d))$ with Lipschitz constant $L$, then $U_n \in \text{Lip}(\mathcal{P}_1(\mathbb{R}^d))$ with Lipschitz constant $CL$, where $C$ is independent of $n$.

- If $U \in C^1(\mathcal{P}_2)$, and $\partial_\mu U$ is uniformly continuous in $(\mathcal{M} \cap \mathcal{P}_2(\mathbb{R}^d)) \times K$ under $\mathcal{W}_1$, where $\mathcal{M} \subset \mathcal{P}_1(\mathbb{R}^d)$ and $K \subset \subset \mathbb{R}^d$, then

$$\lim_{n \to \infty} \sup_{\mu \in \mathcal{M} \cap \mathcal{P}_2(\mathbb{R}^d)} \int_K |\partial_\mu U_n(\mu, x) - \partial_\mu U(\mu, x)| \, dx = 0.$$  

**Remark:** Our smooth mollifier does not keep monotonicity.
Definition (Weak solution)

We say that $V \in C^{0,1,0}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ is a weak solution if for any $0 \leq t_0 \leq t_1 \leq T$ and any initial condition $\mu_0$, the SPDE on $[t_0, t_1]$

\[
\begin{cases}
    d\mu = \left[ \frac{1+\beta^2}{2} \text{tr}(\partial_{xx} \mu) - \text{div}(\mu \partial_p H(x, \partial_x V(t, x, \mu_t))) \right] dt - \beta \partial_x \mu dB_t^0; \\
    \mu(t_0) = \mu_0.
\end{cases}
\]

has a weak solution $\mu$; moreover, for such $\mu$, $u(t, x) = V(t, x, \mu_t)$ is a weak solution to the BSPDE on $[t_0, t_1]$

\[
\begin{cases}
    du = -\left[ \text{tr} \left( \frac{1+\beta^2}{2} \partial_{xx} u \right) + \beta \partial_x \gamma(t, x) + H(x, \partial_x u(t, x)) + F(x, \mu_t) \right] dt \\
    + \gamma(t, x) dB_t^0 \\
    u(T, x) = G(x, \mu(T)).
\end{cases}
\]
Well-posedness for weak solutions

Theorem (Mou-Zhang, 2020)

- Any function is a weak solution of master equation (1) if and only if it is a good solution.
- Consequently, the decoupling field $V$ of the FBSDE (4) is also the unique weak solution of the master equation.
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Nash systems

The Nash system with common noise

\[
\begin{align*}
\partial_t v^{N,i} &+ \frac{1}{2} \text{tr}(\partial_{\vec{x}\vec{x}} v^{N,i}) + \frac{\beta^2}{2} \sum_{j,k} \text{tr}(\partial_{x_j x_k} v^{N,i}) + H(x_i, \partial x_i v^{N,i}) \\
&+ F(x_i, m^{N,i}_{\vec{x}}) + \sum_{j \neq i} \partial_p H(x_j, \partial x_j v^{N,j}) \partial x_j V^{N,i} = 0, \\
V^{N,i}(T, \vec{x}) &= G(x_i, m^{N,i}_{\vec{x}}),
\end{align*}
\]

where

\[
m^{N,i}_{\vec{x}} = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j} \quad \text{for any } \vec{x} = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}.
\]
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Known results

- Cardaliaguet-Delarue-Lasry-Lions, Carmona-Delarue, Delarue-Lacker-Ramanan, Lacker...
For any $i \in \{1, ..., N\}$ and $(t, \vec{x}) \in [0, T] \times \mathbb{R}^{dN}$

$$|V(t, x_i, m^N_{\vec{x},i}) - \nu^N_i(t, \vec{x})| \leq \frac{C}{N}(1 + |x_i|^2 + \frac{1}{N} \sum_j |x^j|^2)^{\frac{1}{2}}.$$

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Let $\xi_i$ are i.i.d. such that $L\xi_i = \mu$.

$X_{i,t} = \xi_i + \int_0^t \partial_p H(X_{i,s}, \partial_x V(s, X_{i,s}, LX_{i,s}|B^0)) + B^i_t + \beta B^0_t$. 

$X^{N}_{i,t} = \xi_i + \int_0^t \partial_p H(X^{N}_{i,s}, \partial_x v^{N,i}(s, X^N_s)) + B^i_t + \beta B^0_t$. 

**Theorem**

*For any $\eta > 0$, there exists a constant $C_\eta > 0$, independent of $N$, such that for any $i\{1, \ldots, N\}$*

$$\mathbb{E}[\sup_{t \in [0,T]} |X_{i,t} - X^{N}_{i,t}|] \leq \frac{C_\eta}{N^{1/\max\{d,2+\eta\}}}.$$
Thank you for your attention!