Primal-dual methods for the mean-field game (MFG) and control (MFC) problems via monotone inclusions

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The problem

We are interested in developing computational methods for

$$\begin{cases} -\phi_t + H(t,x,\nabla\phi,\nabla^2\phi) = f\left(x,\rho(x,t),\int_\Omega K(x,y)\rho(y,t)dy\right) \\ \rho_t - \sum_i \partial_{x_i}(\rho\nabla_{p_i}H) + \sum_{ij}\partial_{x_ix_j}(\rho\partial_{M_{ij}}H) = 0 \\ \rho(x,0) = \rho_0(x),\ \ \phi(x,1) = g\left(x,\rho(x,1),\int_\Omega S(x,y)\rho(y,1)dy\right), \end{cases}$$

- The source term, and the boundary condition of HJB model the interactions between agents.
- The nonlocal interaction terms

$$\int_\Omega K(x,y)\rho(y,t)dy, \quad \int_\Omega S(x,y)\rho(y,1)dy$$

make the problem challenging from computational perspective. Indeed, non-singular K, S yield dense systems on a discrete level.

Existing numerical methods

There are number of general-purpose numerical methods that handle the system above.

- Newton's method [ACD10, Ach13, ACCD13]
- Semi-Lagrangian methods [CS12, CS14, CS15]
- ADMM (Brenier-Benamou) [BC15, BCS17] for *potential* MFG
- PDHG [BnAKS18, BnAKK$^+$19] for *potential* MFG
- HJB in density-space [CLOY19] for *potential* MFG
- Monotone flows [AFG17]

However, these methods yield dense systems on the discrete level when the interactions are nonlocal. Thus, the algorithms become computationally expensive and not amenable to parallelization techniques.

Goal

We aim at developing computational framework that

- yields sparse systems by encoding interactions in a small number of *coefficients*
- yields computational cost that is on par with algorithms for local couplings
- suits well the Lagrangian framework
- is compatible with existing convex optimization techniques and numerical methods when interactions are of mixed type
- extends to the non-potential setting
- provides modeling framework for nonlocal problems

The references for our method are [Nur18, NS18, LJL$^+$20].

The method of coefficients

For concreteness, consider

$$\begin{cases} -\phi_t + H(x, \nabla\phi) = \int_\Omega K(x,y)\rho(y,t)dy \\ \rho_t - \nabla\cdot(\rho\nabla_p H(x,\nabla\phi)) = 0 \\ \rho(x,0) = \rho_0(x),\ \phi(x,1) = g(x) \end{cases}$$

Suppose that $K(x,y) = \sum_{i,j=1}^r k_{ij} f_i(x) f_j(y)$, where $\{f_i\}_{i=1}^r \subset C^2(\Omega)$ is some family of functions. Denote by $\mathbf{K} = (k_{ij})_{i,j=1}^r$. We can assume that $\mathbf{K}$ is invertible.

Theorem

[NS18, Theorem 3.1] If $\mathbf{K}$ is monotone, then the problem is equivalent to finding a zero of a monotone operator $a \mapsto \mathbf{K}^{-1}a - \frac{\delta}{\delta a}\int_\Omega \phi_a(x,0)\rho_0(x)dx$.

Dualize nonlocal variables in a Fourier space

- In the potential case, $\mathbf{K}^\top = \mathbf{K}$, our formulation is

$$\begin{aligned} &\inf_{\alpha} \int_0^1 \mathcal{F}^*(\alpha(\cdot,t))dt - \int_\Omega \phi(x,0)\rho_0(x)dx \\ &\text{s.t.} \ -\phi_t + H(x,\nabla\phi) = \alpha, \quad \phi(x,1) = g(x), \end{aligned}$$

  from [LL07], where $\mathcal{F}(\rho) = \frac{1}{2}\int_\Omega K(x,y)\rho(x)\rho(y)dxdy$.
- Indeed, if $\alpha(x,t) = \sum_{i=1}^r a_i(t) f_i(x)$ then

$$\mathcal{F}^*(\alpha) = \frac{\langle \mathbf{K}^{-1}a, a\rangle}{2}$$

Non-potential case

When $K(x, y)$ is non-symmetric we need to solve the monotone inclusion

$$0 \in Ta - \partial_a \int_\Omega \phi_a(x, 0)\rho_0(x)dx$$

where $Ta = \mathbf{K}^{-1}a$. We apply splitting methods that are extensions of PDHG [Vu13].

Monotone splitting

Consider the following pair of dual monotone inclusions

$$\text{find } s \text{ s.t. } 0 \in Ms + C^*(N(Cs)) \quad \text{(P)}$$
$$\text{find } q \text{ s.t. } q \in N(Cs), \; -C^*q \in Ms, \text{ for some } s \quad \text{(D)}$$

where M, N are maximal monotone, and $C \neq 0$ is linear. When $M = \partial f$, $N = \partial g$ we obtain

$$\inf_s f(s) + g(Cs) = \inf_s \sup_q f(s) + \langle Cs, q \rangle - g^*(q),$$

Monotone splitting

One can solve

$$\begin{aligned} 0 \in Ms + C^*(N(Cs)) \quad &(\mathrm{P}) \\ q \in N(Cs), \; -C^* q \in Ms \quad &(\mathrm{D}) \end{aligned}$$

via

$$\begin{cases} s^{n+1} = J_{\tau_s M}(s^n - \tau_s C^* q^n) \\ \tilde{s}^{n+1} = 2s^{n+1} - s^n \\ q^{n+1} = J_{\tau_q N^{-1}}(q^n + \tau_q C \tilde{s}^{n+1}), \end{cases}$$

where $J_{\tau F} = (I + \tau F)^{-1}$ is the resolvent operator, and $\tau_s, \tau_q > 0$ are such that $\tau_s \tau_q \|C\|^2 < 1$.
When $M = \partial f$, $N = \partial g$ this algorithm reduces to the standard PDHG of Chambolle-Pock [CP11, CP16].

Monotone splitting for MFG

We have that

$$\int_\Omega \phi_a(x,0)\rho_0(x)dx = \inf_{\rho,m} \int_0^1 \int_\Omega \left(L\left(x, \frac{m}{\rho}\right) + \sum_i a_i(t) f_i(x) \right) \rho dx dt + \int_\Omega g(x)\rho(x,1)dx$$

$$\text{s.t. } \rho_t + \nabla \cdot m = 0, \quad \rho(x,0) = \rho_0(x)$$

Therefore, $0 \in Ta - \partial_a \int_\Omega \phi_a(x,0)\rho_0(x)dx$ can be written as

$$0 \in Tz - \partial_z v, \quad v(z) = \inf_{A_1 q = r} u(q) + \langle A_2 q, z \rangle$$

where A_1, A_2 are linear operators, and u is a convex function. So $z \sim a$, and $q \sim (\rho, m)$.

Monotone splitting

Lemma

We have that (z, q) is a solution of

$$0 \in Tz - \partial_z v, \quad v(z) = \inf_{A_1 q = r} u(q) + \langle A_2 q, z \rangle$$

iff (z, λ, q) is a solution of

$$\begin{aligned} 0 \in M(z,\lambda) + C^*(N(C(z,\lambda))) \quad & (P) \\ q \in N(C(z,\lambda)), \ -C^* q \in M(z,\lambda) \quad & (D) \end{aligned}$$

where $M(z,\lambda) = (Tz, r)$, $C(z,\lambda) = -A_2^* z - A_1^* \lambda$, $N(q) = \partial u^*(q)$.

Monotone splitting

Applying the previous algorithm, we can solve

$$0 \in Tz - \partial_z v, \quad v(z) = \inf_{A_1 q = r} u(q) + \langle A_2 q, z \rangle$$

via

$$\begin{cases} z^{n+1} = (I + \tau T)^{-1} (z^n + \tau A_2 q^n) \\ \lambda^{n+1} \in \arg\min_{\lambda} -\langle A_1 q^n - r, \lambda \rangle + \frac{|\lambda - \lambda^n|^2}{2\tau} \\ q^{n+1} \in \arg\min_{q} u(q) + \langle A_2 q, \tilde{z}^{n+1} \rangle + \langle A_1 q - r, \tilde{\lambda}^{n+1} \rangle + \frac{|q - q^n|^2}{2\sigma}, \end{cases}$$

where

$$\tau\sigma \sim \frac{1}{\|A_1\|^2 + \|A_2\|^2}.$$

Preconditioned monotone splitting

The preconditioned version of the previous algorithm is

$$\begin{cases} z^{n+1} = (I+\tau T)^{-1}\left(z^n + \tau A_2 q^n\right) \\ \lambda^{n+1} \in \arg\min\limits_{\lambda} -\langle A_1 q^n - r, \lambda\rangle + \frac{|A_1^*\lambda - A_1^*\lambda^n|^2}{2\tau} \\ q^{n+1} \in \arg\min\limits_{q} u(q) + \langle A_2 q, \tilde{z}^{n+1}\rangle + \langle A_1 q - r, \tilde{\lambda}^{n+1}\rangle + \frac{|q-q^n|^2}{2\sigma}, \end{cases}$$

where

$$\tau\sigma \sim \frac{1}{1+\|A_2\|^2}.$$

Mixed couplings

- Assume that we want to solve

$$\begin{cases} -\phi_t + H(x, \nabla\phi) = f(\rho(x,t)) + \int_\Omega K(x,y)\rho(y,t)dy \\ \rho_t - \nabla \cdot (\rho \nabla_p H(x, \nabla\phi)) = 0 \\ \rho(x,0) = \rho_0(x),\ \phi(x,1) = g(x), \end{cases}$$

where f is some monotone function.

- Then, we introduce parameters $\{a_i(t)\}$ to handle the nonlocal case, and $\alpha(x,t)$ for the local part

$$\begin{cases} -\phi_t + H(x, \nabla\phi) = \alpha(x,t) + \sum_{i=1}^r a_i(t) f_i(x) \\ \phi(x,1) = g(x), \end{cases}$$

Mixed couplings

- Again, we denote by $\phi_{\alpha,a}$ the solution of

$$\begin{cases} -\phi_t + H(x, \nabla\phi) = \alpha(x,t) + \sum_{i=1}^r a_i(t) f_i(x) \\ \phi(x,1) = g(x), \end{cases}$$

  and by $\rho_{\alpha,a}$ the solution of

$$\begin{cases} \rho_t - \nabla \cdot (\rho \nabla_p H(x, \nabla\phi_{\alpha,a})) = 0 \\ \rho(x,0) = \rho_0(x), \end{cases}$$

- The derivative formulas for $J_{\alpha,a} = \int_\Omega \phi_{\alpha,a}(x,0)\rho_0(x)dx$ in this case become

$$\partial_\alpha J = \rho_{\alpha,a}, \quad \partial_{a_i} J = \int_\Omega f_i(x)\rho_{\alpha,a}(x,\cdot)dx$$

Mixed couplings

- Then MFG can be written as

$$\begin{cases} a = \mathbf{K}\partial_a J \\ \alpha = f(\partial_\alpha J) \end{cases} \Leftrightarrow \begin{cases} \mathbf{K}^{-1}a - \partial_a J = 0 \\ (F^*)'(\alpha) - \partial_\alpha J = 0, \end{cases}$$

  where F^* is the convex dual of $F'(\rho) = f(\rho)$. Therefore, we again obtain monotone inclusions and can apply primal-dual algorithms.

- This means that we can handle convex point-wise constraints on ρ mixed with nonlocal couplings.
- For instance, $0 \leq \rho \leq 1$ corresponds to

$$F(\rho) = \mathbf{1}_{0\leq\rho\leq 1}, \quad F^*(\alpha) = \max\{\alpha, 0\}.$$

  Proximal update steps for α are explicit in this case.

Mixed couplings

Assume that

$$\begin{aligned} & f(x,\rho) \\ = & f_1(x,\rho(x)) + f_2(x,\rho(x)) + \cdots + f_m(x,\rho(x)) \\ & + \int K_1(x,y)\rho(y)dy + \int K_2(x,y)\rho(y)dy + \cdots + \int K_n(x,y)\rho(y)dy \end{aligned}$$

Then we split the input of all couplings by introducing dual variables

$$\begin{aligned} \alpha^i(x,t) \quad & \text{for} \quad f_i(x,\rho(x,t)),\ 1 \leq i \leq m \\ a^i(t) \quad & \text{for} \quad \int K_i(x,y)\rho(y,t)dt,\ 1 \leq i \leq n \end{aligned}$$

Indeed, the proximal/resolvent updates for dual variables are decoupled. Moreover, basis functions for K_i-s can be taken distinct.

Thank you for your attention!

Yves Achdou, Fabio Camilli, and Italo Capuzzo-Dolcetta.
Mean field games: convergence of a finite difference method.
SIAM J. Numer. Anal., 51(5):2585–2612, 2013.

Yves Achdou and Italo Capuzzo-Dolcetta.
Mean field games: numerical methods.
SIAM J. Numer. Anal., 48(3):1136–1162, 2010.

Yves Achdou.
Finite difference methods for mean field games.
In *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, volume 2074 of *Lecture Notes in Math.*, pages 1–47. Springer, Heidelberg, 2013.

Noha Almulla, Rita Ferreira, and Diogo Gomes.
Two numerical approaches to stationary mean-field games.
Dyn. Games Appl., 7(4):657–682, 2017.

J.-D. Benamou and G. Carlier.

Augmented Lagrangian methods for transport optimization, mean field games and degenerate elliptic equations.
J. Optim. Theory Appl., 167(1):1–26, 2015.

J.-D. Benamou, G. Carlier, and F. Santambrogio.
Variational mean field games.
In *Active particles. Vol. 1. Advances in theory, models, and applications*, Model. Simul. Sci. Eng. Technol., pages 141–171. Birkhäuser/Springer, Cham, 2017.

L. Briceño Arias, D. Kalise, Z. Kobeissi, M. Laurière, Á. Mateos González, and F. J. Silva.
On the implementation of a primal-dual algorithm for second order time-dependent mean field games with local couplings.
In *CEMRACS 2017—numerical methods for stochastic models: control, uncertainty quantification, mean-field*, volume 65 of *ESAIM Proc. Surveys*, pages 330–348. EDP Sci., Les Ulis, 2019.

L. M. Briceño Arias, D. Kalise, and F. J. Silva.

Proximal methods for stationary mean field games with local couplings.
SIAM J. Control Optim., 56(2):801–836, 2018.

Yat Tin Chow, Wuchen Li, Stanley Osher, and Wotao Yin.
Algorithm for Hamilton–Jacobi equations in density space via a generalized Hopf formula.
J Sci Comput, 80(2):1195–1239, Aug 2019.

Antonin Chambolle and Thomas Pock.
A first-order primal-dual algorithm for convex problems with applications to imaging.
J. Math. Imaging Vision, 40(1):120–145, 2011.

Antonin Chambolle and Thomas Pock.
On the ergodic convergence rates of a first-order primal-dual algorithm.
Math. Program., 159(1-2, Ser. A):253–287, 2016.

Fabio Camilli and Francisco Silva.

A semi-discrete approximation for a first order mean field game problem.
Netw. Heterog. Media, 7(2):263–277, 2012.

E. Carlini and F. J. Silva.
A fully discrete semi-Lagrangian scheme for a first order mean field game problem.
SIAM J. Numer. Anal., 52(1):45–67, 2014.

Elisabetta Carlini and Francisco J. Silva.
A semi-Lagrangian scheme for a degenerate second order mean field game system.
Discrete Contin. Dyn. Syst., 35(9):4269–4292, 2015.

Siting Liu, Matthew Jacobs, Wuchen Li, Levon Nurbekyan, and Stanley J. Osher.
Computational methods for nonlocal mean field games with applications, 2020.

Jean-Michel Lasry and Pierre-Louis Lions.
Mean field games.

Jpn. J. Math., 2(1):229–260, 2007.

Levon Nurbekyan and J. Saúde.
Fourier approximation methods for first-order nonlocal mean-field games.
Port. Math., 75(3-4):367–396, 2018.

Levon Nurbekyan.
One-dimensional, non-local, first-order stationary mean-field games with congestion: a Fourier approach.
Discrete Contin. Dyn. Syst. Ser. S, 11(5):963–990, 2018.

Bang Công Vũ.
A variable metric extension of the forward-backward-forward algorithm for monotone operators.
Numer. Funct. Anal. Optim., 34(9):1050–1065, 2013.