IPAM Culminating Workshop

Hamilton Jacobi Equations

Dima Shlyakhtenko, UCLA

Large-N limit of random matrix models

IPAM Culminating Workshop

Hamilton Jacobi Equations

Dima Shlyakhtenko, UCLA

Large-N limit of random matrix models

Disclaimer: no HJ, barely any PDE in this talk.

RMT $A_1, \ldots, A_d$ random $N \times N$ self-adjoint matrices

Observables: unitarily invariant functions of $A_1, \ldots, A_d$

$A_1, \ldots, A_d \mapsto UA_1U^*, \ldots, UA_dU^*$

These are trace polynomials

$$\mathbb{E}\left(\tau_N(p_1(A_1, \ldots, A_d)) \, \tau_N(p_2(A_1, \ldots, A_d)) \cdots \tau_N(p_k(A_1, \ldots, A_d)) \right)$$

$A_1A_3 + A_3A_2 + A_2A_3 + A_1A_5A_1$

$$\tau_N = \frac{1}{N} \mathrm{Tr}$$

τ_N is trace normalized so $\tau_N(I) = 1$.

$$p(t_1, t_2, t_3) = t_3t_1 + t_1t_3 + t_2^2$$

$\underline{d=1}$ $\quad A_1 / \text{unitary conjugation} = (\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N)$

eigenvalues

$$= \text{random measure } \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j} =: \nu_N$$

$$\tau_N(p_1(A))\,\tau_N(p_2(A)) \dots \tau_N(p_k(A))$$

$$= \prod \int p_j(t)\, d\nu_N(t)$$

RMT

Mainly $d=1$

local eigenvalue statistics

- eigenvalue spacing
- min/max eigenvalue
- Tracy-Widom law
- ...

Especially $d>1$

eigenvalue distribution [limits of $\frac{1}{N}\sum \delta_{x_j}$]

- limit laws
- semicircle, Marchenko-Pastur
- free probability

Examples of RM Models of interest to us

$$(A_1, \dots, A_d) \sim d\mu_N = \frac{1}{Z_N} e^{-N^2 \tau_N (V(A_1, \dots, A_d))} \underbrace{dA_1 \dots dA_d}_{\text{Lebesgue}}$$

normalizing const.

$V(A_1, \dots, A_d)$ e.g. non-commutative polynomial [or limit of]

$$V(A_1, \dots, A_d) = \frac{1}{2} \sum A_j^2 \qquad \mu_N = \text{Gaussian measure}$$

$$V(A_1, \dots, A_d) = \left[\frac{1}{2} \sum A_j^2 + \varepsilon W\right] \times \begin{cases} +\infty & \text{if } \max \|A_j\| < R \\ 1 & \text{if } \max \|A_j\| \geq R \end{cases}$$

$V(A_1, \dots, A_d)$ "convex"

Under these assumptions

$$d\mu_N = \frac{1}{Z_N} e^{-N^2 \tau_N(V(A_1 - A_d))} dA_1 \dots dA_d$$

<u>concentrates</u>

$\hookrightarrow \mathbb{P}(\,(f - \mathbb{E}(f)) > \varepsilon) \sim e^{-N^2}$

if $\|f\|_{Lip} \leq 1$ $\qquad f = \tau_N(A_1^2 A_3 + A_3 A_1)$

$\hookrightarrow$ Trace-polynomials simplify in limit:

$$\lim_{N\to\infty} \mathbb{E}\left(\tau_N(P_1(A_1,\dots,A_d)) \dots \tau_N(P_k(A_1 \dots A_d))\right) = \prod_j \lim_{N\to\infty} \mathbb{E}\left(\tau_N(P_j(A_1 \dots A_d))\right)$$

Thus the $N=\infty$ limit is described:

- $d=1$: deterministic measure

$$\nu = \text{a.sure} \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j}$$

- $d>1$: "non-commutative law"

$$\tau : \mathbb{C}\langle t_1, \dots, t_d \rangle \longrightarrow \mathbb{C}$$

$$\tau(p) = \lim_{N\to\infty} \cancel{\mathbb{E}}\left(\tau_N\left(p\left(A_1, \dots, A_d\right)\right)\right)$$

(ordinary measures: $C_c(\mathbb{R})^*$)

τ is an algebraic object not suitable for analysis

Replace $\mathbb{C}\langle t_1, .., t_d \rangle$ by a larger algebra

$[\ \mathbb{C}\langle t \rangle \rightsquigarrow L^\infty(\mu)\ ]$ $\qquad (t_{i_1} \cdots t_{i_k})^* = t_{i_k} \cdots t_{i_1}$

$$L^2(\tau) = \overline{\mathbb{C}\langle t_1, \dots, t_d \rangle}^{\|\cdot\|_2} \qquad \|x\|_2 = \tau(x^* x)^{1/2}$$

$\hat{A}_j$ = extension of left mult by t_j on $L^2(\tau)$

Get a <u>Von Neumann algebra</u> of operators

on a Hilbert space

($d=1$: $L^\infty(\mu)$ acting on $L^2(\mu)$)

With this it is possible to do analysis at $N=\infty$
$\longrightarrow$ free probability.

Some examples:

1° Notion of independence. (Voiculescu)

(M,τ) von Neumann algebra, $1 \in M_j \subseteq M$ subalgebra

M_j freely independent if $\quad x_1 \in M_1 \; x_2 \in M_3 \; x_3 \in M_1$

$\tau(x_1 \dots x_k) = 0$ if $\tau(x_j) = 0$, $x_j \in M_{i(j)}$, $i(1) \neq i(2)$ $\vdots$ $i(k-1) \neq i(k)$

[analog of $\mathbb{E}(x_1 x_2) = \mathbb{E}(x_1)\,\mathbb{E}(x_2)$ for $x_1 \perp\!\!\!\perp x_2$]

2° Free Probability versions of SDEs

$$\left(\left[dW^t_{ij,1} \right]^N_{i,j=1}, \ldots, \left[dW^t_{ij,d} \right]^N_{i,j=1} \right) \xrightarrow[N\to\infty]{} (dS^t_1, \ldots, dS^t_d)$$

[Biane-Speicher]
[Guionnet et al]

Itô formula [Biane-Speicher]

$$\tau(p(A_1 + \sqrt{t}\,\underbrace{S_1}_{\int_0^t dS_1^t}, \ldots, A_d + \sqrt{t}\,S_d)) = \tau(p(A_1, \ldots, A_d))$$

$$+ \; t \sum_j \tau \otimes \tau \otimes \tau (\partial_j^2 p)$$

$$\partial_j : \mathbb{C}\langle t_1, \ldots, t_d \rangle \to \mathbb{C}\langle t_1, \ldots, t_d \rangle \otimes \mathbb{C}\langle t_1, \ldots, t_d \rangle$$

$$\partial_j(ab) = a\,\partial_j(b) + \partial_j(a)\,b\,; \quad \partial_j t_k = \delta_{j=k}\, 1 \otimes 1$$

∂_j is a "replacement" for $\partial/\partial x_j$

$d=1$: $\partial(p) = \dfrac{p(x)-p(y)}{x-y}$ if identify

$\mathbb{C}\langle t\rangle \otimes \mathbb{C}\langle t\rangle$ with $\mathbb{C}[x,y]$

Fact: limit law of RM model under convexity assumptions

[Guionnet et al]

$$(A_1, \dots, A_d) \sim \frac{1}{Z_N} e^{-N^2 \tau_N(V(A_1, \dots, A_d))}$$

is the unique stationary solution to

$$dX_j^{(t)} = dS_j^t - D_j V(X_1^{(t)}, \dots, X_d^{(t)})\, dt$$

cyclic partial derivative

3° Free version of Fokker–Planck equation

$$dY_j^{(t)} = dS_j^t - \mathcal{D}_j V(Y_1^t, \dots, Y_d^t)\, dt$$

$$\hookrightarrow \quad \partial_t \tau(f(Y_t)) = \tau(\mathbb{L} f)$$

$$\mathbb{L} f = \frac{1}{2} \sum_i \left((1\otimes\tau)\partial_i \circ (1\otimes\tau)\partial_i\right) f - \sum_i \mathcal{D}_i f\, \mathcal{D}_i V$$

$\underline{d=1}$ $$\mathbb{L} f = \frac{1}{2} D^2 f - f' V'$$

$$D f(s) = \int \frac{f(s) - f(t)}{s - t}\, d\mu(t)$$

4° Free independence in limit:

$$V(t_1,\dots,t_d) = V_1(t_1,\dots,t_k) + V_2(t_{k+1},\dots,t_d)$$

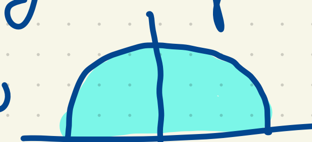
$\Rightarrow$ in the limit

$$A_1,\dots,A_k \quad \& \quad A_{k+1},\dots,A_d$$

are freely independent.

5° $V(t_1,\dots,t_d) = \frac{1}{2}\sum t_j^2$

$\Rightarrow$ in the limit $S_1,\dots,S_d$ are freely indep

& each $S_j \xrightarrow[\text{law}]{}$ semicircular law 

Summary so far:

- Free probability: $N \to \infty$ limit of certain matrix observables
  $\left(\tau_N \left(P_1 (A_1, \dots, A_d) \right) \right)$ [Voiculescu, Biane-Speicher, Guionnet et al]

- Some control of analysis at $N = \infty$

- MFG-like question: how well does analysis at $N = \infty$ control $N < \infty$?

Case study: monotone transport.

<u>Thm</u> [Guionnet + D.S. '12]

τ_0 = law of free semicircular d-tuple $\left[V = \frac{1}{2}\sum t_j^2\right]$

τ_1 = limit law of d-tuple $\left[V = \frac{1}{2}\sum t_j^2 + \varepsilon W\right]$

For $|\varepsilon| << 1$ $\exists$ n.c. power series $F_1, \ldots, F_d$, $G_1, \ldots, G_d$ s.t.:

if $(X_1, \ldots, X_d) \sim \tau_1 \Rightarrow (F_1(X_1, \ldots, X_d), \ldots, F_d(X_1, \ldots, X_d)) \sim \tau_0$

and $F \circ G = id.$ "$F_\# \tau_1 = \tau_0$"

Idea of pf:

look for $F = \mathrm{id} + \varepsilon D\varphi$

where φ is an unknown function.

... get PDE for φ $\leftarrow$ "log of Monge-Ampère"

... solve PDE.

Th2. As before, F s.t. $(F)_\# \tau_1 = \tau_0$

power series.

where $\tau_0 =$ free semicircle $\quad [V = \frac{1}{2}\sum t_j^2]$

$\tau_1 =$ limit law of $\quad V = \frac{1}{2}\sum t_j^2 + \varepsilon W.$

$N = \infty$

$$\tau_1 \xrightarrow{F_\#} \tau_0$$

$F_\#^{(N)} =$ optimal transport map

$N < \infty$

$$\mu_N^{(1)} = \frac{1}{Z_N} e^{-\frac{N^2}{2}\sum \tau_N(A_j^2 + \varepsilon W)} dA_1 \dots dA_d \qquad \mu_N^{(1)} = \frac{1}{Z_N} e^{-\frac{N^2}{2}\sum \tau_N(A_j^2)} dA_1 \dots dA_d$$

$\tilde{F}_\#^{(N)} =$ apply $F_\#$

$\underline{N=\infty}$

$$\tau_1 \xrightarrow{\quad F_\# \quad} \tau_0$$

$N<\infty$

$$\begin{tikzcd}
\mu_N^{(1)} = \frac{1}{Z_N} e^{-\frac{N^2}{2}\sum \tau_N(A_j^2 + \varepsilon W)}\, d\tilde{A}_1 \cdots dA_d \arrow[r, bend left, "F_\#^{(N)} = \text{optimal transport map}"] \arrow[r, bend right, "\tilde{F}_\#^{(N)} = \text{apply } F_\#"'] & \mu_N^{(0)} = \frac{1}{Z_N} e^{-\frac{N^2}{2}\sum \tau_N(A_j^2)}\, dA_1 \cdots dA_d
\end{tikzcd}$$

Then

$$\tilde{F}^{(N)} - F^{(N)} \longrightarrow 0 \quad \text{a.e.}$$

$\tilde{F}^{(N)}$: $N=\infty$ prediction evaluated at $N<\infty$

$F^{(N)}$: Optimal solution at $N<\infty$

Voiculescu's free entropy

$\chi(\tau_1 \| \tau_0)$ [a kind of $H_\infty(\ \|\ )$]

$$\chi(\|) = \lim \left(\frac{1}{N^2} H_N(\ \|\ ) + \alpha_N \right)$$

$\chi(\tau_1 \| \tau_0)$ maximal $\Longleftrightarrow$ $\tau_1 = \tau_0$.

$N=\infty$

$$\tau_1 \xrightarrow{F_\#} \tau_0$$

$N<\infty$

$F_\#^{(N)}$ = optimal transport map

$$\mu_N^{(1)} = \frac{1}{Z_N} e^{-\frac{N^2}{2}\sum \tau_N(A_j^2+\varepsilon W)} d\check{A}_1 \cdots dA_d \qquad \mu_N^{(0)} = \frac{1}{Z_N} e^{-\frac{N^2}{2}\sum \tau_N(A_j^2)} dA_1 \cdots dA_d$$

$\tilde{F}_\#^{(N)}$ = apply $F_\#$

Then $\tilde{F}^{(N)} - F^{(N)} \to 0$ a. e.

$\tilde{F}^{(N)}$: $N=\infty$ prediction evaluated at $N<\infty$

$F^{(N)}$: Optimal solution at $N<\infty$

$(F_N)_\# \mu_N^{(0)} = \mu_N^{(1)}$ $\qquad \mu_N^{(0)} = \frac{1}{Z_N} e^{-\frac{N^2}{2}\tau_N(\sum A_j^2)}$ $\qquad \mu_N^{(1)} = \frac{1}{Z'_N} e^{-\frac{N^2}{2}\tau_N(\sum A_j^2 + W)}$

<u>Pf</u>:

Let $\eta_N(f) = H_N(f_\# \mu_N^{(1)} \,\|\, \mu_N^{(0)})$ $\leftarrow$ entropy / KL divergence.

Then $\eta_N(f)$ is concave in f & reaches max value of $0 \iff f = F_N$.

$f = \nabla g$ free entropy

$$\eta_N(\tilde{F}_N) = H_N(\tilde{F}_{N\#} \mu_N^{(1)} \,\|\, \mu_N^{(0)}) \simeq H_\infty(F_\# \tau_1 \,\|\, \tau_0)$$

$$= 0 \text{ because } F_\# \tau_0 = \tau_1$$

Thus $\eta_N(\tilde{F}_N) \approx 0 \Rightarrow \|\tilde{F}_N - F_N\|_{L^2} \approx 0$

+ concentration : $\tilde{F}_N - F_N \to 0$ a.e.

Summary:

$N = \infty$ free probability analysis on operators

$N < \infty$ RMT analysis on matrices

Other such MFG-like examples?

References

Anderson, Guionet, Zeitouni
An Introduction to Random Matrices

PCMI Random Matrix Summer School
I have some notes on free probability
and RMT

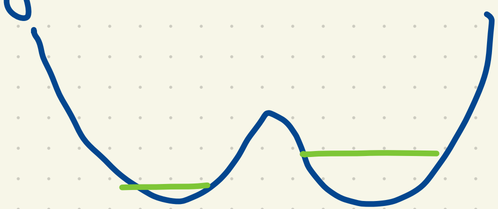
Guionnet, Shlyakhtenko; Free Monotone
Transport, Inv. Math. 2014

See talk by D. Jekel for PDE story

Open questions

1) Non-convex case?

Nice analysis in double-well $d=1$ case



No analog when $d>1$

$\tau_0, \tau_1 : \mathbb{C}\langle t_1, \ldots, t_d \rangle \longrightarrow \mathbb{C}$

$$\inf \sum_{i=1}^{d} \tau\left((t_i - s_i)^2 \right)$$

law

$\tau : \mathbb{C}\langle t_1, \ldots, t_d, s_1, \ldots, s_d \rangle$

$\left[\begin{array}{l} \text{positive} \\ \tau(p^* p) \geq 0 \end{array} \right]$

s.t. $\tau|_{\mathbb{C}\langle t_1, \ldots, t_d \rangle} = \tau_0$

$\tau|_{\mathbb{C}\langle s_1, \ldots, s_d \rangle} = \tau_1$

limit laws abstractly

Vidicke et al.

laws of finite n