Controlling propagation of epidemics: Mean-field SIR games

Stanley Osher

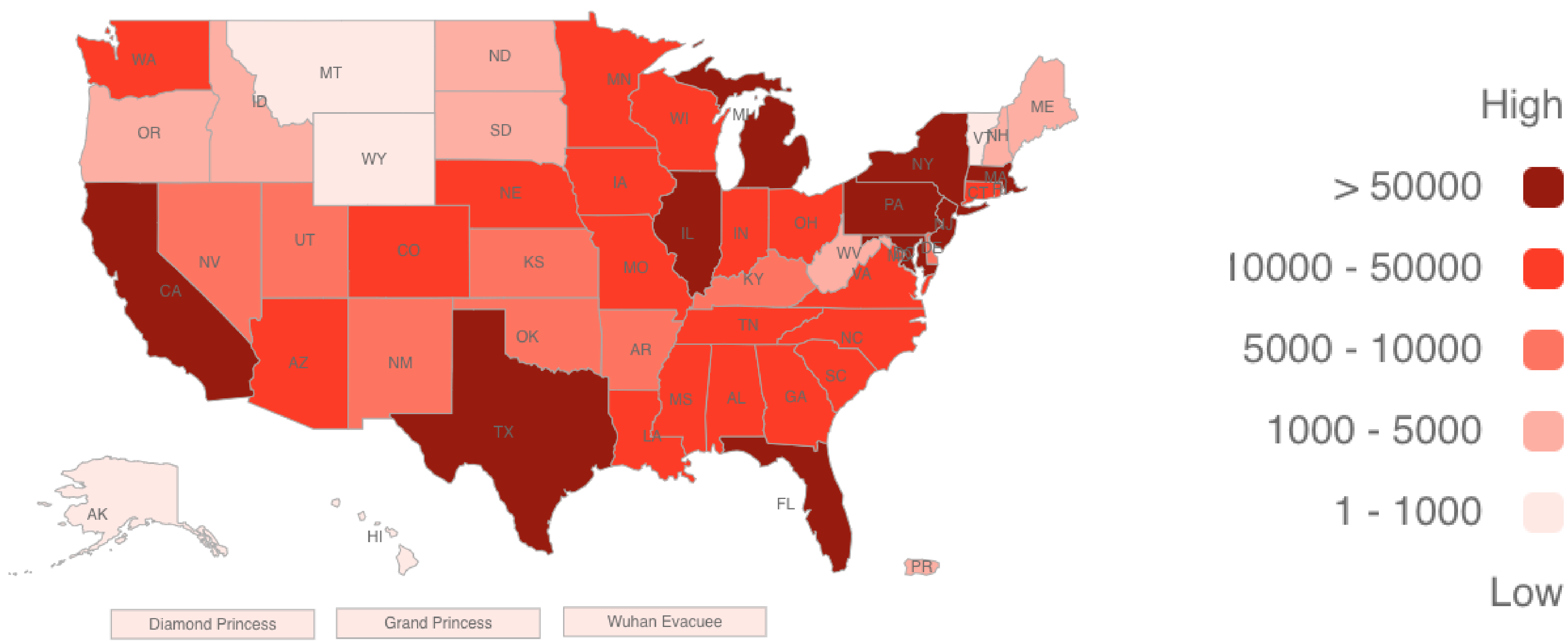
Joint work with Wonjun Lee, Siting Liu, Hamidou Tembine and Wuchen Li

COVID 19

As of May 30, 2020, the total case of COVID 19 has reached:

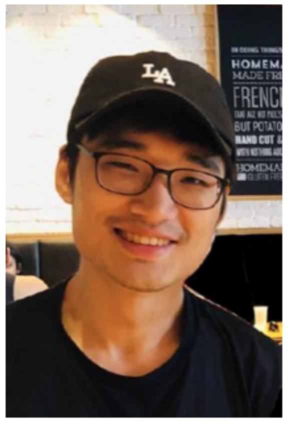
	Cases	Deaths
United States	+2,661 1,789,803	+75 104,375 ⓘ
World	+115,329 5,924,275	+4,559 364,867

COVID 19 in US



Goals

Fight against COVID-19 by optimal transport and mean field games.



W. Lee

S. Liu

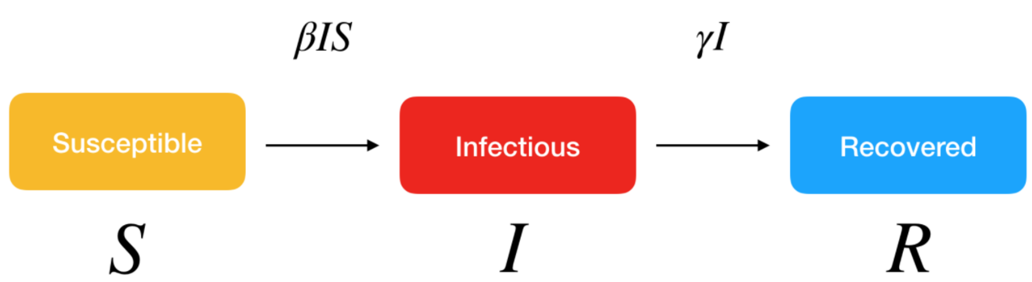
T. Hamidou

W. Li

S. Osher

LLHLO

Classic Epidemic Model

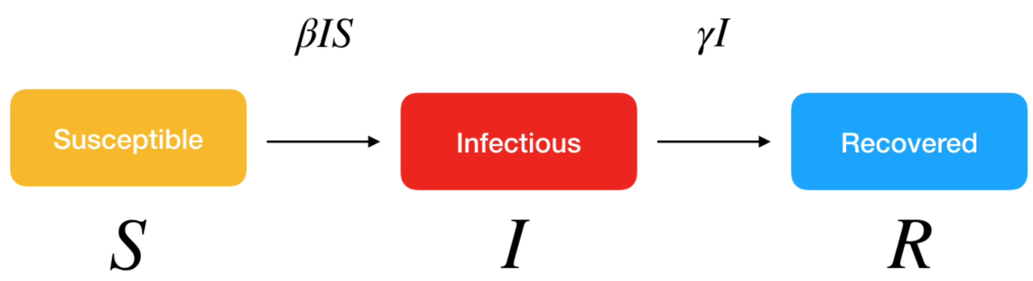


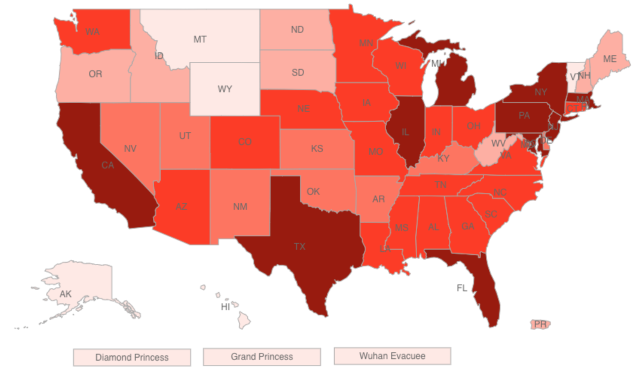
The classical Epidemic model is the SIR model (Kermack and McKendrick, 1927)

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI \\ \dfrac{dI}{dt} = \beta SI - \gamma I \\ \dfrac{dR}{dt} = \gamma I \end{cases}$$

where S, I,$R : [0, T] \to [0, 1]$ represent the density of the susceptible population, infected population, and recovered population, respectively, given time t. The nonnegative constants β and γ represent the rates of susceptible becoming infected and infected becoming recovered.

Spatial SIR





To model the spatial effect of virus spreading ,the spatial SIR model is considered:

$$\begin{cases} \partial_t \rho_S(t,x) + \beta \rho_S(t,x) \int_\Omega K(x,y)\rho_I(t,y)dy - \frac{\eta_S^2}{2}\Delta \rho_S(t,x) = 0 \\ \partial_t \rho_I(t,x) - \beta \rho_I(x) \int_\Omega K(x,y)\rho_S(t,y)dy + \gamma \rho_I(t,x) - \frac{\eta_I^2}{2}\Delta \rho_I(t,x) = 0 \\ \partial_t \rho_R(t,x) - \gamma \rho_I(t,x) - \frac{\eta_R^2}{2}\Delta \rho_R(t,x) = 0 \end{cases}$$

Here Ω is a given spatial domain and $K(x,y)$ is a symmetric positive definite kernel modeling the physical distancing. E.g. $\int K d\rho_I$ is the exposure to infectious agents.

Optimal control of population behaviors

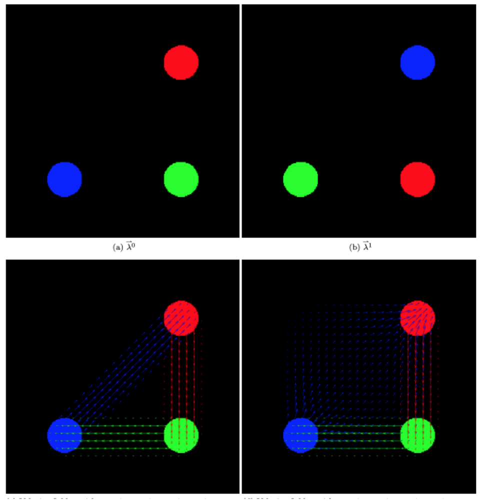
Optimal control of population behaviors have been widely considered in optimal transport and mean field games. Long story short, it refers to an optimal control problem in density space:

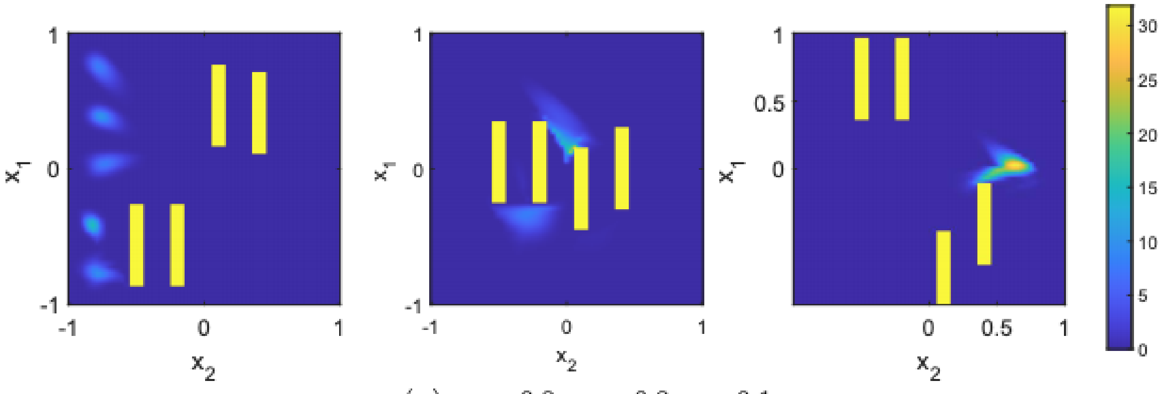
$$\min \ \text{Running cost of a population}$$

s.t.

$$\text{Evolution of population dynamics}$$

E.g.





(A) $\sigma_1 = 0.8, \sigma_2 = 0.8, \mu = 0.1$

Goals: Mean field game spatial SIR model

Questions

To balance the social cost and saving lives under this COVID epidemic daily life, we need to control or allocate S,I, R populations in a spatial domain.

Solutions:

We propose a mean field control problem for spatial SIR models and introduce an efficient numerical scheme.

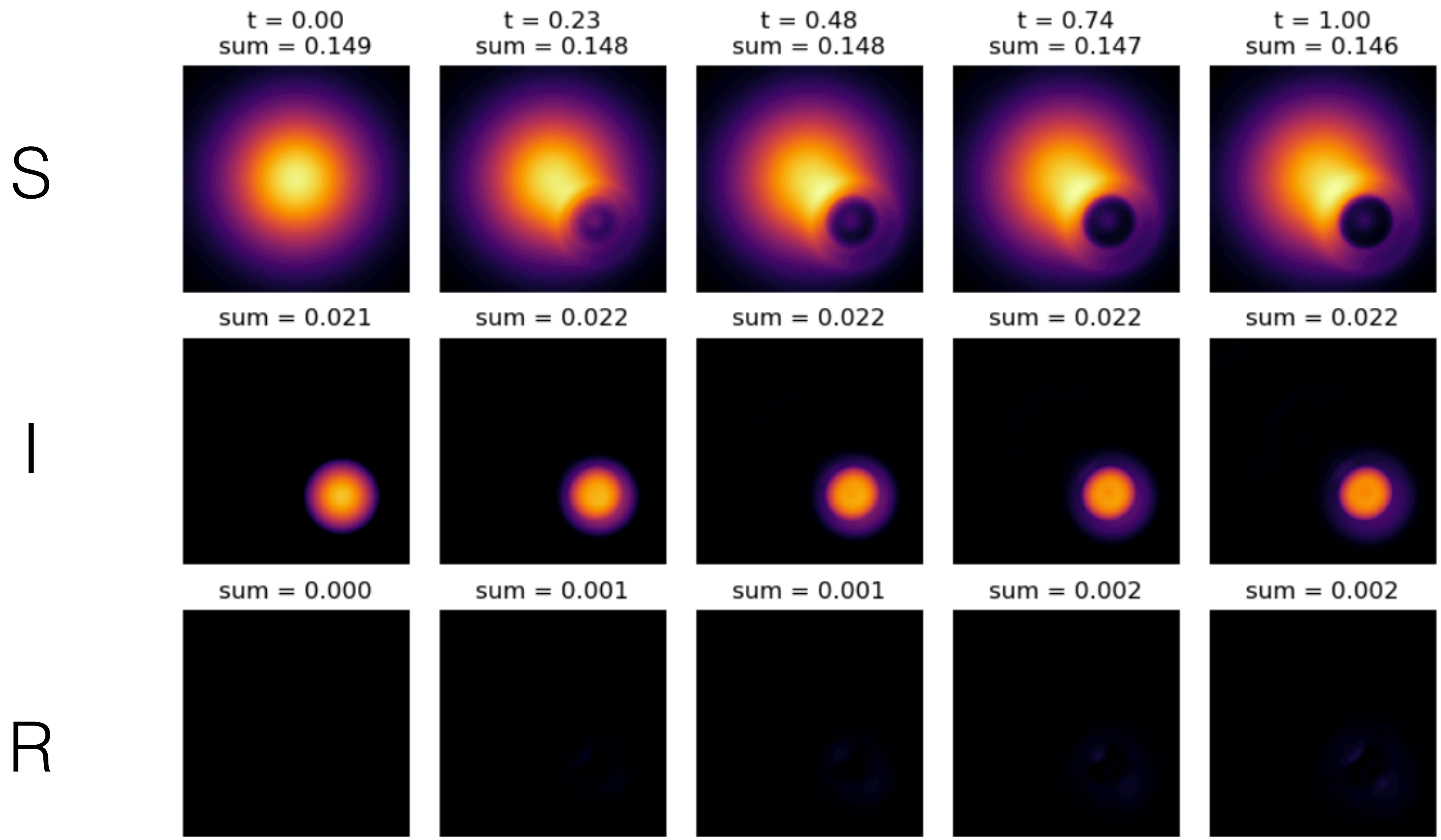
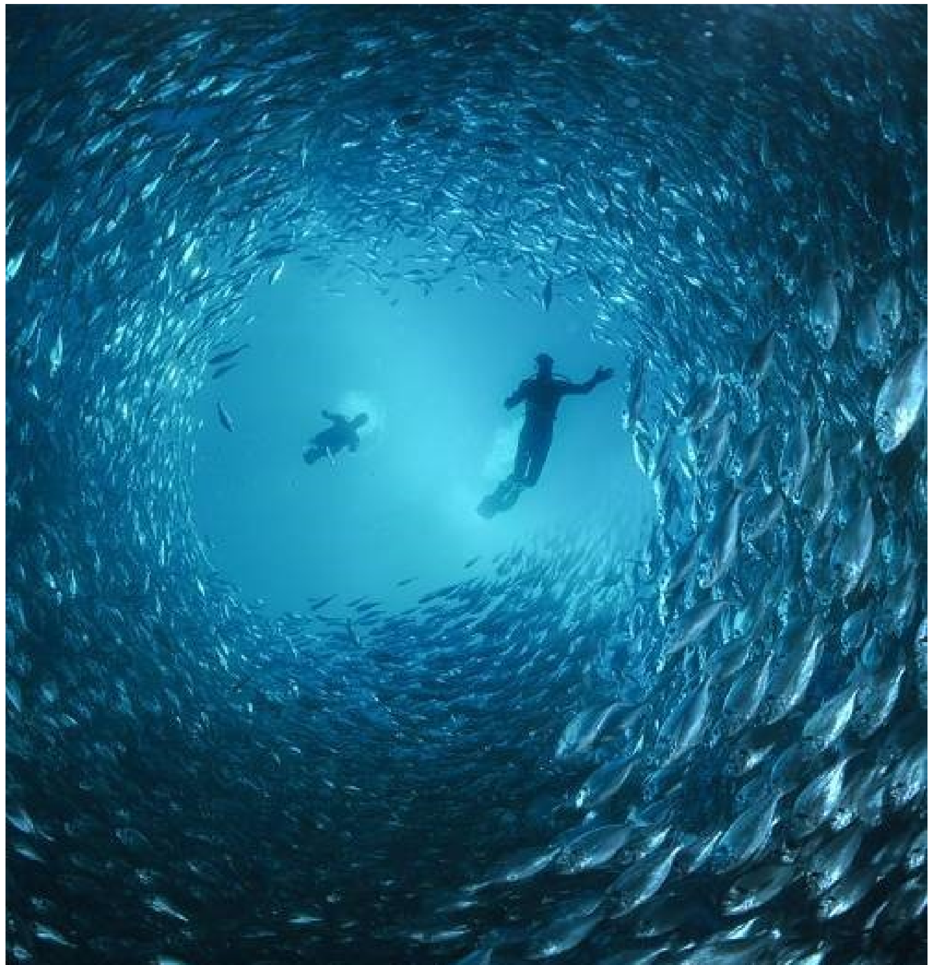


FIGURE 1. Experiment 1. The evolution of populations from $t = 0$ to $t = 1$ with $\beta = 0.7$ and $\gamma = 0.1$. The first row represents susceptible, the second row represents infected, and the last row represents recovered. The solution moves susceptible away from the infected over time.

MFG Related

- Introduced by Jovanovic & Rosenthal[JR88], M. Huang, P. Caines, R. Malhamé [HMC06] and P.-L. Lions, J.-M. Lasry [LL06a, LL06b] to model huge populations of identical agents playing non-cooperative differential games.
- Wide applications to various fields: in economics, Finance, crowd motion, industrial engineering, data science, material dynamics, and more [GNP15, BDFMW13, LLLL16, AL19].
- Computational methods developed to solve high dimensional problems. [BC15, BnAKS18, EHL18, LFL+20, ROL+19, LJL+20].



Related Study on COVID-19

- Study traveling waves to understand the propagation of epidemics. In [BRR20], they introduce a SIRT model to study the effects of the presence of a road on the spatial propagation the epidemic.
- Optimal control with control measures on medicare (vaccination)
- Machine Learning, Data Driven + Epidemic model

Figure: Social Distancing
https://www.wfla.com/news/by-the-numbers/tampa-bay-counties-earn-d-and-f-grades-for-social-distancing/
https://s.hdnux.com/photos/01/12/06/10/19423760/5/1024x1024.jpg

Understand connection between the society **(global)** and the individuals **(local)** .

Spatial SIR variational problems

Construct the following variational problem to balance virus spreading and "social" cost.

$$\min_{\rho_i, v_i} E(\rho_I(T,\cdot)) + \int_0^T \int_\Omega \sum_{i=S,I,R} \frac{\alpha_i}{2}\rho_i \|v_i\|^2 + \frac{c}{2}(\rho_S + \rho_I + \rho_R)^2 dxdt$$

subject to

$$\begin{cases} \partial_t \rho_S + \nabla \cdot (\rho_S v_S) + \beta \rho_S \rho_I - \frac{\eta_S^2}{2}\Delta \rho_S = 0 \\ \partial_t \rho_I + \nabla \cdot (\rho_I v_R) - \beta \rho_S \rho_I + \gamma \rho_I - \frac{\eta_I^2}{2}\Delta \rho_I = 0 \\ \partial_t \rho_R + \nabla \cdot (\rho_R v_R) - \gamma \rho_I - \frac{\eta_R^2}{2}\Delta \rho_R = 0 \\ \rho_S(0,\cdot), \rho_I(0,\cdot), \rho_R(0,\cdot) \text{ are given.} \end{cases}$$

Spatial convolution SIR variation

Consider

$$\min_{\rho_i, v_i} E(\rho_I(T,\cdot)) + \int_0^T \int_\Omega \sum_{i=S,I,R} \frac{\alpha_i}{2} \rho_i \|v_i\|^2 + \frac{c}{2}(\rho_S + \rho_I + \rho_R)^2 dxdt$$

subject to

$$\begin{cases} \partial_t \rho_S + \nabla \cdot (\rho_S v_S) + \beta \rho_S K * \rho_I - \frac{\eta_S^2}{2} \Delta \rho_S = 0 \\ \partial_t \rho_I + \nabla \cdot (\rho_I v_R) - \beta K * \rho_S \rho_I + \gamma \rho_I - \frac{\eta_I^2}{2} \Delta \rho_I = 0 \\ \partial_t \rho_R + \nabla \cdot (\rho_R v_R) - \gamma \rho_I - \frac{\eta_R^2}{2} \Delta \rho_R = 0 \\ \rho_S(0,\cdot), \rho_I(0,\cdot), \rho_R(0,\cdot) \text{ are given.} \end{cases}$$

Here K is the normalized positive definite symmetric convolution kernel. Kendall (1965) introduced this kernel for modeling pandemic dynamics without optimization.

Mean-field game SIR systems

$$
\begin{cases}
\partial_t \phi_S - \dfrac{\alpha_S}{2}|\nabla \phi_S|^2 + \dfrac{\eta_S^2}{2}\Delta \phi_S + c(\rho_S + \rho_I + \rho_R) \\
\qquad\qquad\qquad\qquad + \beta \left(K * (\phi_I \rho_I) - \phi_S K * \rho_I \right) = 0 \\
\partial_t \phi_I - \dfrac{\alpha_I}{2}|\nabla \phi_I|^2 + \dfrac{\eta_I^2}{2}\Delta \phi_I + c(\rho_S + \rho_I + \rho_R) \\
\qquad\qquad + \beta \left(\phi_I K * \rho_S - K * (\phi_S \rho_S) \right) + \gamma \rho (\phi_R - \phi_I) = 0 \\
\partial_t \phi_R - \dfrac{\alpha_R}{2}|\nabla \phi_R|^2 + \dfrac{\eta_R^2}{2}\Delta \phi_R + c(\rho_S + \rho_I + \rho_R) = 0 \\
\partial_t \rho_S - \dfrac{1}{\alpha_S}\nabla \cdot (\rho_S \nabla \phi_S) + \beta \rho_S K * \rho_I - \dfrac{\eta_S^2}{2}\Delta \rho_S = 0 \\
\partial_t \rho_I - \dfrac{1}{\alpha_I}\nabla \cdot (\rho \nabla \phi_I) - \beta \rho_I K * \rho_S + \gamma \rho_I - \dfrac{\eta_I^2}{2}\Delta \rho_I = 0 \\
\partial_t \rho_R - \dfrac{1}{\alpha_R}\nabla \cdot (\rho_R \nabla \phi_R) - \gamma \rho_I - \dfrac{\eta_R^2}{2}\Delta \rho_R = 0.
\end{cases}
$$

Review on PDHG method

Consider a saddle point problem

$$\min_x \sup_y \left\{ L(x,y) := \langle Ax, y \rangle + g(x) - f^*(y) \right\}.$$

Here, f and g are convex functions with respect to a variable x, A is a continuous linear operator. For each iteration, the algorithm finds the minimizer x_* by gradient descent method and the maximizer y_* by gradient ascent method. Thus, the minimizer and maximizer are calculated by iterating

$$\begin{cases} x^{k+1} & = \operatorname{argmin}_x L(x, y^k) + \frac{1}{2\tau}\|x - x^k\|^2 \\ y^{k+1} & = \operatorname{argmax}_y L(x^{k+1}, y) + \frac{1}{2\sigma}\|y - y^k\|^2 \end{cases}$$

where τ and σ are step sizes for the algorithm.

Review on G-Proximal

Here G-Prox PDHG is a modified version of PDHG that solves the minimization problem by choosing the most appropriate norms for updating x and y. Choosing the appropriate norms allows us to choose larger step sizes. Hence, we get a faster convergence rate. In details,

$$\begin{cases} x^{k+1} & = \operatorname{argmin}_x L(x, y^k) + \frac{1}{2\tau}\|x - x^k\|^2_{\mathcal{H}} \\ y^{k+1} & = \operatorname{argmax}_y L(x^{k+1}, y) + \frac{1}{2\sigma}\|y - y^k\|^2_{\mathcal{G}} \end{cases}$$

where $\mathcal{H}$ and $\mathcal{G}$ are some Hilbert spaces with the inner product

$$(u_1, u_2)_{\mathcal{G}} = (Au_1, Au_2)_{\mathcal{H}}.$$

Algorithm: Primal-Dual updates

In particular, we use G-Prox PDHG to solve the variational SIR model by

$$x = (\rho_S, \rho_I, \rho_R, m_S, m_I, m_R), \quad g(x) = F(\rho_i, m_i)_{i=S,I,R},$$

$$f(Ax) = \begin{cases} 0 & \text{if } Ax = (0, 0, \gamma\rho_I) \\ \infty & \text{otherwise.} \end{cases}$$

$$\begin{aligned} Ax = (&\partial_t \rho_S + \nabla \cdot m_S - \frac{\eta_S^2}{2} \Delta \rho_S + \beta \rho_S K * \rho_I, \\ &\partial_t \rho_I + \nabla \cdot m_I - \frac{\eta^2}{2} \Delta \rho_I - \beta \rho_I K * \rho_S + \gamma \rho_I, \\ &\partial_t \rho_R + \nabla \cdot m_R - \frac{\eta^2}{2} \Delta \rho_R). \end{aligned}$$

Variational formulation

Denote $m_i = \rho_i v_i$. Define the Lagrangian functional for Mean field game SIR problem by

$$
\begin{aligned}
&\mathcal{L}((\rho_i, m_i, \phi_i)_{i=S,I,R}) \\
=&P(\rho_i, m_i)_{i=S,I,R} - \int_0^T \int_\Omega \sum_{i=S,I,R} \phi_i \left(\partial_t \rho_i + \nabla \cdot m_i - \frac{\eta_i^2}{2} \Delta \rho_i \right) dxdt \\
&+ \int_0^T \int_\Omega \beta \phi_I \rho_I K * \rho_S - \beta \phi_S \rho_S K * \rho_I + \gamma \rho_I (\phi_R - \phi_I) dxdt.
\end{aligned}
$$

Using this Lagrangian functional, we convert the minimization problem into a saddle problem.

$$
\inf_{(\rho_i, m_i)_{i=S,I,R}} \sup_{(\phi_i)_{i=S,I,R}} \mathcal{L}((\rho_i, m_i, \phi_i)_{i=S,I,R}).
$$

Algorithm

Algorithm: PDHG for mean field game SIR system

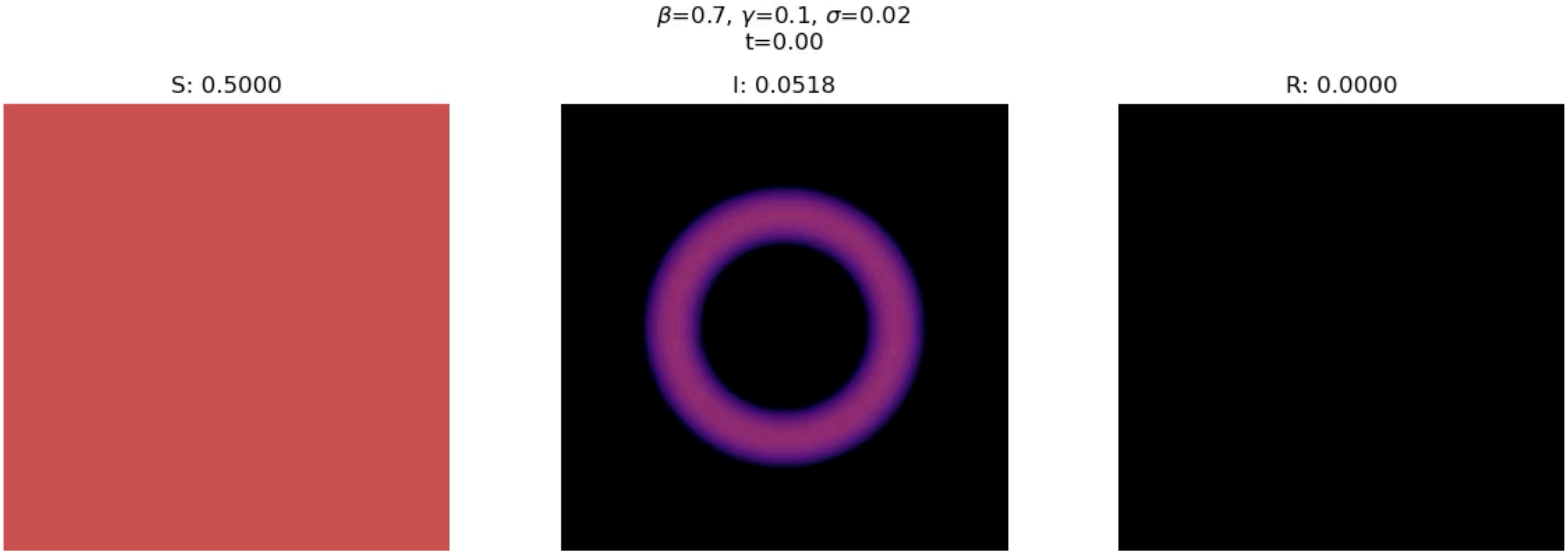
Input: $\rho_i(0,\cdot)$ $(i = S, I, R)$
Output: ρ_i, m_i, ϕ_i $(i = S, I, R)$ for $x \in \Omega$, $t \in [0, T]$

While relative error $>$ tolerance

$$\rho_i^{(k+1)} = \mathrm{argmin}_\rho\, \mathcal{L}(\rho, m_i^{(k)}, \phi_i^{(k)}) + \frac{1}{2\tau_i}\|\rho - \rho_i^{(k)}\|_{L^2}^2$$
$$m_i^{(k+1)} = \mathrm{argmin}_m\, \mathcal{L}(\rho^{(k+1)}, m, \phi_i^{(k)}) + \frac{1}{2\tau_i}\|m - m_i^{(k)}\|_{L^2}^2$$
$$\phi_i^{(k+\frac{1}{2})} = \mathrm{argmax}_\phi\, \mathcal{L}(\rho^{(k+1)}, m_i^{(k+1)}, \phi) - \frac{1}{2\sigma_i}\|\phi - \phi_i^{(k)}\|_{H^2}^2$$
$$\phi_i^{(k+1)} = 2\phi_i^{(k+\frac{1}{2})} - \phi_i^{(k)}$$

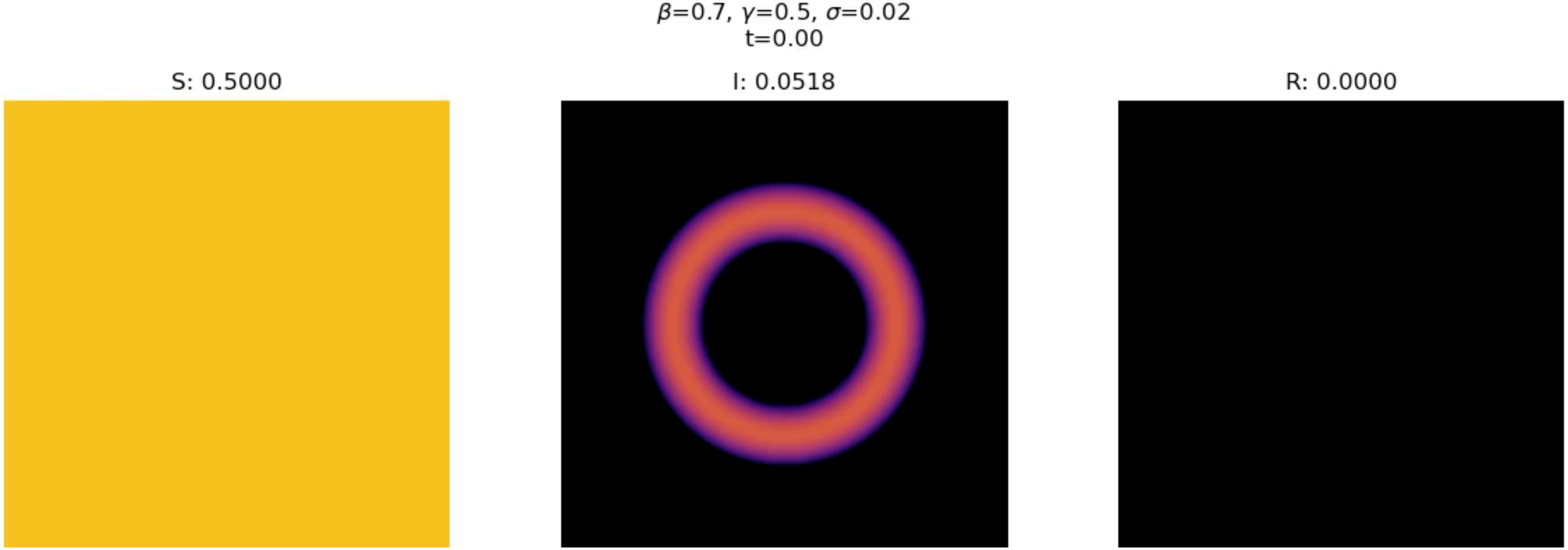
end

Examples I



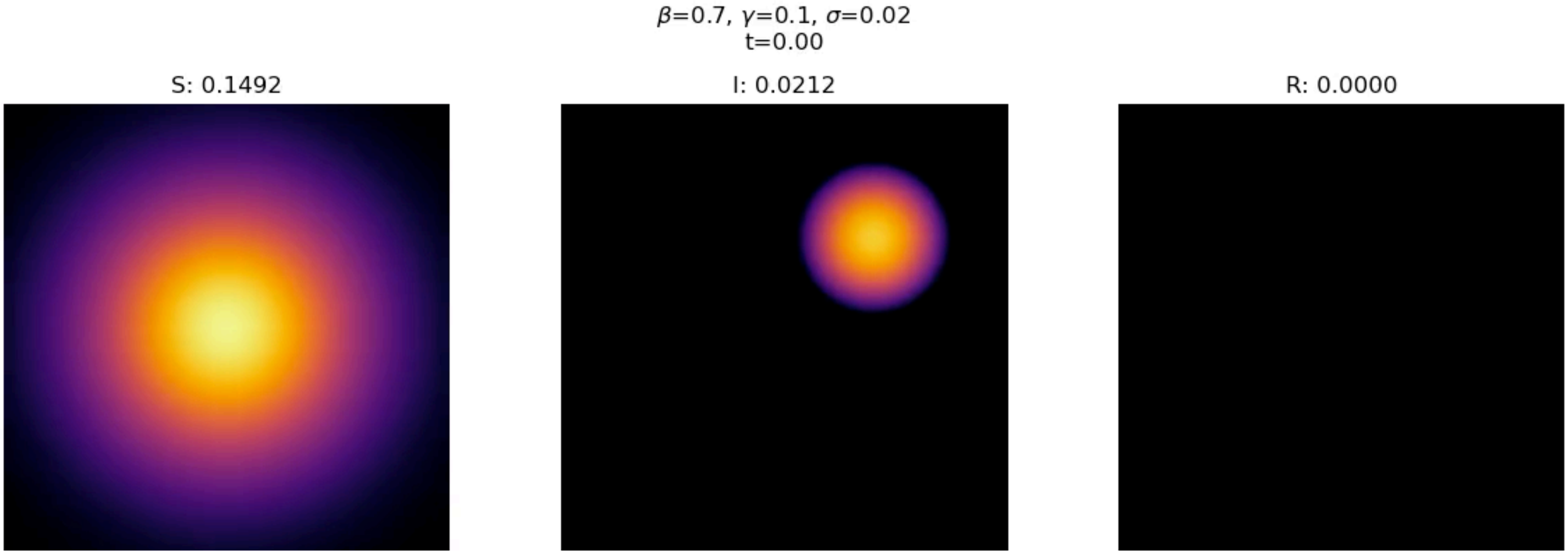
Small recovery rate

Example II



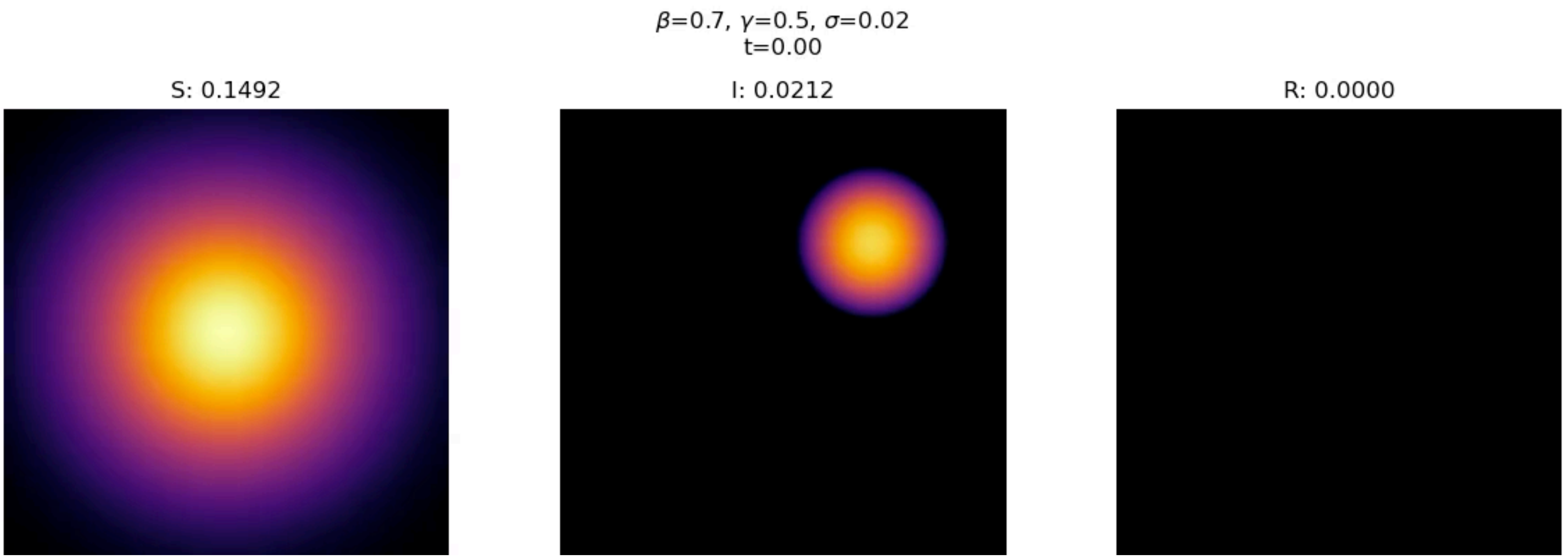
Large recovery rate

Example III



Small recovery rate

Example IV



Large recovery rate

Discussions

Importance of spatial SIR variational problems.

- Consider more status of populations, going beyond S, I, R.
- Construct discrete spatial domain model, including airport, train station, hospital, school etc.
- Propose inverse mean field SIR problems. Learning parameters in the model by daily life data.
- Combine mean field game SIR models with AI and machine learning algorithms, including APAC, Neural variational ODE, Neural Fokker-Planck equations, etc.

References

W. Lee, S. Liu, T. Tembine, W. Li, S. Osher.
Controlling Propagation of epidemics via mean-field games, 2020.

Yves Achdou and Jean-Michel Lasry.
Mean field games for modeling crowd motion.
In *Contributions to partial differential equations and applications*, volume 47 of *Comput. Methods Appl. Sci.*, pages 17–42. Springer, Cham, 2019.

J.-D. Benamou and G. Carlier.
Augmented Lagrangian methods for transport optimization, mean field games and degenerate elliptic equations.
J. Optim. Theory Appl., 167(1):1–26, 2015.

Martin Burger, Marco Di Francesco, Peter Markowich, and Marie-Therese Wolfram.
Mean field games with nonlinear mobilities in pedestrian dynamics.
arXiv preprint arXiv:1304.5201, 2013.

L. M. Briceño Arias, D. Kalise, and F. J. Silva.
Proximal methods for stationary mean field games with local couplings.
SIAM J. Control Optim., 56(2):801–836, 2018.

Henri Berestycki, Jean-Michel Roquejoffre, and Luca Rossi.

Propagation of epidemics along lines with fast diffusion.
arXiv preprint arXiv:2005.01859, 2020.

Weinan E, Jiequn Han, and Qianxiao Li.
A Mean-Field Optimal Control Formulation of Deep Learning.
arXiv:1807.01083 [cs, math], 2018.

Diogo A Gomes, Levon Nurbekyan, and Edgard A Pimentel.
Economic models and mean-field games theory.
IMPA Mathematical Publications. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2015.

M. Huang, R. P. Malhamé, and P. E. Caines.
Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle.
Commun. Inf. Syst., 6(3):221–251, 2006.

Boyan Jovanovic and Robert W. Rosenthal.
Anonymous sequential games.
Journal of Mathematical Economics, 17(1):77 – 87, 1988.

Alex Tong Lin, Samy Wu Fung, Wuchen Li, Levon Nurbekyan, and Stanley J. Osher.
Apac-net: Alternating the population and agent control via two neural networks to solve high-dimensional stochastic mean field games, 2020.

Siting Liu, Matthew Jacobs, Wuchen Li, Levon Nurbekyan, and Stanley J Osher.
Computational methods for nonlocal mean field games with applications.
arXiv preprint arXiv:2004.12210, 2020.

Jean-Michel Lasry and Pierre-Louis Lions.
Jeux à champ moyen. I. Le cas stationnaire.
C. R. Math. Acad. Sci. Paris, 343(9):619–625, 2006.

Jean-Michel Lasry and Pierre-Louis Lions.
Jeux à champ moyen. II. Horizon fini et contrôle optimal.
C. R. Math. Acad. Sci. Paris, 343(10):679–684, 2006.

Aimé Lachapelle, Jean-Michel Lasry, Charles-Albert Lehalle, and Pierre-Louis Lions.

Efficiency of the price formation process in presence of high frequency participants: a mean field game analysis.
Mathematics and Financial Economics, 10(3):223–262, 2016.

Lars Ruthotto, Stanley Osher, Wuchen Li, Levon Nurbekyan, and Samy Wu Fung.
A machine learning framework for solving high-dimensional mean field game and mean field control problems, 2019.