HJB Equations in Wassertein Space and Viscosity Solutions

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Roughly speaking

- Standard HJB : $u(t,x)$, $x \in \mathbb{R}^d$
- HJB on Wassertein space : $V(t,\mu)$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$
- In this talk, $d = 1$.

Outline

1. Motivations/Applications
   - Example 1 : mean field control problem
   - Example 2 : Control under probability distortion
   - Example 3 : Stochastic control with information delay
2. HJB equation in Wasserstein space
   - Basic calculus in Wasserstein space
   - HJB equations for the examples
3. Viscosity solutions
   - Existing approaches
   - Our approach

Some applications

- Mean field game and systemic risk
  - ◇ Caines-Huang-Malhame (2006), Lasry-Lions (2007)
  - ◇ ······
  - ◇ Cardaliaguet, Bensoussan-Frehse-Yam, Carmona-Delarue
- Stochastic control with partial observation
  - ◇ Bandini-Cosso-Fuhrman-Pham (2018, 2019)
  - ◇ Saparito-Z. (2019)
- Time inconsistent problems
  - ◇ Wu-Z. (2020)

Example 1 : Mean field dynamics

- A large controlled interacting system : $i = 1, \cdots, N$,

$$X_t^{i,\alpha^i} = x_i + \int_0^t \sigma(s, X_s^{i,\alpha^i}, \frac{1}{N}\sum_{j=1}^N \delta_{X_s^{j,\alpha^j}}, \alpha_s^i) dB_s^i.$$

  - ◇ Typically we use closed-loop controls α^i

- A limit dynamics as $N \to \infty$: McKean Vlasov SDE

$$X_t^\alpha = X_0 + \int_0^t \sigma(s, X_s^\alpha, \mathcal{L}_{X_s^\alpha}, \alpha_s) dB_s.$$

Example 1 : Mean field control problem

- Mean field control problem (central optimization) :

$$\sup_{\alpha} \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{E}\Big[g\big(X_T^{i,\alpha}, \frac{1}{N}\sum_{j=1}^{N}\delta_{X_T^{j,\alpha}}\big)\Big] \to \sup_{\alpha} \boldsymbol{E}\Big[g\big(X_T^{\alpha}, \mathcal{L}_{X_T^{\alpha}}\big)\Big] =: V_0$$

- Dynamic value : $V(t,\mu) := \sup_{\alpha_{[t,T]}} \boldsymbol{E}\Big[g\big(X_T^{t,\mu,\alpha}, \mathcal{L}_{X_T^{t,\mu,\alpha}}\big)\Big]$.

- DPP : $V_0 = \sup_{\alpha_{[0,t]}} V(t, \mathcal{L}_{X_t^{\alpha}})$.

  ◇ DPP + Ito $\Longrightarrow$ HJB

- Note : mean field game problem has quite different structure

  ◇ The value for mean field control problem is always unique, and typically satisfies the comparison principle

  ◇ Mean field games may have multiple values, and even if it is unique, typically the value does not satisfy the comparison principle

Example 2 : Probability distortion

- Standard expectation : given a r.v. $\xi \geq 0$,

$$\boldsymbol{E}[\xi] := \int_0^\infty \mathbb{P}(\xi \geq y)dy = \int_0^\infty y\, f_\xi(y)dy.$$

- Nonlinear expectation under probability distortion (Zhou etc) :

$$\mathcal{E}[\xi] := \int_0^\infty w\big(\mathbb{P}(\xi \geq y)\big)dy = \int_0^\infty y\, f_\xi(y)w'\big(\mathbb{P}(\xi \geq y)\big)dy.$$

- Distortion function $w : [0,1] \to [0,1]$
  - ◇ strictly increasing, with $w(0) = 0, w(1) = 1$.
  - ◇ typically inverse-S shaped : $w'(0), w'(1) >> 1$

Example 2 : time inconsistency

- Markovian state process : $X_t = x_0 + \int_0^t \sigma(s, X_s) dB_s$

- Conditional expectation under probability distortion : $g \geq 0$,

$$\mathcal{E}\Big[g(X_T)\Big|\mathcal{F}_t\Big] := \int_0^\infty w\Big(\mathbb{P}\big(g(X_T) \geq y\big|\mathcal{F}_t\big)\Big) dy.$$

- Time inconsistency : $\mathcal{E}[\xi] \;\neq\; \mathcal{E}\Big[\mathcal{E}\big[g(X_T)|\mathcal{F}_t\big]\Big]$.

- Note : given the Markovian structure, we have

$$\mathcal{E}\Big[g(X_T)\Big|\mathcal{F}_t\Big] = u(t, X_t),$$

where $u(t,x)$ is deterministic, but does not satisfy a PDE due to the time inconsistency.

Example 2 : Control problem under probability distortion

- Increasing the "dimension" : $U(t,\xi) := \mathcal{E}\big[g(X_T^{t,\xi})\big]$
  - $\diamond$ U is deterministic and law invariant :

$$\mathcal{L}_\xi = \mathcal{L}_{\tilde{\xi}} \Longrightarrow U(t,\xi) = U(t,\tilde{\xi}) \Longrightarrow V(t,\mathcal{L}_\xi) := U(t,\xi).$$

  - $\diamond$ V is time consistent : $V(t_1,\mathcal{L}_{X_{t_1}}) = V(t_2,\mathcal{L}_{X_{t_2}})$.
- Control problem under probability distortion (Wu-Z. (2020)) :

$$V(t,\mathcal{L}_\xi) := \sup_{\alpha_{[t,T]}} \mathcal{E}\Big[g(X_T^{t,\xi,\alpha})\Big], \quad X_s^{t,\xi,\alpha} = \xi + \int_t^s \sigma(r, X_r^{t,\xi,\alpha}, \alpha_r)dB_r.$$

  - $\diamond$ DPP : $V(0,\delta_{x_0}) = \sup_{\alpha_{[0,t]}} V(t,\mathcal{L}_{X_t^{0,x_0,\alpha}})$
  - $\diamond$ Justification of value $V(t,\mathcal{L}_{X_t^\alpha})$ for $t > 0$?

Example 3 : Control with information delay

- Controlled state process : $X_t^\alpha = x_0 + \int_0^t \sigma(s, X_s^\alpha, \alpha_s) dB_s$
  - ◇ delayed information : $\alpha_t \in \mathcal{F}_{t-\delta}$
  - ◇ For simplicity assume $T \le \delta$, then $\alpha \in \mathcal{A}_0$ is deterministic
- Value fun. : $v(t,x) := \sup_{\alpha \in \mathcal{A}_0} \boldsymbol{E}\Big[g(X_T^{t,x,\alpha}) + \int_t^T f(X_s^{t,x,\alpha}, \alpha_s) ds\Big]$.
- DPP fails : $v(0, x_0) \neq \sup_{\alpha \in \mathcal{A}_0} \boldsymbol{E}\Big[v(t, X_t^\alpha) + \int_0^t f(X_s^\alpha, \alpha_s) ds\Big]$
  - ◇ v does not satisfy HJB or any PDE
  - ◇ When $\alpha \in \mathcal{A}$, the value function u satisfies HJB and $\alpha_t^* = \alpha^*(t, X_t^*)$ which is random
  - ◇ An intelligent guess : for $\alpha \in \mathcal{A}_0$, $\alpha_t^* = \alpha^*(t, \mathcal{L}_{X_t^*})$

Example 3 : Stochastic optim. with deterministic control

- The value function (Saparito-Z. (2019)) :

$$V(t,\mu) := U(t,\xi) := \sup_{\alpha\in\mathcal{A}_0} \boldsymbol{E}\Big[g(X_T^{t,\xi,\alpha}) + \int_t^T f(X_s^{t,\xi,\alpha},\alpha_s)ds\Big].$$

- DPP : $V(0,\delta_{x_0}) = \sup_{\alpha\in\mathcal{A}_0} \Big[V(t,\mathcal{L}_{X_t^\alpha}) + \boldsymbol{E}\Big[\int_0^t f(X_s^\alpha,\alpha_s)ds\Big]\Big]$

- V will satisfy an HJB equation in Wasserstein space :

$$\alpha_t^* = \alpha^*(t,\mathcal{L}_{X_t^*}), \quad X_t^* = x_0 + \int_0^t \sigma(s,X_s^*,\alpha^*(s,\mathcal{L}_{X_s^*}))dB_s.$$

  - ◇ Benes and Karatzas (1983)
  - ◇ Bandini-Cosso-Fuhrman-Pham (2018, 2019)

Example 4 : Nonlinear optim. with deterministic control

- Note : $V(0, \delta_{x_0}) = \sup_{\alpha \in \mathcal{A}_0} Y_0^\alpha$, where

$$Y_t^\alpha = g(X_T^\alpha) + \int_t^T f(X_s^\alpha, \alpha_s) ds - \int_t^T Z_s^\alpha dB_s.$$

- Nonlinear BSDE : (assuming $f = f(y)$)

$$Y_t^\alpha = g(X_T^\alpha) + \int_t^T f(Y_s^\alpha) ds - \int_t^T Z_s^\alpha dB_s.$$

- Dynamic version :

$$Y_s^{t,\xi,\alpha} = g(X_T^{t,\xi,\alpha}) + \int_s^T f(Y_r^{t,\xi,\alpha}) dr - \int_s^T Z_r^{t,\xi,\alpha} dB_r.$$

- Candidate value function :
  - ◇ $U(t,\xi) := \sup_{\alpha \in \mathcal{A}_0} Y_t^{t,\xi,\alpha}$: it is random and not law invariant
  - ◇ $V(t,\mu) := U(t,\xi) := \sup_{\alpha \in \mathcal{A}_0} E[Y_t^{t,\xi,\alpha}]$: DPP fails

Example 4 : The solution : path dependence

- Given a process $\xi = \xi_{[0,t]}$: $X_s^{t,\xi,\alpha} := \xi_s$, $s \in [0,t]$,

$$X_s^{t,\xi,\alpha} = \xi_t + \int_t^s \sigma(r, X_r^{t,\xi,\alpha}, \alpha_r) dB_r, \quad s \in [t, T];$$

$$Y_s^{t,\xi,\alpha} = g(X_T^{t,\xi,\alpha}) + \int_s^T f(Y_r^{t,\xi,\alpha}) dr - \int_s^T Z_r^{t,\xi,\alpha} dB_r, \quad s \in [0, T].$$

- Value function : $V(t, \mathcal{L}_{\xi_{[0,t]}}) := U(t, \xi_{[0,t]}) := \sup_{\alpha \in \mathcal{A}_0} Y_0^{t,\xi,\alpha}$
- DPP(Wu-Z. (2020)) : $V(0, \delta_{x_0}) = \sup_{\alpha \in \mathcal{A}_0} V(t, \mathcal{L}_{X^\alpha_{[0,t]}})$.
  - ◇ V satisfies a path dependent HJB in Wasserstein space
  - ◇ $\alpha_t^* = \alpha^*(t, \mathcal{L}_{X^*_{[0,t]}})$, with path dependent McKean-Vlasov SDE :

$$X_t^* = x_0 + \int_0^t \sigma(s, X_s^*, \alpha^*(s, \mathcal{L}_{X^*_{[0,s]}})) dB_s.$$

Outline

From DPP to PDE

- Value function : $u(t,x) := \boldsymbol{E}\big[g(x + B_T - B_t)\big]$
- Flow property (DPP) : $u(t,B_t) = \boldsymbol{E}\big[u(t+\delta, B_{t+\delta})|\mathcal{F}_t\big]$
- Ito formula : $du(t,B_t) = \Big[\partial_t u + \frac{1}{2}\partial_{xx} u\Big](t,B_t)dt + \partial_x u(t,B_t)dB_t$
- DPP (or martingale property) : $\partial_t u + \frac{1}{2}\partial_{xx} u = 0$
- Three ingredients to derive the PDE
  - ◇ DPP or flow property
  - ◇ Appropriate notion of derivatives
  - ◇ Ito formula

Wasserstein derivatives

- Let $V : [0, T] \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$
- $\partial_t V(t,\mu) := \lim_{\varepsilon \downarrow 0} \frac{V(t+\varepsilon,\mu) - V(t,\mu)}{\varepsilon}$.
- $\partial_\mu V : [0, T] \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$: for any $\eta \in \mathbb{L}^2(\mathcal{F}_t)$,

$$\boldsymbol{E}\Big[\partial_\mu V(t,\mu,\xi)\ \eta\ \Big] = \lim_{\varepsilon \to 0} \frac{V(t, \mathcal{L}_{\xi+\varepsilon\eta}) - V(t,\mu)}{\varepsilon}, \quad \mathcal{L}_\xi = \mu.$$

  - ◇ classical result due to Lions, Cardaliaguet
  - ◇ See Wu-Z. (2017) for an elementary proof
  - ◇ See also Gangbo-Tudorascu (2018)

Itô formula

- Assume $V \in C^{1,1,1}([0,T] \times \mathcal{P}_2(\mathbb{R}); \mathbb{R})$: smooth and ....

- For any $dX_t = b_t dt + \sigma_t dB_t$,

$$\frac{d}{dt} V(t, \mathcal{L}_{X_t}) = \partial_t V(t, \mathcal{L}_{X_t})$$
$$+ \mathbb{E}\Big[\partial_\mu V(t, \mathcal{L}_{X_t}, X_t) b_t + \frac{1}{2} \partial_x \partial_\mu V(t, \mathcal{L}_{X_t}, X_t) \sigma_t^2\Big].$$

◇ V is deterministic, so there is no dB_t term

◇ Buckdahn-Li-Peng-Rainer (2017), Chassagneux-Crisan-Delarue (2020)

◇ Wu-Z. (2020) extended it to the path dependent case

Example 1 : mean field control problem

- The control problem :

$$dX_t^\alpha = \sigma(X_t^\alpha, \mathcal{L}_{X_t^\alpha}, \alpha_t)dB_t, \quad V(t,\mu) := \sup_\alpha \boldsymbol{E}[g(X_T^{t,\xi,\alpha}, \mathcal{L}_{X_T^{t,\xi,\alpha}})].$$

- DPP + Ito : denoting $X_s^\alpha := X_s^{t,\xi,\alpha}$ and $\mu_s^\alpha := \mathcal{L}_{X_s^\alpha}$,

$$0 = \sup_\alpha \left[V(t+\delta, \mu_{t+\delta}^\alpha) - V(t,\mu)\right]$$

$$= \sup_\alpha \int_t^{t+\delta} \Big[\partial_t V(s,\mu_s^\alpha) + \frac{1}{2}\boldsymbol{E}\left[\partial_x\partial_\mu V(s,\mu_s^\alpha, X_s^\alpha)\sigma^2(X_s^\alpha, \mu_s^\alpha, \alpha_s)\right]\Big] ds.$$

- HJB : $V(T,\mu) = \boldsymbol{E}[g(\xi,\mu)]$,

$$\partial_t V(t,\mu) + \frac{1}{2}\boldsymbol{E}\Big[\sup_a \big[\partial_x\partial_\mu V(t,\mu,\xi)\sigma^2(\xi,\mu,a)\big]\Big] = 0.$$

$\diamond$ $a^* = a^*(t,\xi,\mu) \implies \alpha_t^* = a^*(t, X_t^*, \mathcal{L}_{X_t^*})$

Example 2 : Control under probability distortion

- The control problem :

$$dX_t^\alpha = \sigma(X_t^\alpha, \alpha_t)dB_t,$$

$$V(t,\mu) := \sup_\alpha \mathcal{E}[g(X_T^{t,\xi,\alpha})] := \sup_\alpha \int_0^\infty w\Big(\mathbb{P}(g(X_T^{t,\xi,\alpha}) \geq y)\Big)dy.$$

- HJB : $V(T,\mu) = \mathcal{E}[g(\xi)]$,

$$\partial_t V(t,\mu) + \frac{1}{2}\boldsymbol{E}\Big[\sup_a \big[\partial_x\partial_\mu V(t,\mu,\xi)\sigma^2(\xi,a)\big]\Big] = 0.$$

◇ The same HJB as in Example 1, but with different $V(T,\mu)$

◇ When there is no control, the nonlinear expectation corresponds to a linear equation with nonlinear terminal $V(T,\mu)$

Example 3 : Stochastic optim. with deterministic control

- The control problem : $\alpha \in \mathcal{A}_0$ deterministic,

$$dX_t^\alpha = \sigma(X_t^\alpha, \alpha_t)dB_t,$$
$$V(t,\mu) := \sup_{\alpha\in\mathcal{A}_0} \boldsymbol{E}\Big[g(X_T^{t,\xi,\alpha}) + \int_t^T f(X_s^{t,\xi,\alpha}, \alpha_s)ds\Big].$$

- HJB : $V(T,\mu) = \boldsymbol{E}[g(\xi)]$,

$$\partial_t V(t,\mu) + \sup_a \frac{1}{2}\boldsymbol{E}\Big[\partial_x\partial_\mu V(t,\mu,\xi)\sigma^2(\xi,a)\Big] = 0.$$

  ◇ Unlike the previous HJB, the $\sup_a$ is outside of $\boldsymbol{E}$ here.

  ◇ $a^* = a^*(t,\mu) \implies \alpha_t^* = a^*(t, \mathcal{L}_{X_t^*})$

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   - Example 3 : Stochastic control with information delay
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   - Existing approaches
   - Our approach

Viscosity solutions of standard PDEs

- Parabolic PDE (with terminal condition) :

$$\mathcal{L}u(t,x) := \partial_t u(t,x) + G(t,x,u,\partial_x u,\partial_{xx} u) = 0.$$

- Test function : $D_\delta(t,x) := [t, t+\delta] \times \overline{O}_\delta(x)$,

$$\underline{\mathcal{A}}u(t,x) := \bigcup_{0<\delta\le T-t} \Big\{\varphi \in C^{1,2}(D_\delta(t,x)) :$$
$$[\varphi - u](t,x) = 0 = \sup_{(t',x')\in D_\delta(t,x)} [\varphi - u](t',x')\Big\}$$

- $u \in C^0([0,T]\times \mathbb{R})$ is a viscosity subsolution of the PDE if

$$\mathcal{L}\varphi(t,x) \ge 0 \quad \text{for all } \varphi \in \underline{\mathcal{A}}u(t,x).$$

- The compactness of $D_\delta(t,x)$ is crucial for the viscosity theory.

The Wasserstein space

- Underlying state space : $\mathbb{R}$
- $\mathcal{P}_2(\mathbb{R})$: square integrable probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$
- State space : $\Theta := [0, T] \times \mathcal{P}_2(\mathbb{R})$
- Wasserstein distance : for $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ and coupling $\mathcal{P}(\mu, \nu)$

$$\mathcal{W}_2(\mu, \nu) := \inf_{\pi \in \mathcal{P}(\mu,\nu)} \left(\int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \pi(dx, dy) \right)^{\frac{1}{2}}$$

  - ◇ $(\mathcal{P}_2(\mathbb{R}), \mathcal{W}_2)$ is Polish, namely complete and separable
  - ◇ $\overline{O}_\delta(\mu) := \{\nu : \mathcal{W}_2(\mu, \nu) \leq \delta\}$ is not compact

Viscosity solutions : naive approach

- HJB equation : $\mathcal{L}V(t,\mu) = 0$
- Test function : $D_\delta(t,\mu) := [t, t+\delta] \times \overline{O}_\delta(\mu)$,

$$\underline{\mathcal{A}}V(t,\mu) := \bigcup_{0<\delta\le T-t} \Big\{ \varphi \in C^{1,1,1}(D_\delta(t,\mu)) : [\varphi - V](t,\mu) = 0 = \sup_{(s,\nu)\in D_\delta(t,\mu)} [\varphi - V](s,\nu) \Big\}$$

- $V \in C^0(\Theta)$ is a viscosity subsolution if

$$\mathcal{L}\varphi(t,\mu) \ge 0 \quad \text{for all } \varphi \in \underline{\mathcal{A}}V(t,\mu).$$

- $D_\delta(t,\mu)$ is not compact, no hope for comparison principle.

An alternative approach : lifting the function

- Lift the function $U(t, \xi) := V(t, \mathcal{L}_\xi)$ for ξ in Hilbert space $\mathbb{L}^2(\mathcal{F})$
- Apply the viscosity theory on Hilbert space (Pham-Wei (2018))
- Good news : both existence and comparison principle hold
- Bad news :

$\diamond$ A classical solution (in Wasserstein space) may not be a viscosity solution (in Hilbert space), see a counterexample by Buckdahn-Li-Peng-Rainer (2017) in 2nd order case

$\diamond$ The theory is not available in the path dependent case (again due to the lack of compactness in the path space, even in finite dimensional case)

Our approach

- Recall DPP + Ito $\Longrightarrow$ HJB

$$\text{DPP}: \quad V(t,\mu) = \sup_{\alpha} V(t+\delta, \mathcal{L}_{X_{t+\delta}^{t,\xi,\alpha}})$$

Apply Ito on $\varphi(s, \mathcal{L}_{X_s^{t,\xi,\alpha}})$, only need $\varphi \le V$ on the set $(s, \mathcal{L}_{X_s^{t,\xi,\alpha}})$ for all $s \in [t, t+\delta]$ and all α.

- A new neighborhood : for $L > 0$,

$$\mathcal{P}_\delta^L(t,\mu) := \Big\{ (s, \mathcal{L}_{X_s}) : s \in [t, t+\delta], |b| \le L, |\sigma| \le L,$$
$$X_s := \xi + \int_t^s b_r dr + \int_t^s \sigma_r dB_r, \quad \mathcal{L}_\xi = \mu \Big\}$$

◇ $\forall \delta > 0$, $\exists \delta' > 0$ such that $\mathcal{P}_{\delta'}^L(t,\mu) \subset \overline{O}_\delta(t,\mu)$

◇ $\mathcal{P}_\delta^L(t,\mu)$ is compact under $\mathcal{W}_2$

Definition

- Test function :

$$\underline{\mathcal{A}}^L V(t,\mu) := \bigcup_{0<\delta\le T-t} \Big\{ \varphi \in C^{1,1,1}(\mathcal{P}^L_\delta(t,\mu)) : [\varphi - V](t,\mu) = 0 = \sup_{(s,\nu)\in\mathcal{P}^L_\delta(t,\mu)} [\varphi - V](s,\nu) \Big\}$$

- $V \in C^0(\Theta)$ is an L-viscosity subsolution if

$$\mathcal{L}\varphi(t,\mu) \ge 0 \quad \text{for all } \varphi \in \underline{\mathcal{A}}^L V(t,\mu).$$

◇ For $L_1 < L_2$, L_1-viscosity subsol. $\Longrightarrow$ L_2-viscosity subsol.

◇ L-viscosity subsol. $\Longrightarrow$ viscosity subsol. in naive sense, so our definition helps for uniqueness

Basic results

- Consistency with classical solution
- Equivalent definition via jets
- Existence by representation
- Stability
- Partial comparison (between viscosity subsolution and classical supersolution)

Comparison principle

• Theorem : If mollified equations have classical solutions, then comparison principle holds.

• Some examples with classical solutions :

◇ Linear equations (with possible nonlinear terminal)

◇ First order conditions (under convexity conditions) : Gangbo-Meszaros (2020), $\cdots$

◇ HJB derived from stochastic optimization with deterministic controls : under convexity conditions (Saparito-Z. (2019))

$$\partial_t V(t,\mu) + \frac{1}{2}\boldsymbol{E}^\mu\Big[\partial_x\partial_\mu V(t,\mu,\xi)\Big] + F\Big(t, \boldsymbol{E}^\mu[\partial_\mu V(t,\mu,\xi)]\Big) = 0.$$

◇ General HJB/Isaacs equations : ? ? ?

Thank you very much for your attention !