The Ising Game

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with Mark Cerenzia (U Chicago)

High Dimensional Hamilton Jacobi Equations.

IPAM (virtual). June 11, 2020

Talk Outline

The Ising game is an attempt to uncover and understand phenomena in MFG in analogy to analysis from statistical physics.

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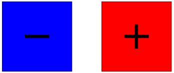
- 'Spin' and discrete MFG
- Ising model interlude
- Spin games and the Ising game
- Ising game MFG system solution
- Ising game master equation solution and concluding remarks

Spins and long time behavior

- 'Spin' markov process, $\Sigma_t \in \{\pm 1\}$ flips according to rate $A(\Sigma_t) > 0$.



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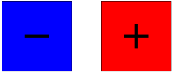

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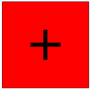
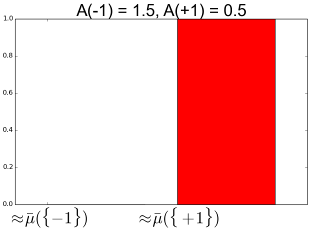


- Ergodic limit

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\Big[\int_0^T f(\Sigma_t)dt\Big] = \sum_{\sigma\in\pm 1} f(\sigma)\bar{\mu}(\{\sigma\})$$

stationary measure $\bar{\mu}$ solves

$$A(1)\bar{\mu}(\{1\}) = A(-1)\bar{\mu}(\{-1\}),$$
$$\bar{\mu}(\{1\}) + \bar{\mu}(\{-1\}) = 1.$$



Discrete N-player Games Literature

- N-spins, $\mathbf{\Sigma} = (\Sigma^1, \ldots, \Sigma^N)$, with *controlled* dynamics, $(A^1, \ldots, A^N)$, in a Nash equilibrium.

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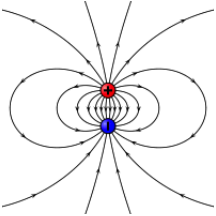
- Bayraktar-Cohen [2017], Cecchin-Dai Pra-Fischer-Pelino [2019] Bayraktar-Zhang [2019] - Nonuniqueness of (time-dependent) mean field limit. Convergence to 'entropy solution' of master equation.

Ising Interlude

- The Ising model [Lenz (1920), Ising (1925)], N magnetic dipoles with 'spin', $\sigma^i \in \{\pm 1\}$. Total energy:

$$\mathcal{H}^N(\boldsymbol{\sigma}) = \sum_{i=1}^{N} \left[-H_i \, \sigma^i - \frac{1}{N} \sum_{j=1}^{N} J_{ij}^N \, \sigma^i \, \sigma^j \right].$$



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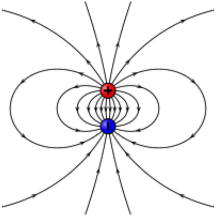

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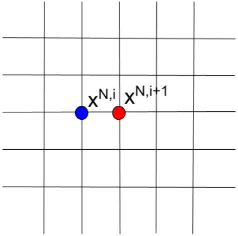
- Fixed positions on a lattice $x^i \in \mathbb{T}^d$, and $J_{ij}^N = J^N(x^i, x^j)$. Most studied when $J^N(x^i, x^j)$ is only nonzero for lattice neighbors.



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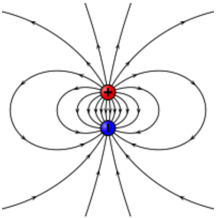

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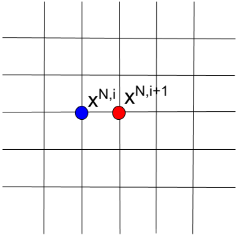

- Thermal equilibrium:

$$\boldsymbol{\mu}^N(\boldsymbol{\sigma}) = \frac{1}{Z^N} \exp\{-\beta\, \mathcal{H}^N(\boldsymbol{\sigma})\}.$$

(Prescribe A^i)

Ising Phase Transitions

- Free energy per particle in thermodynamic limit:

$$f(\beta) = \lim_{N\to\infty} \beta^{-1} N^{-1} \log\Big(\sum_{\boldsymbol{\sigma}\in\{-1,1\}^N} \exp\Big\{ -\beta\,\mathcal{H}^N(\boldsymbol{\sigma})\Big\}\Big)$$

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- The Ising model with nearest neighbor interactions does not have a phase transition for $0 < \beta < +\infty$ when $d = 1$ (Ising 1925) but does when $d \geq 2$ ($d = 2$ solved by Onsager 1944, $d = 3$ unproven).

Ising Mean Field

- Mean field approximation:

$$\frac{1}{N}\sum_{j=1}^{N} J_{ij}^{N}\,\sigma^i\,\sigma^j \approx \sigma^i \int_{\mathbb{T}^d} J(x^i, y)\,s(y)dy,$$

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- We get

$$f(\beta) = \beta^{-1}\int_{\mathbb{T}^d} \log\left(2\cosh\left(\beta\left(H(x) + \int_{\mathbb{T}^d} J(x,y)\, s^*(y)\, dy\right)\right)\right) dx,$$

  where consistency equation becomes

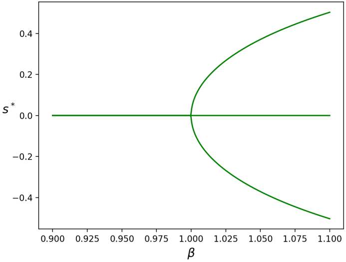
$$s^*(x) = \tanh\left(\beta\left(H(x) + \int_{\mathbb{T}^d} J(x,y)\, s^*(y)\, dy\right)\right).$$

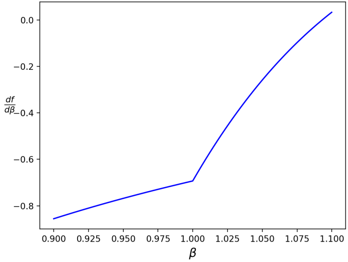
Ising Mean Field Solution

- Simplest model when $H = 0$ and J are constant. Then s is constant and

$$s = \tanh\left(\beta\, J\, s\right).$$

$s = 0$ is always a solution. A bifurcation occurs when $H = 0$ and $\beta\, J = 1$. The ferromagnetic phase exists when $\beta\, J > 1$.





Spin Games

- We now consider that each spin, $\Sigma_t^i \in \{\pm 1\}$ a random variable, corresponds to a player, that chooses the rate the spin flips $A^i(\mathbf{\Sigma}_t)$ to minimize the

$$\text{ergodic cost of player } i: \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\Big[\int_0^T \Big(L(A^i(\mathbf{\Sigma}_t)) + f^{N,i}(\mathbf{\Sigma}_t)\Big)dt\Big].$$

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- Player positions on a lattice, $x^i \in \mathbb{T}^d$. Interaction cost has a 'mean field' formulation for the empirical measure

$$m_{\boldsymbol{\sigma}} = \frac{1}{N}\sum_{i=1}^N \delta_{\sigma^i}\,\delta_{x^i} \in \mathcal{P}(\{\pm 1\}\times\mathbb{T}^d).$$

- Mean field interactions:

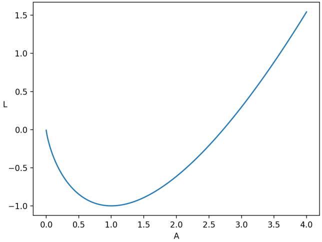
$$f^{N,i}(\boldsymbol{\sigma}) = f(\sigma^i, x^i, m_{\boldsymbol{\sigma}}).$$

The Ising Game

The Ising game control cost is:

- $$L(A) = \beta^{-1} A\big(\log(A) - 1\big).$$

Minimized at $A = 1$.
This incentivises *neutrality*.



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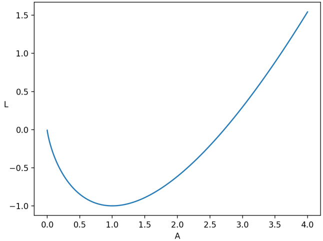

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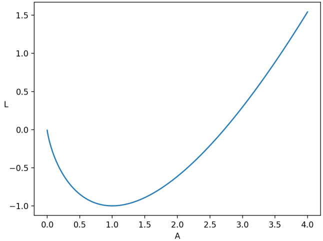

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- Mean field cost, empirical measure $m \in \mathcal{P}(\{-1,1\} \times \mathbb{T}^d)$:

$$f(\sigma, x, m) = -H(x)\,\sigma - \int_{\mathbb{T}^d} J(x,y)\,\sigma\,\Big(m(1,dy) - m(-1,dy)\Big)$$

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$$\begin{aligned} 0 &= l(x) + h(u(-\sigma,x) - u(\sigma,x)) - f(\sigma,x,m) \\ a(\sigma,x) &= h'(\sigma,x,u(-\sigma,x) - u(\sigma,x)) \\ a(-1,x)\hat{m}(-1,x) &= a(1,x)\hat{m}(1,x) \\ 1 &= \hat{m}(1,x) + \hat{m}(-1,x). \end{aligned}$$

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- In the variables $s(x) = \hat{m}(1, x) - \hat{m}(-1, x)$ and $b(x) = u(1, x) - u(-1, x)$ this simplifies to:
$$\begin{aligned} \beta^{-1} \sinh\left(\beta\, b(x)\right) &= H(x) + \int_{\mathbb{T}^d} J(x, y) s(y) dy \\ s(x) &= \tanh\left(\beta\, b(x)\right). \end{aligned}$$

Ising MFG system solution

More explicitly, let $H = 0$ and $J(x, y) = J$, so s and b are constant.

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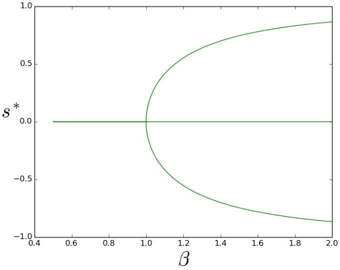
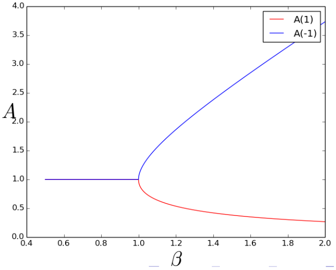
- There is always a solution with $s^* = b^* = 0$, $A(1) = A(-1) = 1$, cost: $I = -\beta^{-1}$.
- The other solutions exist when $\beta\, J > 1$, cost: $I = -J$ and

$$s^* = \pm\sqrt{1 - \beta^{-2} J^{-2}}, \qquad A(\pm 1) = \beta\, J(1 \mp s^*).$$





Ising Master Equation

- The master equation for $v(\sigma, x, m)$ is

$$0 = l(x) + \sum_{\gamma \in \mathbb{S}} \int_{\mathbb{T}^d} \Big[A(\gamma, y, m) \partial v(\sigma, x, m)(\gamma, y) \Big] m(\gamma, dy)$$
$$+ h(v(-\sigma, x, m) - v(\sigma, x, m)) - f(\sigma, x, m),$$

  where ∂v is like a discrete Wasserstein gradient.
  $A(\gamma, y, m) = h'(v(-\gamma, y, m) - v(\gamma, y, m))$.
- Fix H and J constant, $s = \int_{\mathbb{T}^d} m(1, dx) - \int_{\mathbb{T}^d} m(-1, dx)$, this reduces to a nonlinear transport equation for $b(s) = v(1, x, m) - v(-1, x, m)$:

$$0 = -\beta^{-1} \sinh(\beta\, b(s)) + \frac{1}{4} \Big[\sinh(\beta\, b(s)) - s \cosh(\beta\, b(s)) \Big] b'(s) + H + J\, s.$$

Ising Potential Game

- It is easier to solve as a potential game. $v(\sigma, x, m) = DV(m)(\sigma, x)$ where V solves

$$0 = \lambda + \sum_{\sigma \in \{\pm 1\}} \int_{\mathbb{T}^d} h(\partial V(m)(\sigma, x))\, m(\sigma, dx) - F(m)$$

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  and $DF(m)(\sigma, x) = f(\sigma, x, m)$.
- Using that $b(s) = \partial V(m)(\sigma, x)$ we find the solution has the form:

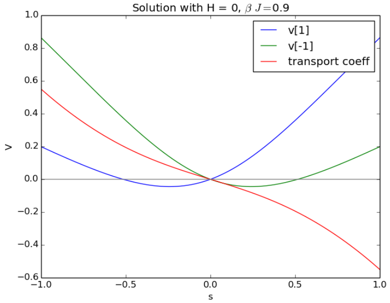
$$\sinh(\beta\, b(s)) = \frac{s\, B \pm \sqrt{B^2 - \beta^{-2}(1 - s^2)}}{\beta^{-1}(1 - s^2)},$$

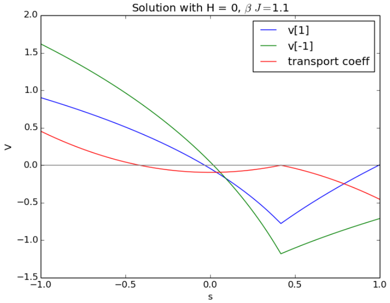
  where $B = \big(- H\, s - \frac{J\, s^2}{2} - \lambda\big)$and λ (the ergodic constant for the potential game) has to be determined.

Analysis of master equation solution

Theorem ("")

*When $H \neq 0$ or $\beta J < 1$ there is a unique continuous solution that is C^1. When $H = 0$ and $\beta J = 1$ there is a unique continuous solution that is not C^1.
When $H = 0$ and $\beta J > 1$ there are two continuous solutions (not C^1) and a unique 'correct' solution that is discontinuous.*





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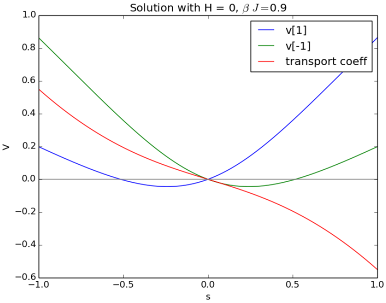

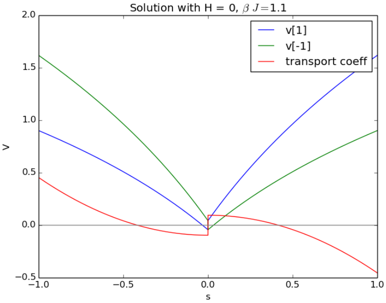

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  (*The Dyson and Coulomb games* [Carmona-Cerenzia-P] finds the "universal local limit" behavior of the N player game can still be relevant to and reflected in the convergence of the equations characterizing Nash optimality)
- Techniques from spin systems (which we have not used) include: transfer matrix, renormalization group, cluster expansion, ...

Thank you!