After Abel

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Braids, Polynomials and Hilbert's 13th Problem
IPAM
February 18, 2019

Find and understand roots of a polynomial

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

Algebraic Functions

18th/19th century perspective

View

$$(a_1, \ldots, a_n) \mapsto \{z \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0\}$$

as a (multi-valued) function in the a_is.

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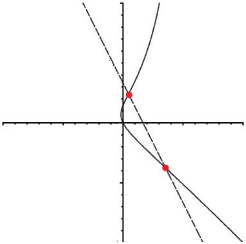
as a (multi-valued) function in the a_is.
More generally: $a_i \in \mathbb{C}(X)$.

Algebraic Functions

Many beautiful examples!



By F. Shen, W. Wang

Algebraic Functions

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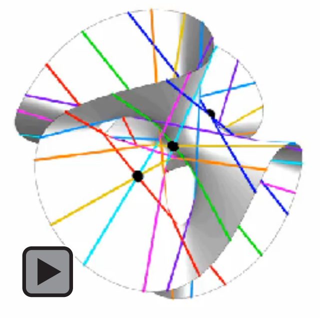

Gif by Greg Egan

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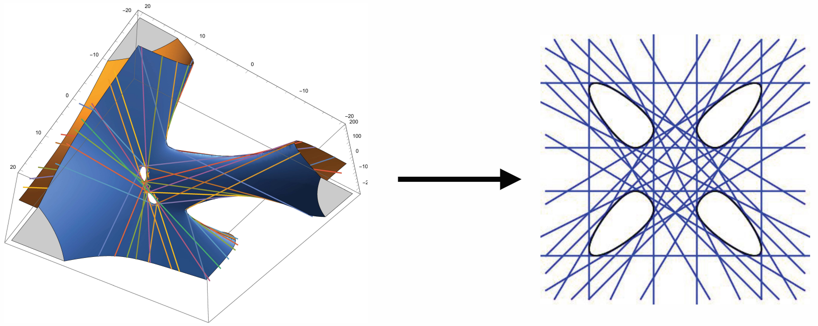

LHS by Shutt-Shioda, RHS by Plauman-Sturmfels-Vinzant

Algebraic Functions

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Gif by Aldoaldoz, Wikimedia Commons

When nature hands you an algebraic function . . .

You want to do two things:

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1. Construct the simplest formula you can.

When nature hands you an algebraic function . . .

You want to do two things:

1. Construct the simplest formula you can.
2. Prove no simpler formula is possible.

Example

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Theorem (Babylonians)

Let $\sqrt{a} := \{z \mid z^2 - a = 0\}$. *Then*

$$U_2(b, c) = \frac{-b + \sqrt{b^2 - 4c}}{2}.$$

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Theorem (Pythagoreans)

There is no formula for the function $U_2(b, c)$ *using only rational functions.*

Example 2

Theorem (Abel, 1824)

There is no formula for the general quintic in radicals.

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There is a formula for the general quintic in $\sqrt{-}$, $\sqrt[3]{-}$, *and*

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Example 3

Theorem (Klein-Burkhardt)

There is a formula for a line on a general smooth cubic surface using only $\sqrt[d]{-}$ *and the 3-variable algebraic function which assigns to a general abelian surface a 3-torsion point.*

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Theorem (Farb-Kisin-W)

The problems of finding a line on a cubic surface and a 3-torsion point on an abelian surface are equivalent.

Example 2 revisited

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Theorem (Kronecker, 1861)

There is no formula for the general quintic using only rational functions of the roots ("natural irrationalities") and algebraic functions of 1-variable.

Solutions of Higher Algebraic Equations

"Formerly the "solution of an algebraic equation" used to mean its solution by radicals. . . . Even at the present time, such ideas are still sometimes found prevailing; and yet, ever since the year 1858, a very different point of view should have been adopted." *(Felix Klein, 1893)*

Algebraic Functions

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Alg. function Φ on $X \rightsquigarrow$

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$$\begin{tikzcd} E_\Phi \arrow[r, "(x{,}z)\mapsto z"] \arrow[d, "(x{,}z)\mapsto x"'] & \mathbb{P}^1 \\ X & \end{tikzcd}$$

Natural v. accessory irrationalities

Rules of the game

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$$\pi_1(E_\Psi) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Mon}(E_\Phi \longrightarrow X)$$

is surjective.

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Corollary (Klein, 1884)

To write a formula for the general quintic in 1-variable functions, you must adjoin an **accessory** *square root.*

Solutions of Higher Algebraic Equations

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1900: Hilbert's Challenge

SUR LES

PROBLÈMES FUTURS DES MATHÉMATIQUES,

PAR M. DAVID HILBERT (Göttingen),

TRADUITE PAR M. L. LAUGEL (1).

Qui ne soulèverait volontiers le voile qui nous cache l'avenir afin de jeter un coup d'œil sur les progrès de notre Science et les secrets de son développement ultérieur durant les siècles futurs? Dans ce champ si fécond et si vaste de la Science mathématique, quels seront

1900: Hilbert's Challenge

Passing to algebra, I shall mention a problem from the theory of equations and one to which the theory of algebraic invariants has led me.

13. Impossibility of the solution of the general equation of the 7th degree by means of functions of only two arguments

Nomography[28] deals with the problem: to solve equations by means of drawings of families of curves depending on an arbitrary parameter. It is seen at once that every root of an equation whose coefficients depend upon only two parameters, that is, every function of two independent variables, can be represented in manifold ways according to the principle lying at the foundation of nomography. Further, a large class of functions of three or more variables can evidently be represented by this principle alone without the use of variable elements, namely all those which can be generated by forming first a function of two arguments, then equating each of these arguments to a function of two arguments, next replacing each of those arguments in their turn by a function of two arguments, and so on, regarding as admissible any finite number of insertions of functions of two arguments. So, for example, every rational function of any number of arguments belongs to this class of functions constructed by nomographic tables; for it can be generated by the processes of addition, subtraction, multiplication and division and each of these processes produces a function of only two arguments. One sees easily that the roots of all equations which are solvable by radicals in the natural realm of rationality belong to this class of functions; for here the extraction of roots is adjoined to the four arithmetical operations and this, indeed, presents a function of one argument only. Likewise the general equations of the 5th and 6th degrees are solvable by suitable nomographic tables; for, by means of Tschirnhausen transformations, which require only extraction of roots, they can be reduced to a form where the coefficients depend upon two parameters only.

1900: Hilbert's Challenge

Problem (Hilbert's 13th)

"Prove that the equation of the seventh degree $f^7 + xf^3 + yf^2 + zf + 1 = 0$ is not solvable with the help of any continuous functions of only two arguments."

1900: Hilbert's Challenge

"I may be allowed to add that I have satisfied myself by a rigorous process that there exist analytical functions of three arguments x, y, z which cannot be obtained by a finite chain of functions of only two arguments."

(David Hilbert, 1900)

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- One of only 10 actually presented at the Universal Exposition! (1, 2, 6, 7, 8, 13, 16, 19, 21, and 22)

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- Looks to topology as the *reason* for impossibility of solutions!

Observation (Riemann, Poincare)

Global structure of alg. functions determined by topology!

Hilbert's 13th Problem

Fundamental insight of 19th century

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$\vdots$

Hilbert's 13th Problem

"At the present time, there appear very many topological papers, but when it comes to fundamental problems, we have hardly gone beyond Poincaré, or strictly speaking, Riemann, this in spite of the fact that such progress would be of great significance, for among other things, the theory of algebraic functions of two variables."

(Max Dehn, 1928)

Resolvent degree

Classical formulation

$$\mathrm{RD}(n) := \min\{d \mid \exists \text{ formula for roots of general deg. } n \text{ poly} \\ \text{in alg. fns of } \leq d \text{ variables}\}$$

Resolvent degree

State of current knowledge

Theorem

1. $\mathrm{RD}(5) = 1$ *(Bring, Klein).*

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5. $\mathrm{RD}(9) \leq 4$ *(Hilbert).*

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Conjecture (Hilbert)

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More generally:

Conjecture

As $n \longrightarrow \infty$, $\mathrm{RD}(n) \longrightarrow \infty$.

Resolvent degree

State of current knowledge

Question

Is $\mathrm{RD}(n) \equiv 1$*?*

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We don't know!

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History hasn't helped.

Theorem (Arnol'd–Kolmogorov, 1957)

Let $f : [0,1]^n \longrightarrow \mathbb{R}$ be a continous function. Then there exist functions $g_i, \phi_{ij} : [0,1] \longrightarrow \mathbb{R}$ such that

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"Continuous functions of more than one variable do not exist."

AK and H13

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"Arnold and Kolmogorov's [theorem] killed Hilbert's problem." *(Jacques Dixmier, 1993)*

AK and H13

"The problem remains open, and by the highest standards, the range of issues is in fact as broad as it was at the beginning of the 20th century."

(Anatolii Georgievich Vitushkin, 2004)

AK and H13

"I think the representation [of Hilbert's degree 7] remains impossible even . . . by non-holomorphic complex functions which are topologically *equivalent to algebraic ones."*

(Vladimir Arnold, 2006)

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Thank you for coming!

Enjoy the workshop!