Inverse problems for the Einstein equation and other non-linear hyperbolic equations

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in collaboration with

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We will consider inverse problems for non-linear wave equations, e.g.

$$\frac{\partial^2}{\partial t^2} u(t,y) - c(t,y)^2 \Delta u(t,y) + a(t,y) u(t,y)^2 = f(t,y).$$

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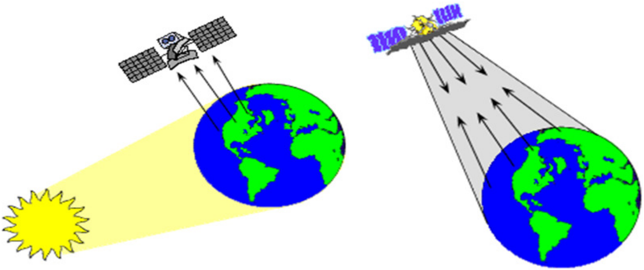
We will show that:

-Non-linearity helps to solve the inverse problem,

-"Scattering" from the interacting wave packets determines the structure of the spacetime.

Outline:

- Passive measurements with point sources
- Inverse problem for non-linear wave equation
- Einstein-scalar field equations
- Ideas of the proofs

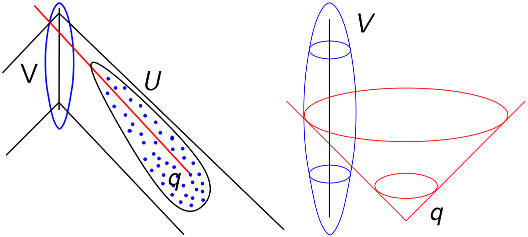


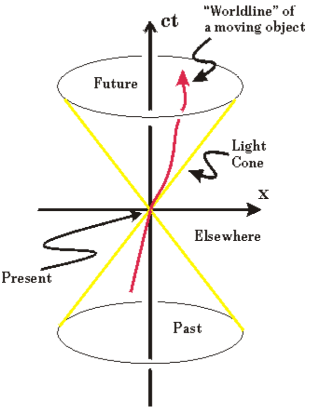
Passive sensors Active sensors

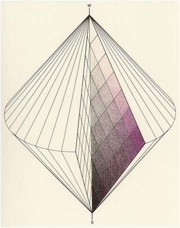
Inverse problem with passive observations

Passive imaging problem: There is a large number of point sources in a subset U of a spacetime M. The light from these point sources are observed in a set V. Do these observations determine the structure of the space time in U?

We will show: Let the sets V and U be as in the figure below. Assume that U contains a dense set of point sources q_j, $j \in \mathbb{Z}_+$. We observe the intersections of the light cones emanating from the points q_j and the set V so that the light coming from different sources can be separated (i.e. the sources q_j have different spectra). Then we can reconstruct the set U as a differentiable manifold and the metric $g|_U$ up to a scalar factor.







Definitions

Let (M, g) be a Lorentzian manifold,

$\gamma_{x,\xi}(t)$ is a geodesic with the initial point (x, ξ)

$\xi \in T_x M$ is time-like if $g(\xi, \xi) < 0$,

$\xi \in T_x M$ is light-like if $g(\xi, \xi) = 0$, $\xi \neq 0$.

$L_x^+ M \subset T_x M$ is the future light cone.

$J^+(p) = \{x \in M |\ x \text{ is in causal future of } p\}$,

$J^-(p) = \{x \in M |\ x \text{ is in causal past of } p\}$.

(M, g) is globally hyperbolic if

there are no closed causal curves and the set

$J^+(p_1) \cap J^-(p_2)$ is compact for all $p_1, p_2 \in M$.

Then M can be represented as $M = \mathbb{R} \times N$.

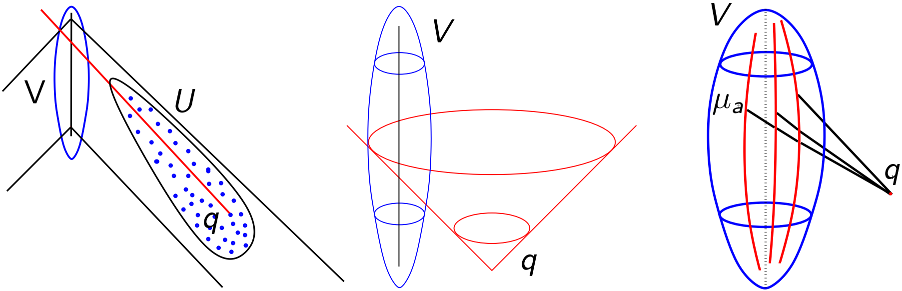
More definitions

Let $A \subset \mathbb{R}^m$ be open and $\mu_a : (-1,1) \to M$, $a \in A$ be a family of time-like geodesics such that $V = \bigcup_{a \in A} \mu_a(-1,1)$ is open. We consider observations in V. Let $p^-, p^+ \in \mu_{a_0}$.

Let $U \subset J^-(p^+) \setminus J^-(p^-)$ be an open, relatively compact set.

The observation time function $F_q : A \to \mathbb{R}$ for a point $q \in U$ is

$$F_q(a) = \inf\{s \in \mathbb{R} \quad ; \quad \text{there is a future-directed light-like geodesic from } q \text{ to } \mu_a(s)\}$$

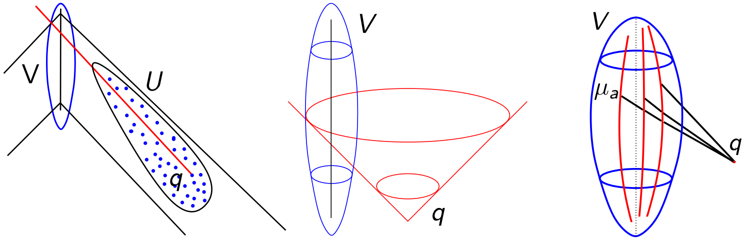


Theorem (Kurylev-L.-Uhlmann 2018 (Arxiv 2014))

Let (M,g) be a globally hyperbolic Lorentzian manifold of dimension $n \geq 3$. Assume that $\mu_a(-1,1) \subset M$, $a \in A \subset \mathbb{R}^m$ are time-like geodesics, $V = \cup_{a \in A} \mu_a$ is open, and $p^-, p^+ \in \mu_{a_0}$. Let $U \subset J^-(p^+) \setminus J^-(p^-)$ be a relatively compact open set. Then $(V, g|_V)$ and the collection of the observation time functions,

$$\mathcal{F}_U = \left\{ F_q : A \to \mathbb{R} \;\middle|\; q \in U \right\} \subset C(A),$$

determine the set U, up to a change of coordinates, and the conformal class of the metric g in U.



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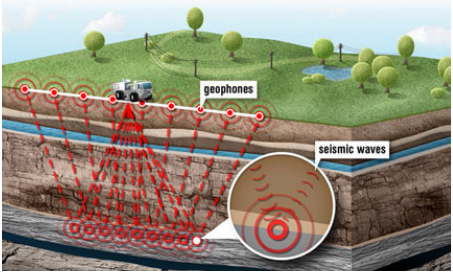
- Einstein-scalar field equations
- Ideas of the proofs

Some results for hyperbolic inverse problems for linear equations:

- Nachman-Sylvester-Uhlmann 1988: Inverse problem for $\Delta + q$.
- Belishev-Kurylev 1992 and Tataru 1995: Reconstruction of a Riemannian manifold with a time-indepedent metric. The used unique continuation fails for non-real-analytic time-depending coefficients (Alinhac 1983).
- Eskin 2017: Wave equation with a time-depending metric that is real-analytic in the time variable.
- Helin-L.-Oksanen 2012: Combining several measurements for together for the wave equation.

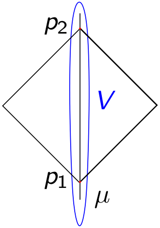


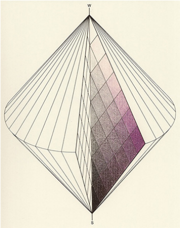
Theorem (Kurylev-L.-Uhlmann '18, L.-Uhlmann-Wang '17)

Let (M, g) be a globally hyperbolic Lorentzian manifold, $\dim(M) = 4$, $m \geq 2$, $\mu \subset M$ be a time-like curve, $p_1, p_2 \in \mu$ and V be a neighbourhood of μ. Let $L_V : f \mapsto u|_V$ be the source-to-solution map for

$$\Box_g u + u^m = f \quad \text{in } (-\infty, T) \times N \subset M,$$
$$u = 0 \quad \text{in } t = x^0 < 0.$$

L_V is defined for small sources f, $supp(f) \subset V$. Then V and L_V determine the manifold $J^+(p_1) \cap J^-(p_2)$ and the conformal class of g on it. If $m \neq 3$, the metric tensor g can be determined.





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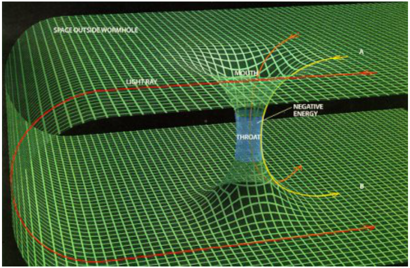
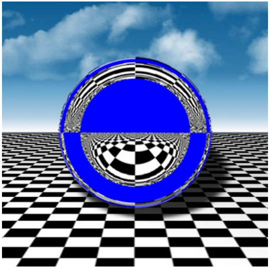




Figures: Anderson institute and Greenleaf-Kurylev-L.-Uhlmann

Einstein equations

The Einstein equation for the $(-,+,+,+)$-type Lorentzian metric g_{jk} of the space time is

$$\mathrm{Ein}_{jk}(g) = T_{jk},$$

where

$$\mathrm{Ein}_{jk}(g) = \mathrm{Ric}_{jk}(g) - \frac{1}{2}(g^{pq}\,\mathrm{Ric}_{pq}(g))g_{jk}.$$

In wave map coordinates, the Einstein equation yields a quasilinear hyperbolic equation and a conservation law,

$$g^{pq}(x)\frac{\partial^2}{\partial x^p \partial x^q} g_{jk}(x) + B_{jk}(g(x), \partial g(x)) = T_{jk}(x),$$
$$\nabla_p(g^{pj}\,T_{jk}) = 0.$$

To consider active measurements, we add matter fields.
We consider the coupled Einstein and scalar field equations with sources,

$$\begin{aligned} &\mathrm{Ein}(g) = T, \quad T = \mathbf{T}(\phi, g) + \mathcal{F}_1, \quad \text{on } (-\infty, t_0) \times N, \\ &\square_g \phi_\ell - m^2 \phi_\ell = \mathcal{F}_2^\ell, \quad \ell = 1, 2, \dots, L, \\ &g|_{t<0} = \widehat{g}, \quad \phi|_{t<0} = \widehat{\phi}. \end{aligned} \tag{1}$$

Here, $\widehat{g}$ and $\widehat{\phi}$ are C^∞-smooth background solutions that satisfy equations (1) with the zero sources. Moreover,

$$\mathbf{T}_{jk}(g, \phi) = \sum_{\ell=1}^{L} \partial_j \phi_\ell \, \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \, \partial_q \phi_\ell - \frac{1}{2} m^2 \phi_\ell^2 g_{jk}.$$

To obtain a physically meaningful model, the stress-energy tensor T needs to satisfy the conservation law

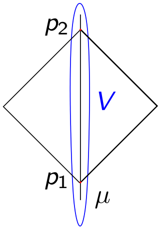
$$\nabla_p (g^{pj} T_{jk}) = 0, \quad k = 1, 2, 3, 4. \tag{2}$$

Let $V \subset M$ be an open neighbourhood of a time-like geodesics μ on $(M, \widehat{g})$ and $L \geq 4$.

Condition A: Assume that at any $x \in V$ the 4×4 matrix

$$A(x) = \left[(\partial_j \widehat{\phi}_\ell(x))_{\ell,j=1}^4 \right]$$

is invertible.



Inverse problem with active measurements:
Assume that Condition A is valid. Do the observations analogous to a source-to-solution map in V determine the manifold $J^+(p^-) \cap J^-(p^+)$ and the metric $\widehat{g}$ in it?

To answer to this question, we have to guarantee that the condition $\nabla_p (g^{pj} T_{jk}) = 0$ is valid for all solutions of the Einstein-scalar field equations (1).

A formulation of measurements with secondary sources

We can formulate the direct problem for the Einstein-scalar field equations so the the conservation law is valid.
Let g and $\phi = (\phi_\ell)_{\ell=1}^{L}$ satisfy

$$
\begin{aligned}
&\mathrm{Ein}_{jk}(g) = P_{jk} + \mathbf{T}_{jk}(g,\phi), \quad \text{on } (-\infty, t_0) \times N,\\
&\square_g \phi_\ell - m^2 \phi_\ell = S_\ell, \quad \ell = 1,2,3,\dots,L,\\
&S_\ell = Q_\ell + \mathcal{S}_\ell^{2nd}(g,\phi,\nabla\phi,Q,\nabla^g Q,P,\nabla^g P),\\
&g\big|_{t<0} = \widehat{g}, \quad \phi\big|_{t<0} = \widehat{\phi}.
\end{aligned}
$$

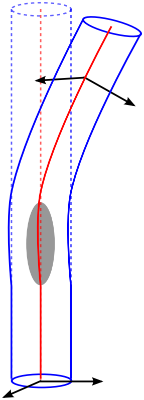
Here Q_ℓ and P_{jk} are considered as the primary sources.
The functions $\mathcal{S}_\ell^{2nd}$ need to be constructed so that the conservation law is satisfied for the solutions (g,ϕ). These functions correspond to a model for a measurement device.
When Condition A is satisfied, secondary source functions $\mathcal{S}_\ell^{2nd}$ can be constructed, for small Q and P, by solving a pointwise system of linear equations.

The data set

We define the measured data in Fermi coordinates associated to a freely falling observer. For $\delta > 0$ small enough, we define

$$\begin{aligned}\mathcal{D}(\delta) \quad = \quad & \{(\Psi_g^* g|_{\widetilde{V}}, \Psi_g^* \phi|_{\widetilde{V}}, \Psi_g^* \mathcal{F}|_{\widetilde{V}}) : \ \mathcal{F} \in C^4(M), \\ & \|\mathcal{F}\|_{C^4(M)} < \delta, \ \ \text{supp}\,(\mathcal{F}) \subset V_g, \\ & (g, \phi, \mathcal{F}) \text{ satisfy the Einstein-scalar equations (1)} \\ & \text{and } \nabla_p(g^{pj} T_{jk}) = 0\},\end{aligned}$$

where $\Psi_g : \widetilde{V} \subset \mathbb{R}^4 \to V_g \subset V$ are the Fermi coordinates of a time-like geodesic $\mu_g([0,1])$ in metric g (a freely falling observe on (M, g)).



Theorem

(Kurylev-L-Oksanen-Uhlmann'16,Uhlmann-Wang'20)

Let $(M^{(i)}, \widehat{g}^{(i)})$, $i = 1, 2$ be 4-dimensional globally hyperbolic Lorentzian manifolds that satisfy the Einstein-scalar field equations with $\widehat{\phi}^{(i)}$ and vanishing sources. Assume condition (A). Let $\widehat{\mu}^{(i)}$ be time-like geodesics and $p_\pm^{(i)} = \widehat{\mu}^{(i)}(s_\pm)$ with $s_- < s_+$ and $V^{(i)}$ be neighborhoods of $\widehat{\mu}^{(i)}$. Consider the Einstein-scalar field systems

$$\begin{aligned} &Ein(g) = T, \quad T = \mathbf{T}(\phi, g) + \mathcal{F}_1, \quad on\ (-\infty, t_0) \times N, \\ &\Box_g \phi_\ell - m^2 \phi_\ell = \mathcal{F}_2^\ell, \quad \ell = 1, 2, \dots, L, \\ &g|_{t<0} = \widehat{g}, \quad \phi|_{t<0} = \widehat{\phi}. \end{aligned} \tag{3}$$

If $\mathcal{D}^{(1)}(\delta) = \mathcal{D}^{(2)}(\delta)$, then there is a diffeomorphism

$$\Psi : J^+_{\widehat{g}^{(1)}}(p_-^{(1)}) \cap J^-_{\widehat{g}^{(1)}}(p_+^{(1)}) \to J^+_{\widehat{g}^{(2)}}(p_-^{(2)}) \cap J^-_{\widehat{g}^{(2)}}(p_+^{(2)}) \tag{4}$$

such that $\Psi^* \widehat{g}^{(2)} = \widehat{g}^{(1)}$ in $J^+_{\widehat{g}^{(1)}}(p_-^{(1)}) \cap J^-_{\widehat{g}^{(1)}}(p_+^{(1)})$.

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Non-linear wave equation in space-time

Let $M = \mathbb{R} \times N$, $\dim(M) = 4$. Consider the equation

$$\Box_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } M_1 = (-\infty, T) \times N,$$
$$u(x) = 0 \quad \text{for } x = (x^0, x^1, x^2, x^3) \in (-\infty, 0) \times N,$$

where

$$\Box_g u = \sum_{p,q=0}^{3} |\det(g(x))|^{-\frac{1}{2}} \frac{\partial}{\partial x^p} \left(|\det(g(x))|^{\frac{1}{2}} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right)$$

and $a(x)$ is a non-vanishing C^∞-smooth function.

Alternative model:

$$\frac{\partial^2}{\partial t^2} u(t,y) - c(t,y)^2 \Delta u(t,y) + a(t,y) u(t,y)^2 = f(t,y), \quad x = (t,y).$$

Inverse problem for non-linear wave equation

Consider the equation

$$\begin{aligned} \square_g u(x) + a(x)\, u(x)^2 &= f(x) \quad \text{on } M_1 = (-\infty, T) \times N, \\ u(x) = 0 \quad &\text{for } x \in (-\infty, 0) \times N, \end{aligned}$$

where the source $f \in C_0^6(V)$ is supported in an open set $V \subset M_1$.

In a neighborhood $\mathcal{W} \subset C_0^6(V)$ of the zero-function we define the measurement operator (source-to-solution operator),

$$L_V : f \mapsto u|_V, \quad f \in \mathcal{W} \subset C_0^6(V).$$

Idea of the proof: Non-linear geometrical optics.

The non-linearity helps in solving the inverse problem.

Let $u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$ satisfy

$$\Box_g u + au^2 = f, \quad \text{on } M_1 = (-\infty, T) \times N,$$
$$u|_{(-\infty,0)\times N} = 0$$

with $f = \varepsilon f_1$, $\varepsilon > 0$.
When $Q = \Box_g^{-1}$, we have

$$\begin{aligned} w_1 &= Qf_1, \\ w_2 &= -Q(a\, w_1\, w_1), \\ w_3 &= 2Q(a\, w_1\, Q(a\, w_1\, w_1)), \\ w_4 &= -Q(a\, Q(a\, w_1\, w_1)\, Q(a\, w_1\, w_1)) \\ &\quad -4Q(a\, w_1\, Q(a\, w_1\, Q(a\, w_1\, w_1))), \\ \|E_\varepsilon\| &\le C\varepsilon^5. \end{aligned}$$

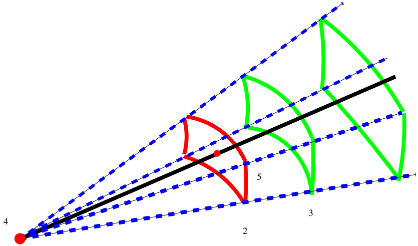
Interaction of waves in Minkowski space $\mathbb{R}^4$

Let x^j, $j = 1, 2, 3, 4$ be coordinates such that

$$K_j = \{x^j = 0\}, \quad j = 1, 2, 3, 4,$$

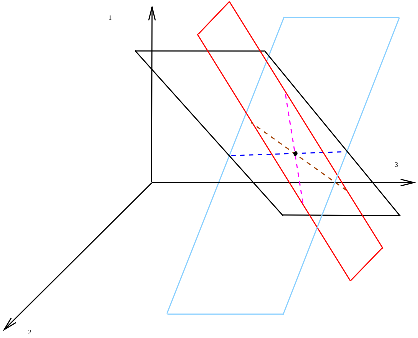
are light-like. We consider plane waves

$$u_j(x) = v \cdot (x^j)_+^m, \quad (s)_+^m = |s|^m H(s), \quad v \in \mathbb{R}, \; j = 1, 2, 3, 4.$$



The interaction of the waves $u_j(x)$ produce new sources on

$$\begin{aligned} K_{12} &= K_1 \cap K_2, \\ K_{123} &= K_1 \cap K_2 \cap K_3 = \text{line}, \\ K_{1234} &= K_1 \cap K_2 \cap K_3 \cap K_4 = \{q\} = \text{one point}. \end{aligned}$$



Interaction of two waves

If we consider sources $f_{\vec{\varepsilon}}(x) = \varepsilon_1 f_1(x) + \varepsilon_2 f_2(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, and the corresponding solution $u_{\vec{\varepsilon}}$ of the nonlinear wave equation, we have

$$W_2(x) = \frac{\partial}{\partial \varepsilon_1} \frac{\partial}{\partial \varepsilon_2} u_{\vec{\varepsilon}}(x) \Big|_{\vec{\varepsilon}=0} = \Box_g^{-1}(a\, u_1 \cdot u_2),$$

where $u_j = \Box_g^{-1} f_j$.

All light-like co-vectors in the normal bundle of $K_1 \cap K_2$ are in $N^* K_1 \cup N^* K_2$.
Thus no interesting singularities are produced by the interaction of two waves. (Greenleaf-Uhlmann '93)

Interaction of three waves

Consider sources

$$f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{3} \varepsilon_j f_j(x), \quad \vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3),$$

and let $u_{\vec{\varepsilon}}$ be the solution of the nonlinear wave equation, with source $f_{\vec{\varepsilon}}$.
We have

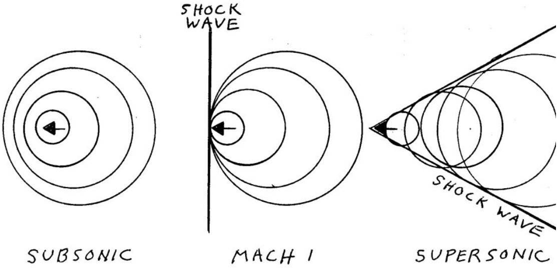
$$\begin{aligned} W_3 &= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec{\varepsilon}}\Big|_{\vec{\varepsilon}=0} \\ &= \square_g^{-1}(a\, u_1 \cdot \square_g^{-1}(a u_2 \cdot u_3)) + \dots \end{aligned}$$

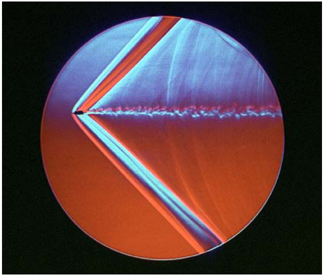
The interaction of the three waves happens on the line $K_{123} = K_1 \cap K_2 \cap K_3$ and produce new singularities.

Similar results in $\mathbb{R}^{1+2}$: Rauch-Reed '82 and Melrose-Ritter '85.

Interaction of waves:

The non-linearity helps in solving the inverse problem.
Artificial sources can be created by interaction of waves using the non-linearity of the wave equation.





The interaction of 3 waves creates a point source in space that seems to move with a speed higher than the speed of light, that is, it appears like a tachyonic point source, and produces a "shock wave" type singularity.

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Interaction of three waves.

Interaction of four waves

Consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{4} \varepsilon_j f_j(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, the corresponding solution $u_{\vec{\varepsilon}}$ of the non-linear wave equation, and

$$W_4 \quad = \quad \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec{\varepsilon}}(x)\Big|_{\vec{\varepsilon}=0}.$$

We have $K_{1234} = \{q\}$. Thus, when the four waves intersect, an artificial point source S_q appears at point q,

$$W_4 = \Box_g^{-1} S_q.$$

Here $S_q = B(x,D)\delta_q + r(x)$, where $B(x,D)$ is a pseudodifferential operator and the wavefront set of $r(x)$ is "small".

Interaction of four waves.

The 3-interaction produces conic waves (only one is shown below).

The 4-interaction produces

a spherical wave from the point q

that determines the

observation times $F_q(a)$.

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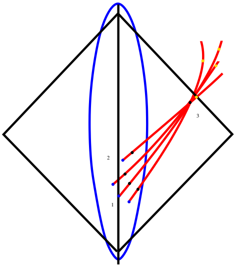

Thank you for your attention!