A New Approach to the Binary Inspiral Problem

IPAM, October 27, 2021

Zvi Bern

**ZB, C. Cheung, R. Roiban, C. H. Shen, M. Solon, M. Zeng,
arXiv:1901.04424 and arXiv:1908.01493.**

**ZB, A. Luna, R. Roiban, C. H. Shen, M. Zeng,
arXiv:2005.03071**

**ZB, J. Parra-Martinez, R. Roiban, E. Sawyer, C.-H. Shen,
arXiv 2010.08559**

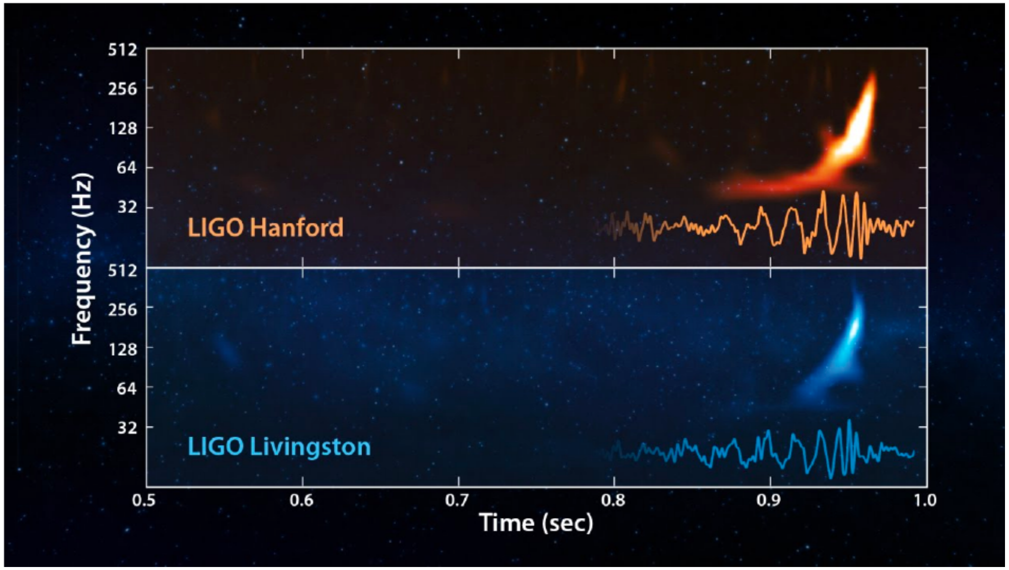
**ZB, J. Parra-Martinez, R. Roiban, M. Ruf. C.-H. Shen,
M. Solon, M. Zeng, arXiv:2101.07254**

UCLA Mani L. Bhaumik Institute for Theoretical Physics

Outline

See talk from Frans Pretorius

Era of gravitational-wave astronomy has begun.



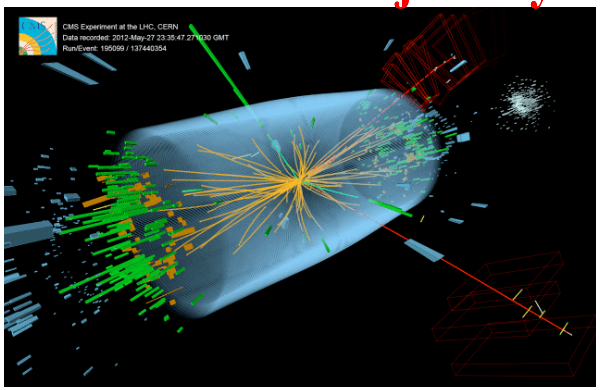
For an instant brighter in gravitational radiation than all the stars in the visible universe are in EM radiation!

How can we in theoretical particle-physics community, help out with core mission of LIGO/Virgo?

Can Particle Theory Help with Gravitational Waves?

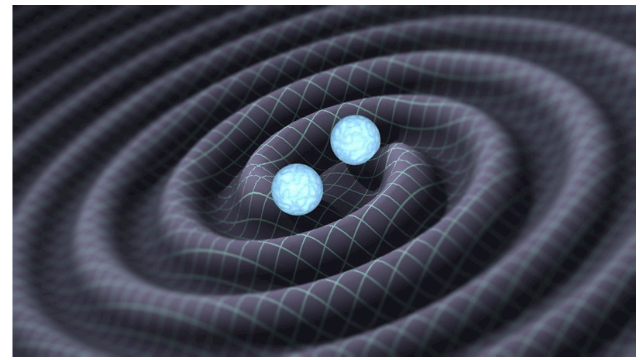
What does particle physics have to do with classical dynamics of astrophysical objects?

unbounded trajectory



gauge theories, QCD, electroweak
quantum field theory

bounded orbit



General Relativity
classical physics

Black holes and neutron stars are point particles as far as long-wavelength radiation is concerned.

Iwasaki (1971); Goldberger, Rothstein (2006), Porto; Vaydia, Foffa , Porto, Rothstein, Sturani; Kol; Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Levi, Steinhoff; Vines etc

Will explain that particle theory is well suited to push state-of-the-art perturbative calculations for gravitational-wave physics.

Approach to General Relativity

Our appoach does *not* start from usual Einstein Field equations.

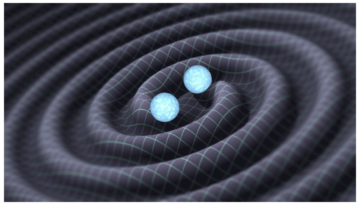
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

~~**geometry**~~

Richard Feynman

Gravitons are spin 2 particles

- **Not suited for all problems, but works well for asymptotically flat space-times in context of perturbation theory.**
- **Well suited for gravitational-wave physics from compact astrophysical objects**



Can Quantum Scattering Help with Gravitational Waves?

**In particle physics we are very good at perturbation theory.
Vast experience with gauge theories and supergravity theories.**

Two serious issues for applying this to gravitational waves:

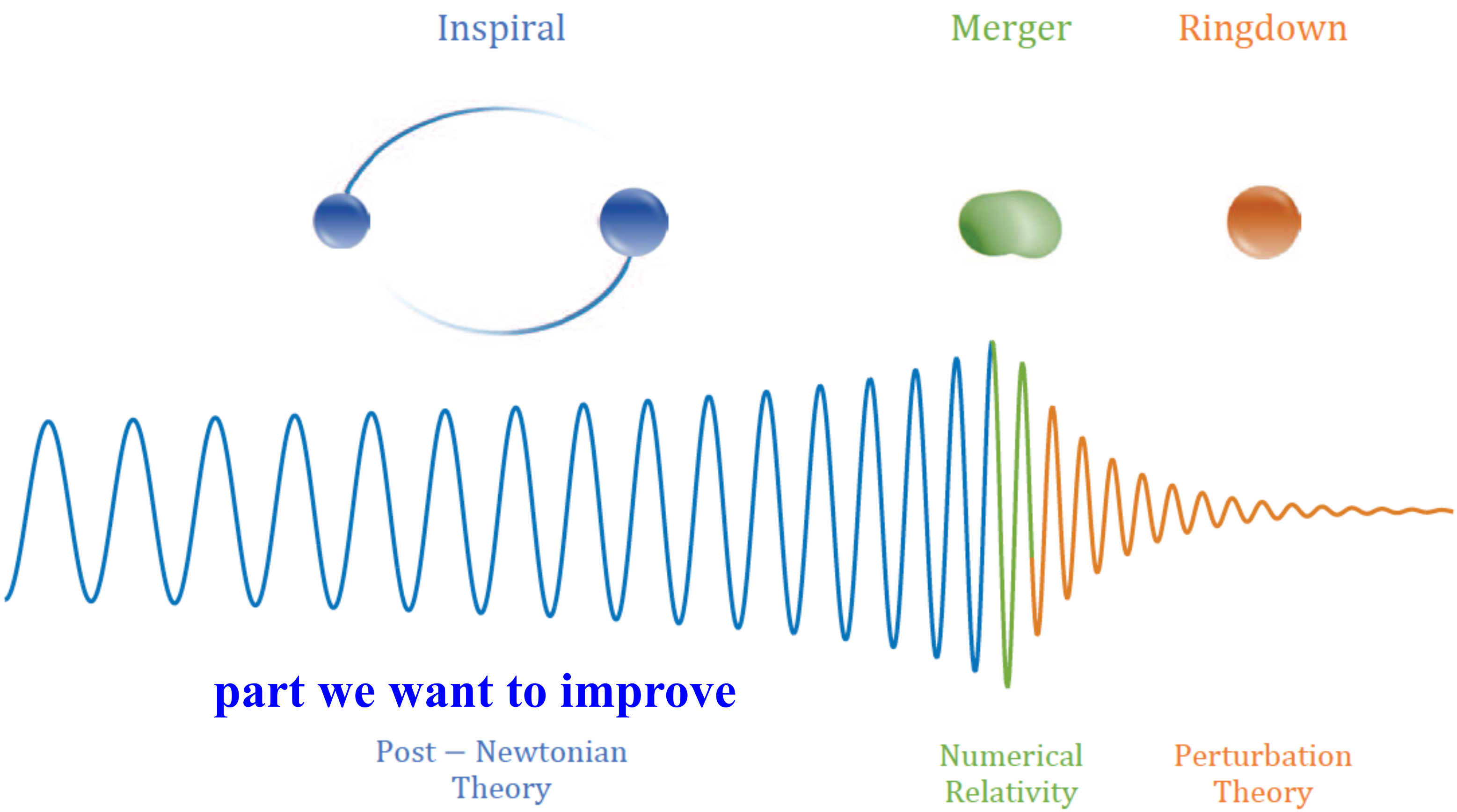
1. **We do quantum *not* classical perturbation theory.**
2. **Scattering process unbounded orbit. Want bounded one for binary black hole gravitational wave emission.**

Two key topics for this talk:

- **Modern approach to perturbative gravity.**
- **How do we effectively deal with the above annoying issues?**

Two Body Problem

From Antelis and Moreno, arXiv:1610.03567



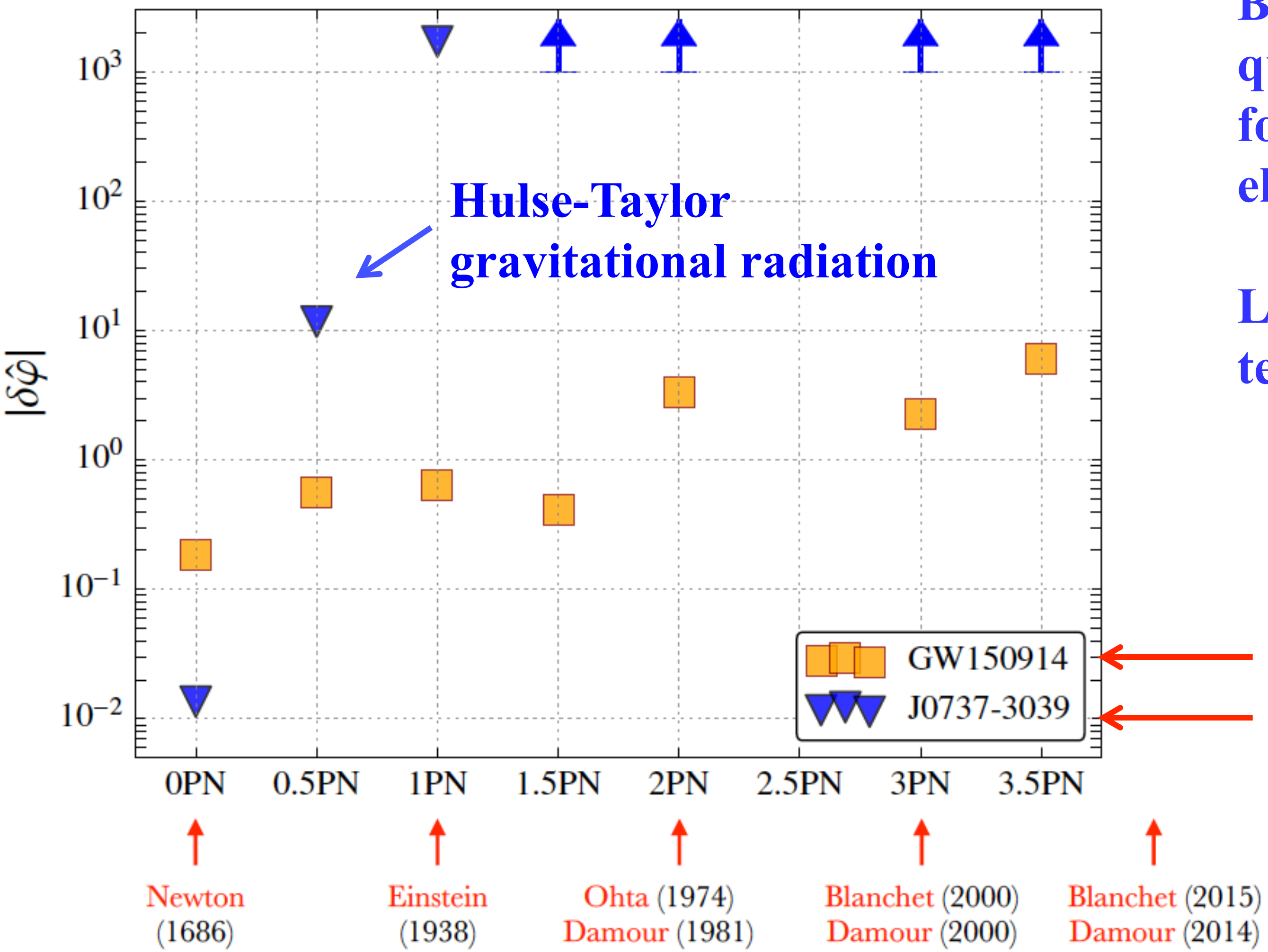
part we want to improve

- **Small errors accumulate. Need for high precision.**
- **Input to EOB or other modeling to reliably approach merger**

Buonanno and Damour

Importance of higher orders for LIGO/Virgo

LIGO/Virgo Collaboration arXiv:1602.03841



Binary pulsar confirms quadrupole radiation formula and not much else.

LIGO/Virgo tests PN terms from GR

LIGO/Virgo sensitive to high PN orders.

Post Newtonian Approximation

For orbital mechanics:

$$v^2 \sim \frac{GM}{R} \ll 1$$

Expand in G and v^2

virial theorem

In center of mass frame:

$$m = m_A + m_B, \quad \nu = \mu/M,$$

$$\mu = m_A m_B/m, \quad P_R = P \cdot \hat{R}$$

$$\frac{H}{\mu} = \frac{P^2}{2} - \frac{Gm}{R} \quad \leftarrow \text{Newton}$$

$$+ \frac{1}{c^2}\left\{ -\frac{P^4}{8} + \frac{3\,\nu\,P^4}{8} + \frac{Gm}{R}\left(-\frac{{P_R}^2\,\nu}{2} - \frac{3\,P^2}{2} - \frac{\nu\,P^2}{2} \right) + \frac{G^2 m^2}{2R^2} \right\}$$

$$+ \ldots$$

1PN: Einstein, Infeld, Hoffmann; Droste, Lorentz

Hamiltonian known to 4PN order.

2PN: Ohta, Okamura, Kimura and Hiida.

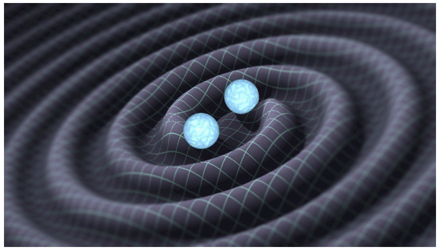
3PN: Damour, Jaranowski and Schaefer; L. Blanchet and G. Faye.

4PN: Damour, Jaranowski and Schaefer; Foffa (2017), Porto, Rothstein, Sturani (2019).

Which problem to solve?

ZB, Cheung, Roiban, Shen, Solon, Zeng

Some problems for (analytic) theorists:



1. Spin.
2. Finite size effects.
3. New physics effects.
4. Radiation.
5. → High orders in perturbation theory. ←

Which problem should we solve?

- Needs to be extremely difficult using standard methods.
- Needs to be of direct importance to LIGO theorists.
- Needs to be in a form that can in principle enter LIGO analysis pipeline.

2-body Hamiltonian at 3rd and 4th post-Minkowskian order

PN versus PM expansion for conservative two-body dynamics

From Buonanno Amplitudes 2018

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}}_{\downarrow} + \frac{1}{c^2}\left[\dots\right] + \frac{1}{c^4}\left[\dots\right] + \dots$$

$$E(v) = -\frac{\mu}{2}v^2 + \cdots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	...
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	...
...						...	...	

(credit: Justin Vines)

$$1 \to Mc^2, \qquad v^2 \to \frac{v^2}{c^2}, \qquad \frac{1}{r} \to \frac{GM}{rc^2}.$$

current known PN results — current known PM results — overlap between PN & PM results — unknown

- PM results (Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep)

High-energy gravitational scattering and the general relativistic two-body problem

Thibault Damour*
Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France
(Dated: October 31, 2017)

A technique for translating the classical scattering function of two gravitationally interacting bod-

tum gra... ...plitude of two particles, and we urge amplitude experts to use their novel techniques to compute the 2-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian

"… and we urge amplitude experts to use their novel techniques to compute the 2-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian."

arXiv:1710.10599v1 [gr-qc] 29 Oct 2017

Hard to resist an invitation with this kind of clarity!

The recent observation [1–4] of gravitational wave signals from inspiralling and coalescing binary black holes has been significantly helped, from the theoretical side, by the availability of a large bank of waveform templates, defined [5, 6] within the analytical effective one-body (EOB) formalism [7–11]. The EOB formalism combines in a su… results… with s… numer… (see [1… system… of bina… sults u… mostly based on the post-Newtonian (PN) approach to the general relativistic two-body interaction. The conservative two-body dynamics was derived, successively, at the second post-Newtonian (2PN) [14, 15], third post-

…ntly intro- … to derive … from the (gauge-invariant) *scattering function* Φ linking (half) the center of mass (c.m.) classical gravitational scattering angle χ to the total energy, $E_{\rm real} \equiv \sqrt{s}$, and the total angular momentum, J, of the system[1]

- **Difficult using standard methods.**
- **Of direct importance to LIGO/Virgo theorists.**
- **Can in principle enter LIGO/Virgo analysis pipeline.**

with

$M \equiv m_1 + m_2;\ \mu \equiv \frac{m_1 m_2}{\ldots};\ \nu \equiv \frac{\mu}{M} = \frac{m_1 m_2}{(m_1+m_2)^2}.$

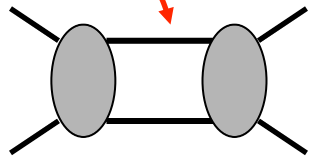
Generalized Unitarity Method

Use simpler tree amplitudes to build higher-order (loop) amplitudes.

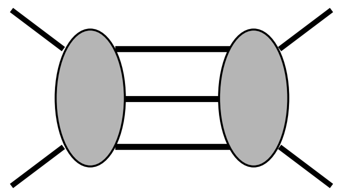
ZB, Dixon, Dunbar and Kosower (1994)

$$E^2 = \vec{p}^{\,2} + m^2$$ ← **on-shell**

Two-particle cut:



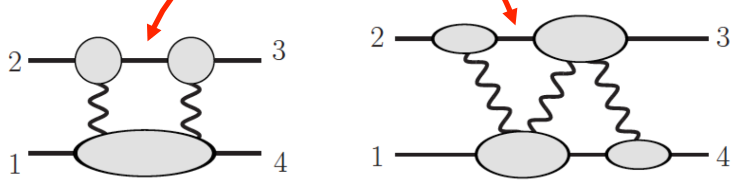
Three-particle cut:



- **Systematic assembly of complete loop amplitudes from tree amplitudes.**
- **Works for any number of particles or loops.**

Generalized unitarity as a practical tool for loops.

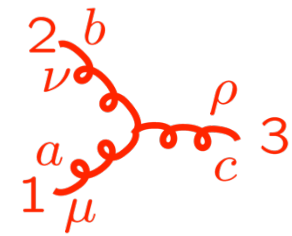
on-shell



ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger;
ZB, Carrasco, Johansson, Kosower
and many others

Idea used in the "NLO revolution" in QCD collider physics.
No gauge fixing in the formalism.

Standard perturbative approach:



Three-gluon vertex:

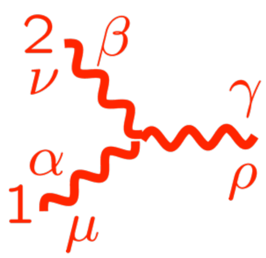
$$V^{abc}_{3\,\mu\nu\sigma} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

Three-graviton vertex:

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) =$$

$$\text{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})$$
$$+ P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})$$
$$+ P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})$$
$$+ 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]$$



About 100 terms in three vertex

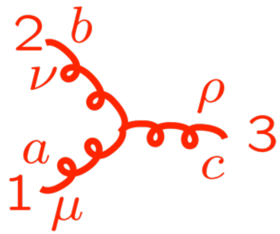
Naïve conclusion: Gravity more complicated than gauge theory.

Simplicity of Gravity Scattering Amplitudes

People were looking at gravity amplitudes the wrong way.

On-shell **three vertices contains all information:** $E_i^2 - \vec{p}_i^{\,2} = 0$

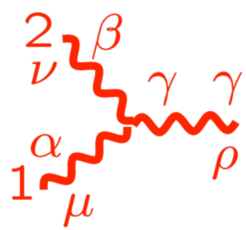
Yang-Mills (QCD) gauge theory:



color factor

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

Einstein gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

Starting from this on-shell vertex any multi-loop amplitude can be constructed via modern methods

Gravitons are like two gluons!

Gravity ~ (gauge theory)^2

KLT Relation Between Gravity and Gauge Theory

KLT (1985)

Kawai-Lewellen-Tye string relations in low-energy limit:

gravity

gauge-theory color ordered

$$M_4^{\rm tree}(1,2,3,4) = -is_{12} A_4^{\rm tree}(1,2,3,4)\, A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = i s_{12} s_{34} A_5^{\rm tree}(1,2,3,4,5)\, A_5^{\rm tree}(2,1,4,3,5) + i s_{13} s_{24} A_5^{\rm tree}(1,3,2,4,5)\, A_5^{\rm tree}(3,1,4,2,5)$$

Inherently gauge invariant!

Generalizes to explicit all-leg form. ZB, Dixon, Perelstein, Rozowsky

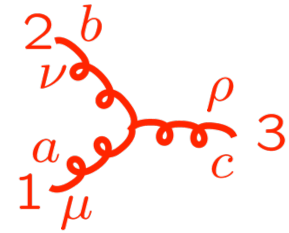
1. **Gravity amplitudes derivable from gauge theory.**
2. **Once gauge-theory amplitude is simplified, so is gravity.**
3. **Standard Lagrangian methods offer no hint why this is possible.**

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (2007)

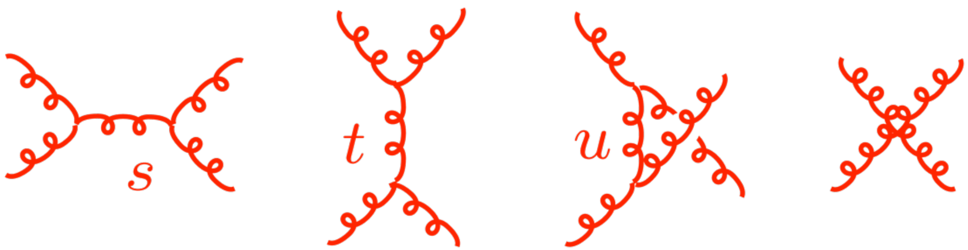
coupling constant → color factor → momentum dependent kinematic factor

$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (BCJ)

Consider five-point tree amplitude:

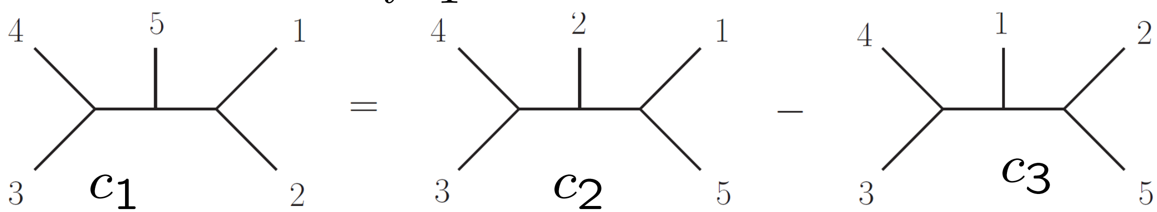
gauge theory

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

color factor (c_i)

kinematic numerator factor (n_i)

Feynman propagators ($\prod_{\alpha_i} p_{\alpha_i}^2$)



$$c_1 = f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} \qquad c_2 = f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5} \qquad c_3 = f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* algebraic constraint equations.

Proven at tree level

ZB, Carrasco, Johansson; Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Cachazo, etc

Gravity from Gauge Theory

ZB, Carrasco, Johansson

gauge theory (QCD):

$$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i \, n_i}{D_i}$$

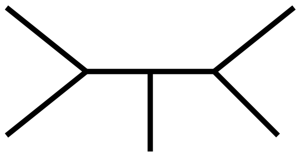
color factor (c_i)

kinematic numerator factor (n_i)

Feynman propagators (D_i)

$$c_k = c_i - c_j$$
$$n_k = n_i - n_j$$

$$c_i \to n_i$$



sum over diagrams with only 3 vertices

Einstein gravity:

$$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

Gravity and gauge theory kinematic numerators are the same!

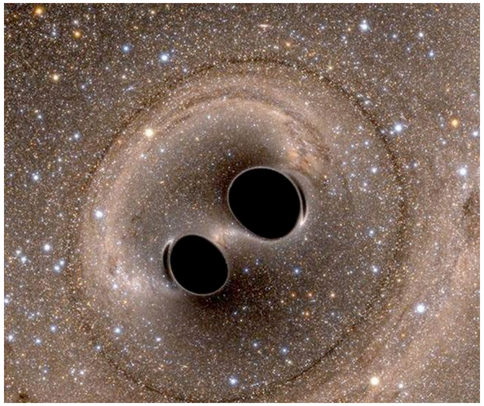
We use this form of double copy in latest calculations.

Double Copy for Classical Solutions

Goal is to formulate gravity solutions directly in terms of gauge theory

Variety of special cases:

- **Schwarzschild and Kerr (spinning) black holes.**
- **Solutions with cosmological constant.**
- **Radiation from accelerating black hole.**
- **Maximally symmetric space times.**
- **Plane wave background.**
- **Gravitational radiation.**

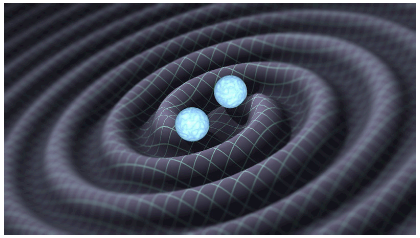


Still no general understanding.
But plenty of examples.

Luna, Monteiro, O'Connell and White;
Luna, Monteiro, Nicholsen, O'Connell and White;
Ridgway and Wise; Carrillo González, Penco, Trodden;
Adamo, Casali, Mason, Nekovar;
Goldberger and Ridgway; Chen;
Luna, Monteiro, Nicholson, Ochirov;
Bjerrum-Bohr, Donoghue, Vanhove;
O'Connell, Westerberg, White; Kosower, Maybee, O'Connell, etc

Can use help from mathematicians to come and clean this up.

Scattering Amplitudes and Gravitational Radiation



A small industry has developed to study this.

- **Connection to scattering amplitudes.**

  Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Luna, Nicholson, O'Connell, White; Guevara; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon; Damour; Bautista, Guevara; Kosower, Maybee, O'Connell; Plefka, Steinhoff, Wormsbecher; Foffa, Mastrolia, Sturani, Sturm; Guevara, Ochirov, Vines; Chung, Huang, Kim, Lee; etc.

- **Worldline approach for radiation and double copy.**

  Goldberger and Ridgway; Goldberger, Li, Prabhu, Thompson; Chester; Shen.

- **Technical issues having to do with keeping right physical states.**

  Luna, Nicholson, O'Connell, White; Johansson, Ochirov; Johansson, Kalin; Henrik Johansson, Gregor Kälin, Mogull.

Key Question: Can we calculate something of direct interest to LIGO/Virgo, decisively *beyond* previous state of the art?

Effective Field Theory Approach

ZB, Cheung, Roiban, Shen, Solon, Zeng

Cheung, Rothstein, Solon (2018)

Amplitudes community

Gravitational Scattering Amplitudes

Effective Field Theory Methods

EFT community

Kawai, Lewellen, Tye
ZB, Dixon, Dunbar and Kosower
ZB, Dixon, Dunbar, Perelstein, Rozowsky
ZB, Carrasco, Johansson; Etc

Beneke, Smirnov (Method of regions)
Goldberger, Rothstein;
Porto; Neill, Rothstein;
Vaydia, Foffa , Porto, Rothstein, Sturani;
Kol, Smolkin, Levi, Steinhoff, etc.

Post Minkowskian Potentials

In a form useful for bound state problem

The EFT directly gives us a two-body Hamiltonian of a form appropriate to enter LIGO analysis pipeline (after importing into EOB or pheno models).

We prefer the EFT matching when pushing into new territory.

EFT is a Clean Approach

No need to re-invent the wheel.
Build EFT from which we can read off potential.

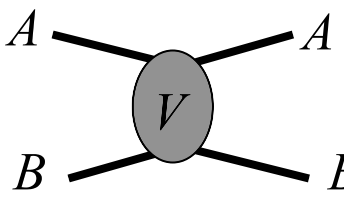
Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

$$L_{\rm kin} = \int_{\boldsymbol{k}} A^\dagger(-\boldsymbol{k})\left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_A^2}\right)A(\boldsymbol{k}) + \int_{\boldsymbol{k}} B^\dagger(-\boldsymbol{k})\left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_B^2}\right)B(\boldsymbol{k})$$

$$L_{\rm int} = -\int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}')\,A^\dagger(\boldsymbol{k}')A(\boldsymbol{k})B^\dagger(-\boldsymbol{k}')B(-\boldsymbol{k})$$

two body potential

***A*, *B* scalars represents spinless black holes**



Match amplitudes of this theory to the full theory in classical limit to extract a potential which can then be directly used for bound state.

The EFT is used to define the potential and 2 body Hamiltonian

EFT Matching

full general relativity
(complicated)

Amplitude methods
double copy

↓

tree amplitude

$\hbar \to 0$

generalized
unitarity

↓

loop integrand

Loop integration
Method of regions

↓

GR loop amplitude

identical
physics

=

effective theory
(simpler)

build
ansatz

↓

potential

Feynman
diagrams

↓

loop integrand

loop
integration

↓

EFT loop amplitude

Roundabout but efficiently determines potential

General Relativity: Unitarity + Double Copy

- **Long-range force: Two matter lines must be separated by on-shell propagators.**
- **Classical potential: 1 matter line per loop is cut (on-shell).**

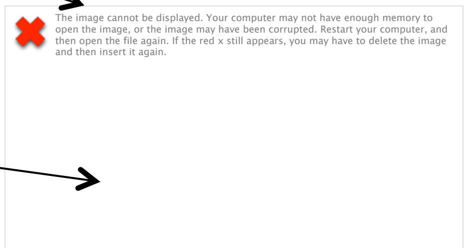
Neill and Rothstein ; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

Only independent unitarity cut for 2 PM 2 body Hamiltonian.

on-shell

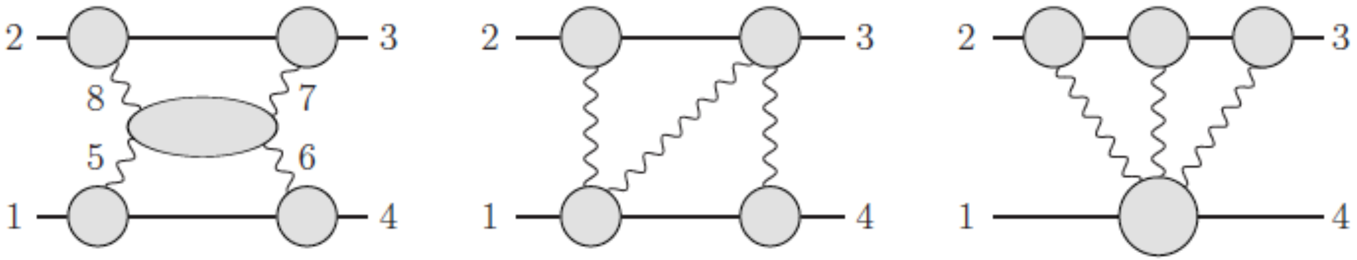
3 vertex, EM tensor

Compton

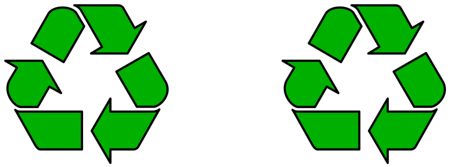


Treat exposed lines on-shell (long range).
Pieces we want are simple!

Independent generalized unitarity cuts for 3 PM.

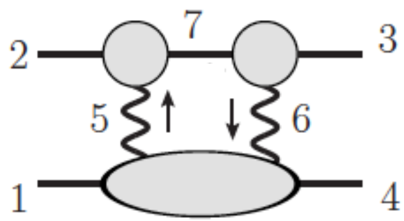


Our amplitude tools fit perfectly with extracting pieces we want.

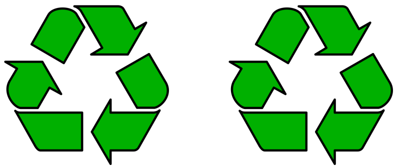


gravity **loops**

Generalized Unitarity Cuts



2^{nd} post-Minkowkian order



KLT relations

$$C_{\rm GR} = \sum_{h_5,h_6=\pm} M_3^{\rm tree}(3^s,6^{h_6},-7^s)\, M_3^{\rm tree}(7^s,-5^{h_5},2^s)\, M_4^{\rm tree}(1^s,5^{-h_5},-6^{-h_6},4^s)$$
$$= \sum_{h_5,h_6=\pm} it[A_3^{\rm tree}(3^s,6^{h_6},-7^s)\, A_3^{\rm tree}(7^s,-5^{h_5},2^s)\, A_4^{\rm tree}(1^s,5^{-h_5},-6^{-h_6},4^s)]$$
$$\times [A_3^{\rm tree}(3^s,6^{h_6},-7^s)\, A_3^{\rm tree}(7^s,-5^{h_5},2^s)\, A_4^{\rm tree}(4^s,5^{-h_5},-6^{-h_6},1^s)]$$

Problem of computing the generalized cuts in gravity is reduced to multiplying and summing gauge-theory tree amplitudes.

$$A_4^{\rm tree}(1^s,2^+,3^+,4^s) = i\frac{m^2\,[2\,3]}{\langle 2\,3\rangle\,\tau_{12}} \qquad A_4^{\rm tree}(1^s,2^+,3^-,4^s) = i\frac{\langle 3|\,1\,|2]^2}{s_{23}\tau_{12}}$$

$$\tau_{12} = 2p_1\cdot p_2$$
$$s_{23} = (p_1+p_2)^2$$

- **For spinless case, same logic works to all orders: KLT and BCJ work for massless *n*-point in *D*-dimension. Dimensional reduction gives massive case**
- **Unwanted states (dilaton) easy to remove with physical state projectors.**

Amplitude in Conservative Classical Potential Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (BCRSSZ)

To make story short. The O(G^3) or 3PM conservative terms are:

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2 \xi}\Bigg[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} - \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\Bigg] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi}\Big[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)F_1 - 32m^2\nu^2\left(1-2\sigma^2\right)^3 F_2\Big]$$

$$m = m_1 + m_2 \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$

$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

- **Amplitude remarkably compact.**
- **Arcsinh and the appearance of a mass singularity is new and robust feature. Cancels mass singularity of real radiation. No surprise.**
  Di Vecchia, Heissenberg, Russo, Veneziano; Damour
- **IR finite parts of amplitude directly connected to scattering angle.**
  Expanded on by Kälin, Porto; Bjerrum-Bohr, Cristofoli, Damgaard
- **Derived conservative scattering angle has simple mass dependence.**
  Observed by Antonelli, Buonanno, Steinhoff, van de Meent, Vines (1901.07102)
  Comprehensive understanding: Damour

Conservative O(G^3) 2-body Hamiltonian

BCRSSZ

The O(G^3) 3PM Hamiltonian:

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2+m_1^2}+\sqrt{\boldsymbol{p}^2+m_2^2}+V(\boldsymbol{p},\boldsymbol{r})$$

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{3} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i,$$

Newton in here

$$c_1 = \frac{\nu^2 m^2}{\gamma^2\xi}\left(1-2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2\xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$c_3 = \frac{\nu^2 m^4}{\gamma^2\xi}\left[\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\right.$$
$$- \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2}$$
$$\left.+\frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\right],$$

$$m = m_1+m_2 \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$
$$\xi = E_1E_2/E^2, \qquad E = E_1+E_2, \qquad \sigma = p_1\cdot p_2/m_1m_2,$$

- **Expanding in velocity gives infinite sequence of terms in PN expansion.**
- **Can be put into EOB form.** Antonelli, Buonannom Steinhoff,van de Meent, Vines

How do we know it is right?

Original check:

Compared to 4PN Hamiltonians after canonical transformation

Damour, Jaranowski, Schäfer; Bernard, Blanchet, Bohé, Faye, Marsat

Thibault Damour seriously questioned correctness.
Specific corrections proposed. Damour, arXiv:1912.02139v1

Saint Julien 2011

Subsequent calculations confirm our 3PM result:

1. **Papers confirming our result in 6PN overlap.**
   Blümlein, Maier, Marquard, Schäfer;
   Bini, Damour, Geralico
2. **Subsequent calculations reproducing our 3PM result.**
   Cheung and Solon; Kälin, Liu, Porto
3. **Scattering angle checks.** ZB, Ita, Parra-Martinez, Ruf
4. **Adding real radiation removes mass singularity.**
   Di Vecchia, Heissenberg, Russo, Veneziano; Damour

3PM results have passed highly nontrivial checks and careful scrutiny.

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$\widehat{H}_{\rm N}(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r},$$

$$c^2\,\widehat{H}_{\rm 1PN}(\mathbf{r},\mathbf{p}) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\Big\{(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r} + \frac{1}{2r^2},$$

$$c^4\,\widehat{H}_{\rm 2PN}(\mathbf{r},\mathbf{p}) = \frac{1}{16}\left(1-5\nu+5\nu^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\Big\{\left(5-20\nu-3\nu^2\right)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r}$$
$$+\frac{1}{2}\Big\{(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r^2} - \frac{1}{4}(1+3\nu)\frac{1}{r^3},$$

$$c^6\,\widehat{H}_{\rm 3PN}(\mathbf{r},\mathbf{p}) = \frac{1}{128}\left(-5+35\nu-70\nu^2+35\nu^3\right)(\mathbf{p}^2)^4 + \frac{1}{16}\Big\{\left(-7+42\nu-53\nu^2-5\nu^3\right)(\mathbf{p}^2)^3$$
$$+(2-3\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n}\cdot\mathbf{p})^6\Big\}\frac{1}{r}$$
$$+\Big\{\frac{1}{16}\left(-27+136\nu+109\nu^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r^2}$$
$$+\Big\{\Big(-\frac{25}{8} + \Big(\frac{\pi^2}{64} - \frac{335}{48}\Big)\nu - \frac{23\nu^2}{8}\Big)\mathbf{p}^2 + \Big(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4}\Big)\nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r^3} + \Big\{\frac{1}{8} + \Big(\frac{109}{12} - \frac{21}{32}\pi^2\Big)\nu\Big\}\frac{1}{r^4},$$

G^4

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

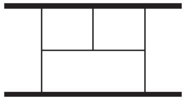
$$
\begin{aligned}
c^8 \widehat{H}^{\rm local}_{\rm 4PN}(\mathbf{r},\mathbf{p}) &= \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right)(\mathbf{p}^2)^5 \\
&+ \Bigg\{\frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4\,\nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2\right)\nu^2 \\
&+ \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 + \frac{35}{256}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^3 \\
&+ \left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 - \frac{35}{128}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^4\Bigg\}\frac{1}{r} \\
&+ \Bigg\{\frac{13}{8}(\mathbf{p}^2)^3 + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{369}{160}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu \\
&+ \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + \frac{9475}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - \frac{1151}{128}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^2 \\
&+ \left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{10353}{1280}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^3\Bigg\}\frac{1}{r^2} \\
&+ \Bigg\{\frac{105}{32}(\mathbf{p}^2)^2 + \Bigg(\left(\frac{2749\pi^2}{8192} - \frac{589189}{19200}\right)(\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)(\mathbf{n}\cdot\mathbf{p})^4\Bigg)\nu \\
&+ \Bigg(\left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800}\right)(\mathbf{p}^2)^2 + \left(-\frac{127}{3} - \frac{4035\pi^2}{2048}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384}\right)(\mathbf{n}\cdot\mathbf{p})^4\Bigg)\nu^2 \\
&+ \left(-\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - \frac{381}{128}(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^3\Bigg\}\frac{1}{r^3} \\
&+ \Bigg\{\frac{105}{32}\mathbf{p}^2 + \Bigg(\left(\frac{185761}{19200} - \frac{21837\pi^2}{8192}\right)\mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576}\right)(\mathbf{n}\cdot\mathbf{p})^2\Bigg)\nu \\
&+ \Bigg(\left(\frac{672811}{19200} - \frac{158177\pi^2}{49152}\right)\mathbf{p}^2 + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840}\right)(\mathbf{n}\cdot\mathbf{p})^2\Bigg)\nu^2\Bigg\}\frac{1}{r^4} \quad \longleftarrow G^4 \\
&+ \Bigg\{-\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400}\right)\nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45}\right)\nu^2\Bigg\}\frac{1}{r^5}. \quad \longleftarrow G^5
\end{aligned}
$$

Mess is partly due to gauge choice and also expansion in velocity.

Ours is all orders in *p* at G^3

After a canonical transformation this matches our result in overlap

Conservative Potential Contribution O(G^4)



O(G^4) amplitude

test particle — **1st self force** — **Iteration. No need to compute**

$$\mathcal{M}_4(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{\frac{1}{3}}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^{\rm p} + \nu\left(\frac{\mathcal{M}_4^{\rm t}}{\epsilon} + \mathcal{M}_4^{\rm f}\right)\right] + \int_\ell \frac{\tilde{I}_{r,1}^4}{Z_1 Z_2 Z_3} + \int_\ell \frac{\tilde{I}_{r,1}^2 \tilde{I}_{r,2}}{Z_1 Z_2} + \int_\ell \frac{\tilde{I}_{r,1}\tilde{I}_{r,3}}{Z_1} + \int_\ell \frac{\tilde{I}_{r,2}^2}{Z_1}$$

tail effect (IR divergent)

$$D = 4 - 2\epsilon$$

$$\mathcal{M}_4^{\rm p} = -\frac{35\left(1 - 18\sigma^2 + 33\sigma^4\right)}{8\left(\sigma^2 - 1\right)}, \qquad \mathcal{M}_4^{\rm t} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2 - 1}},$$

$$\begin{aligned}
\mathcal{M}_4^{\rm f} = {} & h_4 + h_5 \log\left(\tfrac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2 - 1} + h_9 \left[\operatorname{Li}_2\left(\tfrac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\tfrac{\sigma+1}{2}\right)\right] \\
& + h_{10}\left[\operatorname{Li}_2\left(\tfrac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\operatorname{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) - \operatorname{Li}_2\left(\tfrac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\operatorname{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \operatorname{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right] \\
& + \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\operatorname{Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - \operatorname{Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 5\operatorname{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - 5\operatorname{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) + 2\log\left(\tfrac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] \\
& + h_{12}\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{13}\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{14}\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right)
\end{aligned}$$

← elliptic

$$\nu = m_1 m_2/(m_1+m_2)^2 \qquad \sigma = p_1 \cdot p_2 / m_1 m_2,$$

$$h_1 = \frac{1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6}{12(\sigma^2-1)}$$

$$h_2 = \frac{1}{2}\left(5 - 76\sigma + 150\sigma^2 - 60\sigma^3 - 35\sigma^4\right)$$

$$h_3 = \sigma\frac{\left(-3+2\sigma^2\right)}{4(\sigma^2-1)}\left(11 - 30\sigma^2 + 35\sigma^4\right)$$

$$\begin{aligned} h_4 = {} & \frac{1}{144(\sigma^2-1)^2\sigma^7}(-45 + 207\sigma^2 - 1471\sigma^4 + 13349\sigma^6 \\ & - 37566\sigma^7 + 104753\sigma^8 - 12312\sigma^9 - 102759\sigma^{10} - 105498\sigma^{11} \\ & + 134745\sigma^{12} + 83844\sigma^{13} - 101979\sigma^{14} + 13644\sigma^{15} + 10800\sigma^{16}) \end{aligned}$$

$$\begin{aligned} h_5 = {} & \frac{1}{4(\sigma^2-1)}(1759 - 4768\sigma + 3407\sigma^2 - 1316\sigma^3 + 957\sigma^4 \\ & - 672\sigma^5 + 341\sigma^6 + 100\sigma^7) \end{aligned}$$

$$\begin{aligned} h_6 = {} & \frac{1}{24(\sigma^2-1)^2}(1237 + 7959\sigma - 25183\sigma^2 + 12915\sigma^3 + 18102\sigma^4 \\ & - 12105\sigma^5 - 9572\sigma^6 + 2973\sigma^7 + 5816\sigma^8 - 2046\sigma^9) \end{aligned}$$

$$h_7 = 2\sigma\frac{\left(-852 - 283\sigma^2 - 140\sigma^4 + 75\sigma^6\right)}{3(\sigma^2-1)}$$

$$\begin{aligned} h_8 = {} & \frac{\sigma}{8(\sigma^2-1)^2}(-304 - 99\sigma + 672\sigma^2 + 402\sigma^3 - 192\sigma^4 - 719\sigma^5 \\ & - 416\sigma^6 + 540\sigma^7 + 240\sigma^8 - 140\sigma^9) \end{aligned}$$

$$h_9 = \frac{1}{2}\left(52 - 532\sigma + 351\sigma^2 - 420\sigma^3 + 30\sigma^4 - 25\sigma^6\right)$$

$$h_{10} = 2\left(27 + 90\sigma^2 + 35\sigma^4\right)$$

$$h_{11} = 20 + 111\sigma^2 + 30\sigma^4 - 25\sigma^6$$

$$h_{12} = \frac{834 + 2095\sigma + 1200\sigma^2}{2(\sigma^2-1)}$$

$$h_{13} = -\frac{1183 + 2929\sigma + 2660\sigma^2 + 1200\sigma^3}{2(\sigma^2-1)}$$

$$h_{14} = \frac{7\left(169 + 380\sigma^2\right)}{4(\sigma-1)}$$

Read O(G^4) radial action off directly from scattering amplitude:

$$I_{r,4}(J) = -\frac{G^4 M^7 \nu^2 \pi \boldsymbol{p}^2}{8EJ^3}\left(\frac{4\tilde{\mu}^2 e^{2\gamma_E} J^2}{\boldsymbol{p}^2}\right)^{4\epsilon}\left[\mathcal{M}_4^{\rm p} + \nu\left(\frac{\mathcal{M}_4^{\rm t}}{\epsilon} + \mathcal{M}_4^{\rm f} - 14\mathcal{M}_4^{\rm t}\right)\right]$$

$$\chi = -\frac{\partial I_r}{\partial J}$$

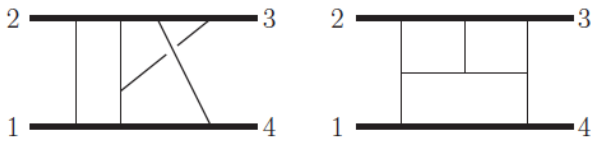
Scattering angle (χ); radial action (I_r); angular momentum (J)

- **Here radiation contributions to conservative tail effect not included. New to O(G^4).**
- **IR divergence will cancel once radiation is included. Working on this.**
- **High-energy limit has mass singularity. Presumably, cancels against real radiation.**

Integration

ZB, Parra-Martinez, Roiban, Ruf. Shen, Solon, Zeng

Integration more challenging than at 2 loops.



Developed a new hybrid approach that combines ideas from various methods:

1. **Method of regions to separate potential and radiation.** Beneke and Smirnov
2. **Nonrelativistic integration. Velocity expand and then mechanically integrate. Get first few orders in velocity. Boundary conditions.** Cheung, Rothstein, Solon
3. **Integration by parts and differential equations. Imported from QCD.** Chetyrkin, Tkachov; Laporta; Kotikov, Bern, Dixon and Kosower; Gehrmann, Remiddi.
   Single scale integrals! Parra-Martinez, Ruf, Zeng

IBP: $$0 = \int \prod_i^L \frac{d^D \ell_i}{(2\pi)^D} \frac{\partial}{\partial \ell_i^\mu} \frac{N^\mu(\ell_k, p_M)}{Z_1 \ldots Z_n}$$

Solve linear relations between integrals in terms of master integrals.

DEs: $$\frac{\partial}{\partial s_i} I_j^{\text{master}} = \text{simplified via IBP}$$

Solve DEs either as series or basis of functions.

- **Many tools available: We use FIRE6, which is more than sufficient.** Smirnov, Chuharev
- **Elliptic integrals make an appearance. At end just a minor annoyance.**

So far we have not even used close to full power of IBP.

O(G^4) Two-Body Hamiltonian

$$H^{\rm iso} = \sqrt{\mathbf{p}^2 + m_1} + \sqrt{\mathbf{p^2} + m_2} + \sum_{i=1}^{4} \frac{G^n}{r^n} c_n(\mathbf{p^2})$$

Isotropic gauge Hamiltonian

$$c_4 = \frac{M^7\nu^2}{4\xi E^2}\left[\mathcal{M}_4^{\rm p} + \nu\left(\frac{\mathcal{M}_4^{\rm t}}{\epsilon} + \mathcal{M}_4^{\rm f} - 10\mathcal{M}_4^{\rm t}\right)\right] + \mathcal{D}^3\left[\frac{E^3\xi^3}{3}c_1^4\right] + \mathcal{D}^2\left[\left(\frac{E^3\xi^3}{\boldsymbol{p}^2} + \frac{E\xi(3\xi-1)}{2}\right)c_1^4 - 2E^2\xi^2c_1^2c_2\right]$$
$$+\left(\mathcal{D} + \frac{1}{\boldsymbol{p}^2}\right)\left[E\xi(2c_1c_3 + c_2^2) + \left(\frac{4\xi-1}{4E} + \frac{2E^3\xi^3}{\boldsymbol{p}^4} + \frac{E\xi(3\xi-1)}{\boldsymbol{p}^2}\right)c_1^4 + \left((1-3\xi) - \frac{4E^2\xi^2}{\boldsymbol{p}^2}\right)c_1^2c_2\right],$$

$$\xi = E_1E_2/E^2, \qquad E = E_1 + E_2, \qquad \nu = \mu/m, \qquad \mathcal{D} = \frac{d}{dp^2}$$

As for amplitude, radiation effects on conservative dynamics *not* included here.

Above divergent part of tail effect gives radiated energy.

Bini, Damour Geralico

$$\Delta E = \frac{G^3M^7\nu^3\pi\boldsymbol{p}^2}{4E^2J^3}\mathcal{M}_4^{\rm t}$$

Matches direct calculation

Herrmann, Parra-Martinez, Ruf, Zeng

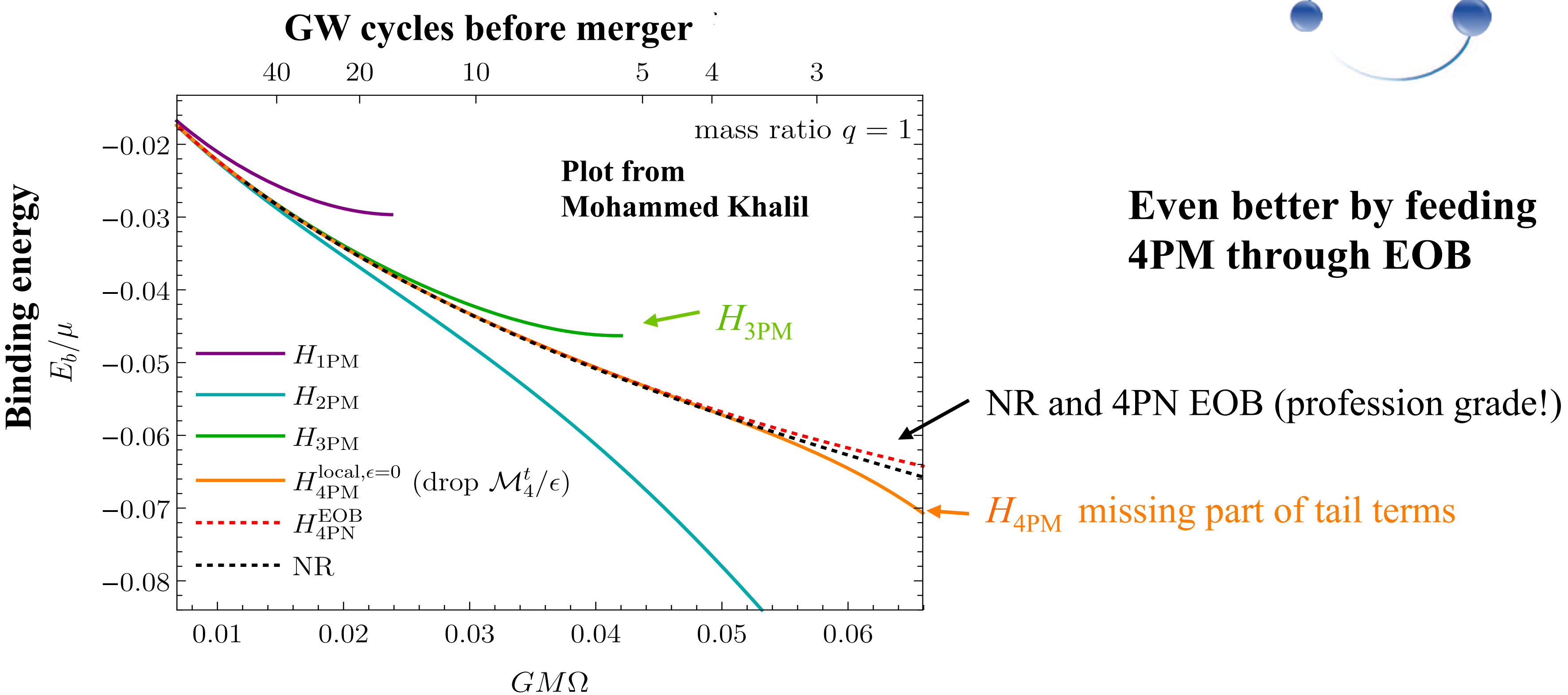
Results agree with all known overlap PN results through 6PN.

Blumlein, Maier, Marquard, Schafer; Bini, Damour and Geralico

Preliminary O(G^4) Binding Energy

Khalil, Buonanno, Vines, Steinhoff

Even though missing radiation contributions, good to look at binding energy to see if we are on a right track.



Even better by feeding 4PM through EOB

- ***Not* conclusive (missing piece), but very encouraging.**
- **Motivates us to finish radiation tail contributions!**

Outlook

Amplitude methods have a lot of promise and their use has already been tested for a variety of problems.

- **Pushing state of the art for high orders in G.** ZB, Cheung, Roiban, Parra-Martinez, Ruf, Shen, Solon, Zeng
- **Radiation.** Cristofoli, Gonzo, Kosower, O'Connell; Herrmann, Parra-Martinez, Ruf, Zeng; Di Vecchia, Heissenberg, Russo, Veneziano
- **Finite size effects.** Cheung and Solon; Haddad and Helset; Kälin, Liu, Porto; Cheung, Shah, Solon; ZB, Parra-Martinez, Roiban, Sawyer, Shen
- **Spin.** Vaidya; Geuvara, O'Connell, Vines; Chung, Huang, Kim, Lee; ZB, Luna, Roiban, Shen, Zeng; Kosmopoulos, Luna, etc

The standard quantities of interest for the inspiral phase can be computed in this formalism.

Summary

- Elementary particle physics provide a new useful way to think about problems of direct interest to gravitational-wave community.
- Scattering amplitudes are independent of gauges, coordinates and field variables, making it simpler to identify useful new structures.
  — Double copy
- Methods work on a variety of problems. Spin, tidal, high orders:
  — Pushed two-body Hamiltonian to (G^3) and now O(G^4): potential-mode contributions complete. Radiation contributions to conservative part in progress.
- Most exciting part is that the methods are not close to exhausted.

In the coming years we can expect new advances, not only in gravitational-wave physics, but also in understanding gravity and its relation to gauge theory.