New Structures in Gravitational Radiation

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MATHEMATICAL AND NUMERICAL ASPECTS OF GRAVITATION

Part of the Program: Mathematical and Computational Challenges in the Era of Gravitational Wave Astronomy

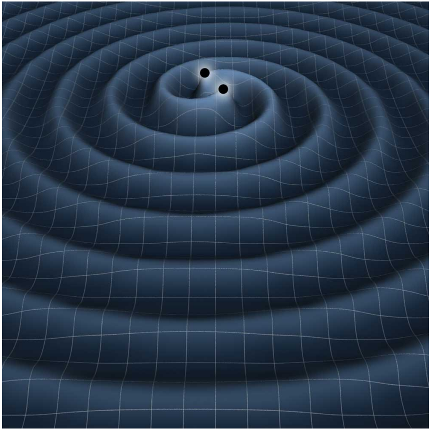
IPAM

10/25/2021 - 10/29/2021

Overview

- Spacetimes and Radiation
- Cauchy Problem for the Einstein Equations
- Gravitational Waves and Memory
- New Structures
- Dynamics of General Asymptotically-Flat Systems
- Outlook

Investigate the Cauchy problem for the Einstein equations for physical systems to gain information on gravitational radiation.



Photos: Courtesy of NASA; EHT.

Measuring Gravitational Waves

- LIGO detected gravitational waves from a binary black hole merger for the first time in September 2015.
- Several times since then.
- LIGO and VIRGO together observed gravitational waves from a binary neutron star merger in 2017. At the same time, several telescopes registered data.

Einstein Equations

$$\mathbf{R}_{\mu\nu} - \frac{\mathbf{1}}{\mathbf{2}}\, \mathbf{g}_{\mu\nu}\, \mathbf{R} = \mathbf{8}\pi\, \mathbf{T}_{\mu\nu}\ , \tag{1}$$

with

$\mathbf{R}_{\mu\nu}$ the Ricci curvature tensor,

$\mathbf{R}$ the scalar curvature tensor,

$\mathbf{g}$ the metric tensor and

$\mathbf{T}_{\mu\nu}$ the energy-momentum tensor.

Investigate dynamics of spacetimes (M, g), where M a 4-dimensional manifold with Lorentzian metric g solving Einstein's equations (1).

Einstein Vacuum Equations

For the main parts of the discussion we concentrate on the Einstein-Vacuum equations.

Solutions of the Einstein-Vacuum (EV) equations:

$$R_{\mu\nu} \quad = \quad 0 \; . \tag{2}$$

Spacetimes (M, g), where M is a four-dimensional, oriented, differentiable manifold and g is a Lorentzian metric obeying (2).

Foliations of the Spacetime

Let t denote a maximal time function foliating the spacetime into complete Riemannian hypersurfaces H_t.

Let u be an optical function foliating the spacetime into null hypersurfaces C_u.

$$S_{t,u} = H_t \cap C_u$$

Evolution Equations, Constraints and Lapse

Evolution equations of a maximal foliation:

$$\begin{aligned} \frac{\partial \bar{g}_{ij}}{\partial t} &= -2\Phi k_{ij} \\ \frac{\partial k_{ij}}{\partial t} &= -\nabla_i \nabla_j \Phi + (\bar{R}_{ij} - 2 k_{im} k^m_{\ j}) \Phi \end{aligned}$$

Constraint equations of a maximal foliation:

$$\begin{aligned} trk &= 0 \\ \nabla^i k_{ij} &= 0 \\ \bar{R} &= \mid k \mid^2 \end{aligned}$$

Lapse equation of a maximal foliation:

$$\triangle \Phi - \mid k \mid^2 \Phi = 0$$

Given an outgoing null vectorfield L, we define a conjugate (incoming) null vectorfield $\underline{L}$ by requiring that

$$g(L, \underline{L}) = -2 \ .$$

L and $\underline{L}$ are orthogonal to $S_{t,u}$.

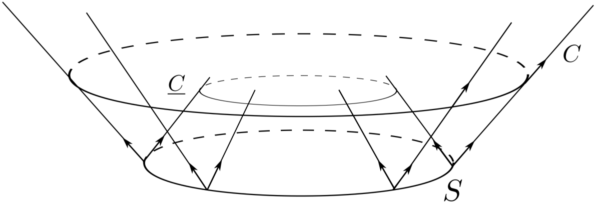
For further purposes denote L by e_4 and $\underline{L}$ by e_3.
Complement e_4 and e_3 with an orthonormal frame e_1, e_2 on $S_{t,u}$
$\Rightarrow$ We obtain a null frame.

The null decomposition of a tensor relative to a null frame e_4, e_3, e_2, e_1 is obtained by taking contractions with the vectorfields e_4, e_3.

Shears and Expansion Scalars

Viewing S as a hypersurface in C, respectively $\underline{C}$:

- Denote the second fundamental form of S in C by χ, and the second fundamental form of S in $\underline{C}$ by $\underline{\chi}$.
- Their traceless parts are the shears and denoted by $\hat{\chi}$, $\underline{\hat{\chi}}$ respectively.
- The traces $tr\chi$ and $tr\underline{\chi}$ are the expansion scalars.
- Null Limits of the Shears:
  $\lim_{C_u, t \to \infty} r^2 \hat{\chi} = \Sigma(u)$ (in (A) spacetimes) and
  $\lim_{C_u, t \to \infty} r \underline{\hat{\chi}} = \Xi(u)$.



Foliation of Null Infinity

Future null infinity $\mathcal{I}^+$ is defined to be the endpoints of all future-directed null geodesics along which $r \to \infty$. It has the topology of $\mathbb{R} \times \mathbb{S}^2$ with the function u taking values in $\mathbb{R}$.

Thus a null hypersurface C_u intersects $\mathcal{I}^+$ at infinity in a 2-sphere $S_{\infty,u}$.

Here we observe gravitational radiation.

General Spacetimes

Theorem [L. Bieri (2007)]

Every asymptotically flat initial data obeying appropriate smallness assumptions (controlled via weighted Sobolev norms) gives rise to a globally asymptotically flat solution of the Einstein vacuum equations that is causally geodesically complete.

Small data ensures existence.

Large data

Main behavior along null hypersurfaces towards future null infinity

$\Rightarrow$ Largely independent from the smallness.

Pioneering stability results:

Global Result by S. Klainerman and D. Christodoulou, 1991, proving the global nonlinear stability of Minkowski spacetime.

Semiglobal Result: [H. Friedrich (1986)]

Asymptotic Flatness

(B) (Most general asymptotically-flat spacetimes.) Asymptotically flat initial data set in the sense of (B): an asymptotically flat initial data set $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\bar{g}_{ij} = \delta_{ij} + o_3\,(r^{-\frac{1}{2}}) \tag{3}$$

$$k_{ij} = o_2\,(r^{-\frac{3}{2}})\,. \tag{4}$$

(D Christodoulou-Klainerman) Strongly asymptotically flat initial data set in the sense of (D): an initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighbourhood of infinity such that, as $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, $\bar{g}_{ij}$ and k_{ij} are:

$$\bar{g}_{ij} = (1 + \frac{2M}{r})\,\delta_{ij} + o_4\,(r^{-\frac{3}{2}}) \tag{5}$$

$$k_{ij} = o_3\,(r^{-\frac{5}{2}})\,, \tag{6}$$

where M denotes the mass.

Asymptotic Flatness

Situation (H). Consider initial data of the asymptotic type

$$\begin{aligned} \bar{g}_{ij} - \delta_{ij} &= l_{ij} + O\left(r^{-1-\varepsilon}\right) && (7) \\ k_{ij} &= O\left(r^{-2-\varepsilon}\right), && (8) \end{aligned}$$

with l_{ij} being homogeneous of degree -1.

Situation (C). Consider initial data of the asymptotic type

$$\begin{aligned} \bar{g}_{ij} - \delta_{ij} &= O\left(r^{-\frac{1}{2}-\varepsilon}\right) && (9) \\ k_{ij} &= O\left(r^{-\frac{3}{2}-\varepsilon}\right), && (10) \end{aligned}$$

with $0 < \varepsilon < \frac{1}{2}$.

Situation (A). As in (B) but with big O instead of o.

Theorems for Large Data

Stability proofs that established the relevant properties of the spacetimes:

(D) D. Christodoulou and S. Klainerman: 1993

(B) L. Bieri: 2007

Stability Theorems: For data as in definition (B) under a smallness condition $\Rightarrow$ established global existence and decay theorem for the Einstein vacuum equations.

Large data: It follows easily by a corollary that there exists a *complete domain of dependence of the complement of a sufficiently large compact subset of the initial hypersurface*. Thus, we have a solution spacetime with a portion of future null infinity corresponding to all values of the retarded time u not greater than a fixed constant.

$\Rightarrow$ This provides the solid foundation to investigate the asymptotic behavior at future null infinity for large data for (B) spacetimes, and to prove theorems on the nature of gravitational radiation.

Naturally, our investigations will extend to these spacetimes coupled to neutrinos via a null fluid.

Data of type (B): total energy finite, total angular momentum diverges.
Data of type (A):
• total energy no longer finite,
• no existence theorem is known for a development which includes a portion of future null infinity.
• Study of type (A) gives conjectures furnished with supporting evidence.

Definition

We define the null components of the Weyl curvature W as follows:

$$\underline{\alpha}_{\mu\nu}(W) = \Pi_\mu{}^\rho\, \Pi_\nu{}^\sigma\, W_{\rho\gamma\sigma\delta}\, e_3^\gamma\, e_3^\delta \tag{11}$$

$$\underline{\beta}_\mu(W) = \frac{1}{2}\, \Pi_\mu{}^\rho\, W_{\rho\sigma\gamma\delta}\, e_3^\sigma\, e_3^\gamma\, e_4^\delta \tag{12}$$

$$\rho(W) = \frac{1}{4}\, W_{\alpha\beta\gamma\delta}\, e_3^\alpha\, e_4^\beta\, e_3^\gamma\, e_4^\delta \tag{13}$$

$$\sigma(W) = \frac{1}{4}\, {}^*W_{\alpha\beta\gamma\delta}\, e_3^\alpha\, e_4^\beta\, e_3^\gamma\, e_4^\delta \tag{14}$$

$$\beta_\mu(W) = \frac{1}{2}\, \Pi_\mu{}^\rho\, W_{\rho\sigma\gamma\delta}\, e_4^\sigma\, e_3^\gamma\, e_4^\delta \tag{15}$$

$$\alpha_{\mu\nu}(W) = \Pi_\mu{}^\rho\, \Pi_\nu{}^\sigma\, W_{\rho\gamma\sigma\delta}\, e_4^\gamma\, e_4^\delta\,. \tag{16}$$

Thus we have the following with the capital indices taking the values $1, 2$:

$$W_{A3B3} = \underline{\alpha}_{AB} \tag{17}$$
$$W_{A334} = 2\ \underline{\beta}_{A} \tag{18}$$
$$W_{3434} = 4\ \rho \tag{19}$$
$$^{*}W_{3434} = 4\ \sigma \tag{20}$$
$$W_{A434} = 2\ \beta_{A} \tag{21}$$
$$W_{A4B4} = \alpha_{AB} \tag{22}$$

`Notation: Hodge duals` $^{*}W$ `and` W^{*} `defined as`

$$
\begin{aligned}
^{*}W_{\alpha\beta\gamma\delta} &= \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta} \\
W^{*}{}_{\alpha\beta\gamma\delta} &= \frac{1}{2}W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta}
\end{aligned}
$$

Let $\tau_-^2 = 1 + u^2$ and $r(t,u)$ is the area radius of the surface $S_{t,u}$.

Weyl curvature components

(D)

$$\begin{aligned}
\underline{\alpha}(W) &= O\left(r^{-1}\ \tau_-^{-\frac{5}{2}}\right)\\
\underline{\beta}(W) &= O\left(r^{-2}\ \tau_-^{-\frac{3}{2}}\right)\\
\rho(W) &= O\left(r^{-3}\right)\\
\sigma(W) &= O\left(r^{-3}\ \tau_-^{-\frac{1}{2}}\right)\\
\alpha(W),\ \beta(W) &= o\left(r^{-\frac{7}{2}}\right)
\end{aligned}$$

(B)

$$\begin{aligned}
\underline{\alpha} &= O\left(r^{-1}\ \tau_-^{-\frac{3}{2}}\right)\\
\underline{\beta} &= O\left(r^{-2}\ \tau_-^{-\frac{1}{2}}\right)\\
\rho,\ \sigma,\ \alpha,\ \beta &= o\left(r^{-\frac{5}{2}}\right)
\end{aligned}$$

Correspondingly, obtain decay rates for cases (A) and (C).

Structures in (B) Spacetimes

$$\hat{\chi} = o\left(r^{-\frac{3}{2}}\right) \quad (23)$$

$$\underline{\hat{\chi}} = O\left(r^{-1}\tau_{-}^{-\frac{1}{2}}\right) \quad (24)$$

$$\zeta = o\left(r^{-\frac{3}{2}}\right) \quad (25)$$

$$tr\chi = \frac{2}{r} + l.o.t. \quad (26)$$

$$tr\underline{\chi} = -\frac{2}{r} + l.o.t. \quad (27)$$

Further, we have

$$k_{AB} = \eta_{AB} \qquad \hat{\eta} = O(r^{-1}\tau_{-}^{-\frac{1}{2}})$$

$$k_{AN} = \varepsilon_{A} \qquad \varepsilon = o(r^{-\frac{3}{2}})$$

$$k_{NN} = \delta \qquad \delta = o(r^{-\frac{3}{2}})$$

Here, ζ is the torsion-one-form. Ricci rotation coefficients of the null frame are:

$$\chi_{AB} = g(D_A e_4, e_B)\ ,\ \underline{\chi}_{AB} = g(D_A e_3, e_B)\ ,\ \underline{\xi}_A = \frac{1}{2}g(D_3 e_3, e_A)\ ,\ \zeta_A = \frac{1}{2}g(D_3 e_4, e_A)$$

$$\underline{\zeta}_A = \frac{1}{2}g(D_4 e_3, e_A)\ ,\ \nu = \frac{1}{2}g(D_4 e_4, e_3)\ ,\ \underline{\nu} = \frac{1}{2}g(D_3 e_3, e_4)\ ,\ \varepsilon_A = \frac{1}{2}g(D_A e_4, e_3)$$

The Bianchi equations for $\not{D}_3\rho$ as well as $\not{D}_3\sigma$ are

$$\not{D}_3\rho \ + \ \frac{3}{2}tr\underline{\chi}\rho \quad = \quad -\not{div}\,\underline{\beta} - \frac{1}{2}\hat{\chi}\underline{\alpha} \ + \ (\varepsilon - \zeta)\underline{\beta} \ + \ 2\underline{\xi}\beta \qquad (28)$$
$$+\frac{1}{4}(D_3R_{34} - D_4R_{33})$$

$$\not{D}_3\sigma \ + \ \frac{3}{2}tr\underline{\chi}\sigma \quad = \quad -\not{curl}\,\underline{\beta} - \frac{1}{2}\hat{\chi}^*\underline{\alpha} + \varepsilon^*\underline{\beta} - 2\zeta^*\underline{\beta} - 2\underline{\xi}^*\beta \qquad (29)$$
$$+\frac{1}{4}(D_\mu R_{3\nu} - D_\nu R_{3\mu})\varepsilon^{\mu\nu}{}_{34}$$

For small as well as large data, the following is a consequence of the relations between the shears, the shear and curvature, and the stability proof (B).

$$\hat{\chi} = [r^{-\frac{3}{2}}] + \{r^{-2}\tau_-^{+\frac{1}{2}}\} + l.o.t. \tag{30}$$

$$\underline{\hat{\chi}} = \{r^{-1}\tau_-^{-\frac{1}{2}}\} + [r^{-\frac{3}{2}}] + l.o.t. \tag{31}$$

Notation: In (B) spacetimes, we denote the part of $\hat{\chi}$ with decay $o(r^{-\frac{3}{2}})$ and which is non-dynamical (i.e. which does not evolve with u) by $[r^{-\frac{3}{2}}]$. Denote the leading order dynamical part of $\hat{\chi}$ (i.e. which evolves with u) by $\{r^{-2}\tau_-^{+\frac{1}{2}}\}$. More generally, for any of the non-peeling curvature components and any of the Ricci coefficients which have a leading order non-dynamical part, let $[\cdot]$ denote the leading order non-dynamical part (thus not evolving in u) of this component; and let $\{\cdot\}$ denote its leading order dynamical part (thus evolving in u).

By the proof (B) and the smallness conditions therein for the e_3-derivative of ρ, respectively σ:

$$\int_H r^4 |\rho_3|^2 \leqslant c\varepsilon$$

$$\int_H r^4 |\sigma_3|^2 \leqslant c\varepsilon$$

it is a consequence that

$$\rho_3 = O(r^{-3}\tau_-^{-\frac{1}{2}}) \quad , \quad \sigma_3 = O(r^{-3}\tau_-^{-\frac{1}{2}})$$

For small data it follows that

$$\rho = [r^{-\frac{5}{2}}] + \{r^{-3}\tau_-^{+\frac{1}{2}}\} + \{r^{-3}\} + \{r^{-3}\tau_-^{+\beta}\} + O(r^{-3}\omega^{-\alpha}) \quad (32)$$

and

$$\sigma = [r^{-\frac{5}{2}}] + \{r^{-3}\tau_-^{+\frac{1}{2}}\} + \{r^{-3}\} + \{r^{-3}\tau_-^{+\beta}\} + O(r^{-3}\omega^{-\alpha}) \quad (33)$$

with ω denoting r or τ_- and $\alpha > 0$, $0 < \beta < \frac{1}{2}$.
For large data, there are more terms present with a variety of decay, including terms in ρ, respectively σ, of the order $r^{-\frac{5}{2}}\tau_-^{-\alpha}$ with $\alpha > 0$.

Limits at null infinity $\mathcal{I}^+$

More general phenomenon. Several quantities, which are defined locally on the surface $S_{t,u}$, do not attain corresponding limits on a given null hypersurface C_u as $t \to \infty$. However, the difference of their values at corresponding points on S_u and S_{u_0} does tend to a limit.

For instance, consider $\hat{\chi}$ defined locally on $S_{t,u}$. Recall (23). Even though $r^2 \hat{\chi}$ does not have a limit as $r \to \infty$ on a given C_u, the difference at corresponding points on S_u in C_u and on S_{u_0} in C_{u_0} does have a limit. In particular, these points being joined by an integral curve of e_3, the said difference attains the limit

$$\int_{u_0}^{u} \not{D}_3 \hat{\chi} \, du'$$

The part of $\hat{\chi}$ with slow decay of order $o(r^{-\frac{3}{2}})$ is non-dynamical, that is, it does not evolve with u. We see that this part does not tend to any limit at null infinity $\mathcal{I}^+$. Similarly, the components of the curvature that are not peeling have leading order terms that are non-dynamical (and do not attain corresponding limits at $\mathcal{I}^+$). Taking off these pieces gives us the dynamical parts of these (non-peeling) curvature components.

Theorem [L. Bieri (2007)]

For the spacetimes of types (B), the normalized curvature components $r\underline{\alpha}(W)$, $r^2\underline{\beta}(W)$ have limits on C_u as $t \to \infty$:

$$\lim_{C_u, t\to\infty} r\underline{\alpha}(W) = A_W(u,\cdot), \qquad \lim_{C_u, t\to\infty} r^2\underline{\beta}(W) = \underline{B}_W(u,\cdot),$$

where the limits are on S^2 and depend on u. These limits satisfy

$$|A_W(u,\cdot)| \leqslant C(1+|u|)^{-3/2} \qquad |\underline{B}_W(u,\cdot)| \leqslant C(1+|u|)^{-1/2}.$$

Moreover, the following limit exists

$$-\frac{1}{2}\lim_{C_u, t\to\infty} r\underline{\widehat{\chi}} = \lim_{C_u, t\to\infty} r\hat{\eta} = \Xi(u,\cdot)$$

Further, it follows that

$$\frac{\partial \Xi}{\partial u} = -\frac{1}{4}A_W \tag{34}$$

$$\underline{B} = -2 \,\mathrm{d\!/\!iv}\, \Xi \tag{35}$$

Curvature Components ρ, σ and Derivatives ρ_3, σ_3

$$\rho_3 - \underbrace{A_\rho(r,u,\cdot)}_{\text{will cancel in Bianchi equ.}} = \underbrace{\rho_{\frac{1}{2}}(r,u,\cdot) + \rho_\beta(r,u,\cdot) + B_\rho(r,u,\cdot)}_{\text{will impact gravitational radiation, more structures}} + l.o.t.$$

$$\sigma_3 - \underbrace{A_\sigma(r,u,\cdot)}_{\text{will cancel in Bianchi equ.}} = \underbrace{\sigma_{\frac{1}{2}}(r,u,\cdot) + \sigma_\beta(r,u,\cdot) + B_\sigma(r,u,\cdot)}_{\text{will impact gravitational radiation, more structures}} + l.o.t.$$

Theorem [L. Bieri (2020)]

For (B) spacetimes the following holds for the domain of dependence of the complement of a sufficiently large compact subset of the initial hypersurface. The quantities $r^3\rho_{\frac{1}{2}}, r^3\rho_\beta, r^3\sigma_{\frac{1}{2}}, r^3\sigma_\beta$ have limits on any null hypersurface C_u as $t \to \infty$. Namely, for $0 < \beta < \frac{1}{2}$,

$$\lim_{C_u, t\to\infty} (r^3\rho_{\frac{1}{2}}) = \mathcal{R}_{\frac{1}{2}}(u,\cdot) \quad , \qquad \lim_{C_u, t\to\infty} (r^3\rho_\beta) = \mathcal{R}_\beta(u,\cdot)$$

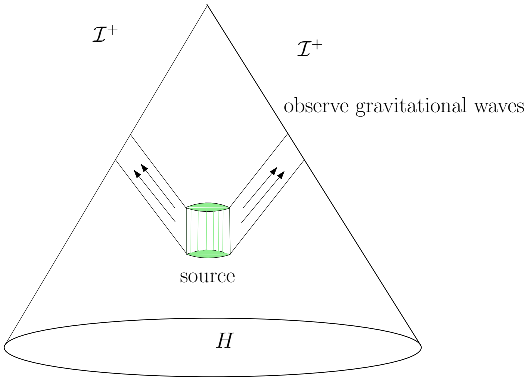
$$\lim_{C_u, t\to\infty} (r^3\sigma_{\frac{1}{2}}) = \mathcal{S}_{\frac{1}{2}}(u,\cdot) \quad , \qquad \lim_{C_u, t\to\infty} (r^3\sigma_\beta) = \mathcal{S}_\beta(u,\cdot)$$

$$\left|\mathcal{R}_{\frac{1}{2}}(u,\cdot)\right| \leqslant C(1+|u|)^{-1/2} \ , \ |\mathcal{R}_\beta(u,\cdot)| \leqslant C(1+|u|)^{-1+\beta}$$

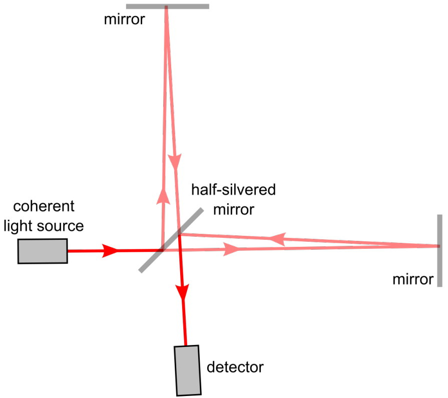
$$\left|\mathcal{S}_{\frac{1}{2}}(u,\cdot)\right| \leqslant C(1+|u|)^{-1/2} \ , \ |\mathcal{S}_\beta(u,\cdot)| \leqslant C(1+|u|)^{-1+\beta}$$

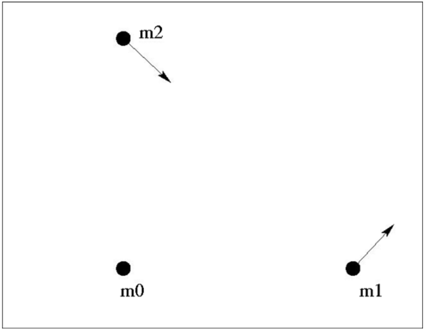
Gravitational Radiation

Gravitational waves travel from their source along null hypersurfaces to future null infinity $\mathcal{I}^+$. At $\mathcal{I}^+$ the detectors observe these waves.



Gravitational Radiation: Geodesic Equation and Detector

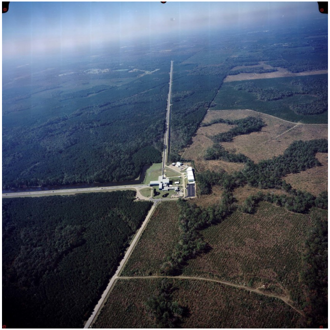




INSTANTANEOUS DISPLACEMENTS (while the wave packet is traveling through, measured by LIGO/VIRGO, first LIGO 2015).

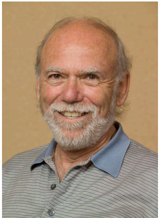
PERMANENT DISPLACEMENTS, (cumulative, stays after wave packet passed, expected to be measured in the near future). This is called the memory effect of gravitational waves. Two types of this memory.

Measurements - Beginning of a New Era



LIGO Facility at Livingston LA

The Nobel Prize in Physics 2017 was awarded with one half to Rainer Weiss, and the other half jointly to Barry C. Barish, and Kip S. Thorne. LIGO/VIRGO collaboration. Photos: Weiss and Thorne, Courtesy of Gruber Foundation; Barish, public domain.



Memory Effect of Gravitational Waves

- Ordinary (formerly called "linear") effect
  => in the slow motion limit [Ya.B. Zel'dovich, A.G. Polnarev 1974]
- Null (formerly called "nonlinear") effect
  => in the fully nonlinear case [D. Christodoulou 1991].
- Early Works on Memory: T. Damour, L. Blanchet, V. B. Braginsky, L. P. Grishchuk, C. M. Will , A. G. Wiseman, K. S. Thorne, J. Frauendiener.
- Other Related Early Works: [A. Ashtekar and various co-authors (1970s and 1980s)] Studies of asymptotic symmetries in GR and infrared problems in quantum field theory.
- 2016: A paper by P. Lasky, E. Thrane, Y. Levin, J. Blackman and Y. Chen suggests a method for detecting gravitational wave memory with aLIGO by stacking events.

Memory - Continued - Isolated Systems

Recent results and new memory effects:

- Contribution from electro-magnetic field to null effect
  => was found by [L. Bieri, P. Chen, S.-T. Yau 2010 and 2011].

- Contribution from neutrino radiation to null effect
  => was found by [L. Bieri, D. Garfinkle 2012 and 2013].

- For the first time outside of GR, for pure Maxwell equations:
  We find an electromagnetic analog of gravitational wave memory.
  EM Memory. [L. Bieri, D. Garfinkle 2013]
  $\Rightarrow$ charged test masses observe a residual kick.

- Recent works on memory include Wald, Tolish, Favata, Flanagan, Nichols, Strominger, Winicour, Loutrel, Mädler, Yunes, Hawking, Perry, Zhiboedov, Pasterski, Prabhu, Satishchandran, Hollands, Ishibashi, and more.
- A growing field of research....

Memory - Permanent Displacement

Structures at $\mathcal{I}^+$. Intricate local structures have implications at $\mathcal{I}^+$. Certain geometric quantities take well-defined limits at $\mathcal{I}^+$ and obey specific equations.

The permanent displacement $\triangle x$ of geodesics (marked by test masses in a detector) is related to the difference $(Chi^- - Chi^+)$ at $\mathcal{I}^+$:

$$\triangle x = -\frac{d_0}{r}\left(Chi^- - Chi^+\right) , \tag{36}$$

where d_0 denotes the initial distance between the test masses, and Chi the null limit of a geometric quantity related to the shear (in spacetimes with stronger fall off it is the limit of the shear).

Contributions to the permanent displacement $\triangle x$:

AF systems with $O(r^{-1})$ fall off: "Simple" structure. The ordinary memory is sourced by the change in the radial component of the electric part of the Weyl tensor. The null memory is sourced by F, the energy per unit solid angle radiated to infinity (including shear and component of energy-momentum tensor).

(B) spacetimes: NEW and rich structures. Let's investigate these now.

Parity of Gravitational Waves and Memory

(M, g) denote our solution spacetimes.

The Weyl tensor $W_{\alpha\beta\gamma\delta}$ is decomposed into its electric and magnetic parts, which are defined by

$$E_{ab} := W_{atbt} \tag{37}$$

$$H_{ab} := \frac{1}{2}\varepsilon^{ef}{}_{a}W_{efbt} \tag{38}$$

Here ε_{abc} is the spatial volume element and is related to the spacetime volume element by $\varepsilon_{abc} = \varepsilon_{tabc}$. The electric part of the Weyl tensor is the crucial ingredient in the equation governing the distance between two objects in free fall. In particular, their spatial separation denoted by Δx^a:

$$\frac{d^2 \Delta x^a}{dt^2} = -E^a{}_b \Delta x^b \tag{39}$$

In this decomposition, it is

$$E_{NN} = \rho \ \ , \ \ H_{NN} = \sigma \ .$$

Electric and Magnetic Memory

Memory effect caused by the electric part of the curvature tensor
$\Rightarrow$ called *electric parity memory (i.e. electric memory)*.

Memory effect caused by the magnetic part of the curvature tensor
$\Rightarrow$ called *magnetic parity memory (i.e. magnetic memory)*.

So far

AF systems with $O(r^{-1})$ decay towards infinity
$\Rightarrow$ only electric parity memory, no magnetic memory occurs.

New (B, 2020 and 2021)

AF spacetimes of slower decay like (B) spacetimes
$\Rightarrow$ magnetic memory occurs naturally.

Overall memory is growing and new structures arise.

Shown for the Einstein vacuum equations and Einstein-null-fluid equations describing neutrino radiation. The new results hold as well for the Einstein equations coupled to other fields of slow decay towards infinity and obeying other corresponding properties.

“Unusual” Examples Versus Generic. New Structures

• G. Satishchandran and R. Wald (2019): An interesting and “unusual” example of stress-energy of an expanding shell in linearized gravity gives rise to an ordinary magnetic memory.

• T. Mädler and J. Winicour (2016): In a linearized setting, they showed that the special case of homogeneous, source-free gravitational waves coming in from past null infinity gives a magnetic memory. We can think of this as putting in magnetic memory by hand by placing these incoming waves at past null infinity.

• In both these examples (Satishchandran-Wald and Mädler-Winicour), magnetic memory is put in through the data.

New (B): We find *magnetic* memory and a wealth of *new structures contributing to electric and magnetic* memory for the Einstein vacuum as well as for the Einstein-null-fluid equations describing neutrino radiation. *It is not included in the initial data, but emerges in the evolution.* We show that this new magnetic memory occurs naturally for slowly decaying AF spacetimes. The new effects grow. In (A) spacetimes we find yet another new type of magnetic memory due to a $c\not{u}rl\, T$ term.

Further results in a different direction: Lower order structures generate further interesting dynamics. [L. Bieri (2021)].

Outgoing radiation is dominated by $\underline{\hat{\chi}}$.

Incoming radiation is dominated by $\hat{\chi}$.

Next, we are going to derive electric and magnetic parity memory for

1) the Einstein vacuum equations and

2) the Einstein-null-fluid equations describing neutrino radiation.

Main Theorem and Proof

Recall from above that $(Chi^- - Chi^+)$ is related to permanent displacement.

Simplified and first version of the main result:

$(Chi^- - Chi^+)$ *determined by equations at* $\mathcal{I}^+$
• *including terms sourced by "electric part of curvature" (always present)*
• *including terms sourced by "magnetic part of curvature" (only for slow fall-off)*

On S^2 *at* $\mathcal{I}^+$*: Let* $Z := \not{d}iv\,(Chi^- - Chi^+)$. *Equations for* Z *involve*

$$\begin{aligned} \not{d}iv\, Z &= \{\textit{structures involving electric part of curvature}\} \\ \not{c}url\, Z &= \{\textit{structures involving magnetic part of curvature}\} \\ &\quad \textit{plus further new structures} \end{aligned}$$

Next:

- Ideas and main steps of the proof of the main theorem.
- Includes intermediate theorems leading up to the main theorem.
- Presented as a "flow", focussing on the main structures.
- Official Version of the Main Theorem

Derivation of Electric Memory

Einstein vacuum equations:

Consider the Bianchi equation for $\not{D}_3 \rho$.

Notation $\rho_3 := \not{D}_3 \rho \ + \ \frac{3}{2} tr\underline{\chi}\rho$.

In the Bianchi equation for $\not{D}_3 \rho$

$$\not{D}_3 \rho \ + \ \frac{3}{2} tr\underline{\chi}\rho \quad = \quad -\not{div}\,\underline{\beta} - \frac{1}{2}\hat{\chi}\underline{\alpha} \ + \ (\varepsilon - \zeta)\underline{\beta} \ + \ 2\underline{\xi}\beta \quad (40)$$

we focus on the higher order terms,

$$\rho_3 \ = \ - \quad \underbrace{\not{div}\,\underline{\beta}}_{=O(r^{-3}\tau_-^{-\frac{1}{2}})} \quad - \quad \underbrace{\frac{1}{2}\hat{\chi} \cdot \underline{\alpha}}_{=O(r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}})} \quad + \ l.o.t.$$

A short computation shows that

$$\rho_3 = - \underbrace{\not{d}iv\,\underline{\beta}}_{=O(r^{-3}\tau_-^{-\frac{1}{2}})} - \underbrace{\frac{\partial}{\partial u}(\hat{\chi}\cdot\underline{\hat{\chi}})}_{=O(r^{-\frac{5}{2}}\tau_-^{-\frac{3}{2}})} + \underbrace{\frac{1}{4}tr\chi|\underline{\hat{\chi}}|^2}_{=O(r^{-3}\tau_-^{-1})} + l.o.t.$$

Thus it is

$$\rho_3 + \frac{\partial}{\partial u}(\hat{\chi}\cdot\underline{\hat{\chi}}) = -\not{d}iv\,\underline{\beta} + \frac{1}{4}tr\chi|\underline{\hat{\chi}}|^2 = O(r^{-3}\tau_-^{-\frac{1}{2}}) \quad (41)$$

Structures:

For large data, various terms of order $r^{-\frac{5}{2}}\tau_-^{-1-\alpha}$ with $\alpha \geqslant 0$ on the left hand side of (41), but these cancel.

Limit at $\mathcal{I}^+$ of the left hand side of (41)

$\Rightarrow$ leading order term originates from ρ_3 and is of order $O(r^{-3}\tau_-^{-\frac{1}{2}})$.

Future Null Infinity and Electric Memory

Notation for the corresponding limit of the LHS of (41):

$$\mathcal{P}_3 := \lim_{C_u, t\to\infty} r^3 \big(\rho_3 + \frac{\partial}{\partial u}(\hat{\chi} \cdot \underline{\hat{\chi}})\big) \tag{42}$$

$$\mathcal{P} := \int_u \mathcal{P}_3 \, du \tag{43}$$

Note that $\mathcal{P}$ is defined on $S^2 \times \mathbb{R}$ up to an additive function $C_{\mathcal{P}}$ on S^2 (thus the latter is independent of u). Later, when taking the integral $\int_{-\infty}^{+\infty} \mathcal{P}_3 \, du$, the term $C_{\mathcal{P}}$ will cancel.

Taking the limit of $\big(r^3 \text{ (41)}\big)$ on C_u as $t \to \infty$, each term on the right hand side takes a well-defined limit. This yields

$$\mathcal{P}_3 = -\operatorname{d\!\!/iv} \underline{B} + 2|\Xi|^2 \tag{44}$$

Details for $\mathcal{P}$ in (43):
Taking into account all these structures we derive:
$\mathcal{P}$ has the following structure for $0 < \beta < \frac{1}{2}$ and $\gamma > 0$,

$$\mathcal{P} \quad = \quad \underbrace{\{\tau_-^{+\frac{1}{2}}\} \ + \ \{\tau_-^{\beta}\}}_{=\mathcal{P}_{\rho_1}} + \underbrace{\{\mathcal{F}(u,\cdot)\}}_{=\mathcal{P}_{\rho_2}-\frac{1}{2}D} + \{\tau_-^{-\gamma}\} \ + \ C_{\mathcal{P}} \qquad (45)$$

where $\mathcal{F}(u,\cdot) \leqslant C$. Again $\{\cdot\}$ as explained above. And $C_{\mathcal{P}}$ is an additive function on S^2 introduced above. Terms of order $O(\tau_-^{\alpha})$ with $0 < \alpha \leqslant \frac{1}{2}$ originate from the integral of the limits of the ρ_3 part. Denote this part by $\mathcal{P}_{\rho_1}$. In (45), the quantity $\mathcal{F}(u,\cdot)$ has pieces that are sourced by ρ_3 and pieces that are sourced by $\frac{\partial}{\partial u}(\hat{\chi} \cdot \underline{\hat{\chi}})$, we denote the former by $\mathcal{P}_{\rho_2}$ and the latter by $-\frac{1}{2}D$.

Next, we define

$$
\begin{aligned}
Chi_3 &:= \lim_{C_u, t \to \infty} \left(r^2 \frac{\partial}{\partial u} \hat{\chi} \right) && (46) \\
Chi &:= \int_u Chi_3 \; du && (47)
\end{aligned}
$$

We have (see before)

$$
\underline{B} = -2 d\not{i}v \, \Xi \quad , \quad Chi_3 = -\Xi \tag{48}
$$

Using these with the above we obtain

$$
\mathcal{P}_3 \quad = \quad -2 d\not{i}v \, d\not{i}v \, Chi_3 + 2|\Xi|^2 \tag{49}
$$

Integrating (49) with respect to u gives

$$
(\mathcal{P}^- - \mathcal{P}^+) - \int_{-\infty}^{+\infty} |\Xi|^2 \; du \;=\; d\not{i}v \, d\not{i}v \, (Chi^- - Chi^+) \tag{50}
$$

In $(\mathcal{P}^- - \mathcal{P}^+)$ an abundance of new terms, leading order $|u|^{+\frac{1}{2}}$.

Derivation of Magnetic Memory

Consider the Bianchi equation for $\not{D}_3\sigma$.

Notation $\sigma_3 = \not{D}_3\sigma + \frac{3}{2}tr\underline{\chi}\sigma$. In the Bianchi equation for σ_3

$$\sigma_3 \;=\; -c\not{u}rl\,\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\;{}^*\underline{\alpha} + \varepsilon\;{}^*\underline{\beta} - 2\zeta\;{}^*\underline{\beta} - 2\underline{\xi}\;{}^*\beta$$

we concentrate on the higher order terms

$$\sigma_3 \quad = \quad -c\not{u}rl\,\underline{\beta} - \frac{1}{2}\hat{\chi}\cdot\;{}^*\underline{\alpha} + l.o.t. \qquad (51)$$

A short computation yields

$$\sigma_3 + \frac{\partial}{\partial u}(\hat{\chi}\wedge\underline{\hat{\chi}}) \quad = \quad -c\not{u}rl\,\underline{\beta} \;=\; O(r^{-3}\tau_-^{-\frac{1}{2}}) \qquad (52)$$

For $\hat{\chi}\wedge\underline{\hat{\chi}}$ the orders of the terms are at the level of $\hat{\chi}\cdot\underline{\hat{\chi}}$ above.

Multiply the left hand side of (52) by r^3 and and take the limit on each C_u for $t \to \infty$ denoting this limit by $\mathcal{Q}_3$. Then introduce $\mathcal{Q}$ as follows:

$$\mathcal{Q}_3 := \lim_{C_u, t \to \infty} r^3 \big(\sigma_3 + \frac{\partial}{\partial u} (\hat{\chi} \wedge \underline{\hat{\chi}}) \big) \tag{53}$$

$$\mathcal{Q} := \int_u \mathcal{Q}_3 \, du \tag{54}$$

Note that $\mathcal{Q}$ is defined on $S^2 \times \mathbb{R}$ up to an additive function $C_{\mathcal{Q}}$ on S^2 (thus the latter is independent of u). Later, when taking the integral $\int_{-\infty}^{+\infty} \mathcal{Q}_3 \, du$, the term $C_{\mathcal{Q}}$ will cancel.

Taking the limit of $\big(r^3$ (52)$\big)$ on C_u as $t \to \infty$, the term on the right hand side takes a well-defined limit. This yields

$$\mathcal{Q}_3 = -c\not{u}rl \, \underline{B} \tag{55}$$

Details for $\mathcal{Q}$ in (54):
Taking into account all these structures we derive:
$\mathcal{Q}$ has the following structure for $0 < \beta < \frac{1}{2}$ and $\gamma > 0$,

$$\mathcal{Q} = \underbrace{\{\tau_-^{+\frac{1}{2}}\} + \{\tau_-^{\beta}\}}_{=\mathcal{Q}_{\sigma_1}} + \underbrace{\{\mathcal{F}(u,\cdot)\}}_{=\mathcal{Q}_{\sigma_2}-\frac{1}{2}G} + \{\tau_-^{-\gamma}\} + C_{\mathcal{Q}} \quad (56)$$

where $\mathcal{F}(u,\cdot) \leqslant C$. Again $\{\cdot\}$ as explained above. And $C_{\mathcal{Q}}$ is an additive function on S^2 introduced above. Terms of order $O(\tau_-^{\alpha})$ with $0 < \alpha \leqslant \frac{1}{2}$ originate from the integral of the limits of the σ_3 part. Denote this part by $\mathcal{Q}_{\sigma_1}$. In (56), the quantity $\mathcal{F}(u,\cdot)$ has pieces that are sourced by σ_3 and pieces that are sourced by $\frac{\partial}{\partial u}(\hat{\chi} \wedge \underline{\hat{\chi}})$, we denote the former by $\mathcal{Q}_{\sigma_2}$ and the latter by $-\frac{1}{2}G$.

Continue to compute using equation (55):

Consider (55) and employ the derived relations between $\hat{\chi}$, $\underline{\hat{\chi}}$ and $\underline{\beta}$ as well as the corresponding limits (35) and (48) to compute

$$\mathcal{Q}_3 \ = \ -2 \ c\not{u}rl \, di\not{v} \, Chi_3 \tag{57}$$

Integrating (57) with respect to u yields

$$(\mathcal{Q}^- - \mathcal{Q}^+) \ = \ c\not{u}rl \, di\not{v} \, (Chi^- - Chi^+) \tag{58}$$

In $(\mathcal{Q}^- - \mathcal{Q}^+)$ an abundance of new terms, leading order $|u|^{+\frac{1}{2}}$.

We obtain

$$(\mathcal{Q}^-_{\sigma_1} - \mathcal{Q}^+_{\sigma_1}) \ + \ (\mathcal{Q}^-_{\sigma_2} - \mathcal{Q}^+_{\sigma_2}) \ - \ \frac{1}{2}(G^- - G^+) \\ = \ c\not{u}rl \, di\not{v} \, (Chi^- - Chi^+) \tag{59}$$

- Behavior of $(\mathcal{Q}^- - \mathcal{Q}^+)$ as well as $c\not{u}rl \, di\not{v} \, (Chi^- - Chi^+)$:
  Fix a point on the sphere S^2 at fixed u_0 and consider $\mathcal{Q}(u_0)$. Next, take $\mathcal{Q}(u)$ at the corresponding point for some value of $u \neq u_0$. Keep u_0 fixed and let u tend to $+\infty$, respectively to $-\infty$. Then the difference $\mathcal{Q}(u) - \mathcal{Q}(u_0)$ is no longer finite, but it grows with $|u|^{+\frac{1}{2}}$. A corresponding argument holds for $Chi(u) - Chi(u_0)$.

- $(G^- - G^+)$ is finite. Contributions rooted in magnetic Weyl curvature and shears (shears: sourced by $\int_u \frac{\partial}{\partial u}(\hat{\chi} \wedge \underline{\hat{\chi}}) \, du$).

- In AF systems with fall-off $O(r^{-1})$ towards infinity, each term in the above equation is identically zero.

- $\mathcal{Q}$ part features terms of diverging order $|u|^{+\frac{1}{2}}$, $|u|^{+\beta}$ for $0 < \beta < \frac{1}{2}$. Rooted in magnetic Weyl curvature.

Gravitational Waves: New Structures

Gravitational Wave Memory: Electric and Magnetic

The above gives the main ingredients in the proof of the following theorem.

Theorem [L. Bieri (2020)]

The following holds for (B) spacetimes.
$(Chi^- - Chi^+)$ is determined by equation (63) on S^2 where Ψ solves (64) and Φ solves (65).

Electric and Magnetic Parts

Next, we are going to COMBINE the two parts.

Electric and Magnetic Memory

There exist functions Φ and Ψ such that

$d\!\!\!/iv\,(Chi^{-} - Chi^{+}) = \nabla\!\!\!/\,\Phi + \nabla\!\!\!/^{\perp}\Psi$.

Let $Z := d\!\!\!/iv\,(Chi^{-} - Chi^{+})$. Note that then the following holds:

$$d\!\!\!/iv\, Z \;=\; \triangle\!\!\!/\,\Phi \quad , \qquad c\!\!\!/url\, Z \;=\; \triangle\!\!\!/\,\Psi \quad .$$

We obtain the system on S^2

$$\begin{aligned} d\!\!\!/iv\,(Chi^{-} - Chi^{+}) &= \nabla\!\!\!/\,\Phi + \nabla\!\!\!/^{\perp}\Psi & (60) \\ c\!\!\!/url\, d\!\!\!/iv\,(Chi^{-} - Chi^{+}) &= \triangle\!\!\!/\,\Psi \\ &= (\mathcal{Q} - \bar{\mathcal{Q}})^{-} - (\mathcal{Q} - \bar{\mathcal{Q}})^{+} & (61) \\ d\!\!\!/iv\, d\!\!\!/iv\,(Chi^{-} - Chi^{+}) &= \triangle\!\!\!/\,\Phi \\ &= (\mathcal{P} - \bar{\mathcal{P}})^{-} - (\mathcal{P} - \bar{\mathcal{P}})^{+} \\ &\quad -2(F - \bar{F}) & (62) \end{aligned}$$

Taking into account the detailed structures, we have the following system on S^2, that is solved by Hodge theory.

$$\begin{aligned}
\not{d}i\not{v}\,(Chi^- - Chi^+) &= \not{\nabla}\,\Phi + \not{\nabla}^{\perp}\Psi && (63)\\
c\not{u}rl\,\not{d}i\not{v}\,(Chi^- - Chi^+) &= \not{\triangle}\,\Psi \\
&= (\mathcal{Q}_{\sigma_1} - \bar{\mathcal{Q}}_{\sigma_1})^- - (\mathcal{Q}_{\sigma_1} - \bar{\mathcal{Q}}_{\sigma_1})^+ \\
&\quad +(\mathcal{Q}_{\sigma_2} - \bar{\mathcal{Q}}_{\sigma_2})^- - (\mathcal{Q}_{\sigma_2} - \bar{\mathcal{Q}}_{\sigma_2})^+ \\
&\quad -\frac{1}{2}(G - \bar{G})^- + \frac{1}{2}(G - \bar{G})^+ && (64)\\
\not{d}i\not{v}\,\not{d}i\not{v}\,(Chi^- - Chi^+) &= \not{\triangle}\,\Phi \\
&= (\mathcal{P}_{\rho_1} - \bar{\mathcal{P}}_{\rho_1})^- - (\mathcal{P}_{\rho_1} - \bar{\mathcal{P}}_{\rho_1})^+ \\
&\quad (\mathcal{P}_{\rho_2} - \bar{\mathcal{P}}_{\rho_2})^- - (\mathcal{P}_{\rho_2} - \bar{\mathcal{P}}_{\rho_2})^+ \\
&\quad -2(F - \bar{F}) && (65)\\
&\quad -\frac{1}{2}(D - \bar{D})^- + \frac{1}{2}(D - \bar{D})^+
\end{aligned}$$

For the more general spacetimes of slow decay (like (B)) we conclude:

1. There is the new magnetic memory effect growing with $|u|^{\frac{1}{2}}$ sourced by $\mathcal{Q}$, rooted in the magnetic Weyl curvature and finite contributions rooted in curvature and the shears.

2. $\mathcal{Q}$ has further diverging terms at lower order.

3. There is the electric memory, previously established. This electric part is growing with $|u|^{\frac{1}{2}}$ sourced by $\mathcal{P}$, further lower-order growing terms and finite contributions from $\mathcal{P}$ and from F (the latter may be unbounded for systems of decay $O(r^{-\frac{1}{2}})$).

4. $c\not{u}rl \; di\not{v} \, (Chi^- - Chi^+)$ being non-trivial allows for the magnetic structures to appear in gravitational radiation and to enter the permanent changes of the spacetime. Thus, these more general spacetimes generate memory of magnetic type.

Points 1, 2, 4 are NEW.

Point 3, the leading order behavior as well as the null memory were established in (B, 2018). The finer structures are new.

Adding Neutrinos

(B 2020) Einstein-null-fluid equations describing neutrino radiation:

$$R_{\mu\nu} \;=\; 8\pi \, T_{\mu\nu} \; .$$

Describe the neutrinos in this equation, represented via the energy-momentum tensor given by

$$T^{\mu\nu} \;=\; \mathcal{N} K^{\mu} K^{\nu} \qquad (66)$$

with K being a null vector and $\mathcal{N} = \mathcal{N}(\theta_1, \theta_2, r, \tau_-)$ a positive scalar function depending on r, τ_-, and the spherical variables θ_1, θ_2.

When coupled to the Einstein equations in the most general settings, the energy-momentum tensor $T^{\mu\nu}$ obeys those loose decay laws. No symmetry nor other restrictions imposed.

In particular, we do not have stationarity outside a compact set, but instead a distribution of neutrinos decaying very slowly towards infinity.

"Geometric terms": same growth rate as in EV case.

"T" terms: growing at rate $\sqrt{|u|}$.

Outlook

Outlook

- Cosmological setting: Study the corresponding problem.
- Dark matter of certain types may behave as described here. Investigate dark matter, including dark matter halos of galaxies.
- Couple Einstein equations to other types of matter-energy to investigate similar questions.
- Many more fascinating questions....

Thank you!