The black hole stability problem

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This talk is about the stability of Kerr-de Sitter (KdS) black holes, as solutions of Einstein's equation. While it matters little, I will use the $(+,-,\ldots,-)$ signature convention for the Lorentzian metric, so in 4 dimensions Einstein's equation in vacuum is an equation for the metric tensor of the form

$$\mathrm{Ric}(g) + \Lambda g = 0,$$

where Λ is the cosmological constant, and $\mathrm{Ric}(g)$ is the Ricci curvature of the metric. If there were matter present, there would be a non-trivial right hand side of the equation, given by (a modification of) the matter's stress-energy tensor.

E.g. the Minkowski metric solves this with $\Lambda = 0$. Another solution, with $\Lambda > 0$, is de Sitter space, which is the one-sheeted hyperboloid in one higher dimensional Minkowski space. We will mostly be interested in $\Lambda > 0$, which corresponds to the current understanding of the universe.

In local coordinates, the Ricci curvature is a non-linear expression in up to second derivatives of g; thus, this is a partial differential equation. The type of PDE that Einstein's equation is most similar to (with issues!) are (tensorial, non-linear) wave equations. The typical formulation of such a wave equation is that one specifies 'initial data' at a (spacelike) Cauchy hypersurface Σ_0.

For linear wave equations $\Box u = f$ on globally hyperbolic spaces like Minkowski space and de Sitter space, the solution u for given data exists globally and is unique. For nonlinear equations this is not automatic. In both cases another natural question is one perturbs the Cauchy data for a solution u_0, does the solution u asymptote to u_0? (Asymptotic stability.)

Since Ric is diffeomorphism invariant, if Ψ is a diffeomorphism, and g solves Einstein's equation, then so does $\Psi^* g$. This means that if there is one solution, there are many (even with same IC). In practice (duality) this means that it may not be easy to solve the equation at all!

This already indicates that Einstein's equation is not quite a wave equation, but it can be turned into one by imposing extra gauge conditions. One version is the harmonic/wave/DeTurck's gauge: one fixes a background metric $\mathfrak{g}_0$, and requires that the identity map $(M, g) \to (M, \mathfrak{g}_0)$ be a wave map (solve a wave equation)...

One version is the harmonic/wave/DeTurck's gauge: one fixes a background metric $\mathfrak{g}_0$, and requires that the identity map $(M, g) \to (M, \mathfrak{g}_0)$ be a wave map (solve a wave equation). This is implemented using the second key property, the 2nd Bianchi identity, $\delta_g G_g \mathrm{Ric}(g) = 0$ for all g, where δ_g is the (negative) divergence (adjoint of the symmetric gradient δ_g^*), and $G_g r = r - \frac{1}{2}(\mathrm{tr}_g\, r)g$. An implementation is

$$\mathrm{Ric}(g) + \Lambda g - \Phi(g, \mathfrak{g}_0) = 0,$$

where

$$\Phi(g, \mathfrak{g}_0) = \delta_g^* \Upsilon(g),\ \Upsilon(g) = g\mathfrak{g}_0^{-1} \delta_g G_g \mathfrak{g}_0.$$

The point is that this *is* a (quasilinear) wave-type equation, so the problems with diffeomorphism invariance have been eliminated, thus at least one has local existence and uniqueness near the initial surface Σ_0! That this relates to Einstein's equation is assured by the 2nd Bianchi identity which gives rise to $\Box^{\mathrm{CP}} = 2\delta_g G_g \delta_g^*$, a one-form wave operator. This enabled Choquet-Bruhat to show local well-posedness: $\Upsilon = 0$.

The first stability results were obtained for Minkowski space and de Sitter space, respectively, and are due to Christodoulou and Klainerman (1990s), later simplified by Lindblad and Rodnianski (2000s) (and extended by Bieri and Zipser, and in a different direction by Hintz and V.), resp. Friedrich (1980s).

Our result with Peter Hintz is the stability of Kerr-de Sitter (KdS) black holes (slowly rotating). These are family of metrics depending on two parameters, called mass m and angular momentum a. The $a = 0$ members of the family are called the Schwarzschild-de Sitter (SdS) black holes; $h_{\mathbb{S}^2}$ the metric on $\mathbb{S}^2$:

$$g = \mu(r)\,dt^2 - \mu(r)^{-1}\,dr^2 - r^2\,h_{\mathbb{S}^2},\ \ \mu(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3},$$

- $\Lambda = 0$ gives the Schwarzschild metric, discovered in about a month after Einstein's 1915 paper.
- $m = 0$ gives the de Sitter metric.

Recall:

$$g = \mu(r)\,dt^2 - \mu(r)^{-1}\,dr^2 - r^2\,h,\ \ \mu(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}.$$

- $\mu(r) = 0$ has two positive solutions r_+, r_- if $m, \Lambda > 0$; if $\Lambda = 0$ the only root is $2m$, if $m = 0$, the only root is $\sqrt{3/\Lambda}$.
- In this form the metric makes sense where $\mu > 0$: $\mathbb{R}_t \times (r_-, r_+)_r \times \mathbb{S}^2$.
- However, $r = r_\pm$ are coordinate singularities; with $c_\pm$ smooth,

  $$t_* = t - F(r),\ \ F'(r) = \pm(\mu(r)^{-1} + c_\pm(r)) \text{ near } r = r_\pm$$

  desingularizes them and extends the metric to

  $$\mathbb{R}_{t_*} \times (0, \infty)_r \times \mathbb{S}^2_\omega,$$

- $r = r_-$ is called the *event horizon*, $r = r_+$ the *cosmological horizon*; they are very similar for the geometry.
- ∂_t is a Killing vector field, i.e. translation in t preserves the metric, and it is spherically symmetric.

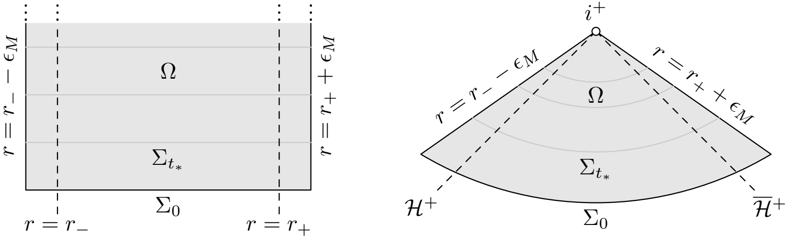

Figure: Setup for the initial value problem for perturbations of a Schwarzschild–de Sitter spacetime (M, g_{b_0}), showing the Cauchy surface Σ_0 of Ω and a few translates (level sets of the nonsingular time t_*) Σ_{t_*}; here $\epsilon_M > 0$ is small. *Left:* Product-type picture, illustrating the stationary nature of g_{b_0}. *Right:* Penrose diagram of the same setup. The event horizon is $\mathcal{H}^+ = \{r = r_-\}$, the cosmological horizon is $\overline{\mathcal{H}}^+ = \{r = r_+\}$, and the (idealized) future timelike infinity is i^+.

Without specifying the general KdS metric, we just mention that the underlying manifold is a $\mathbb{R}_{t_*} \times (0,\infty)_r \times \mathbb{S}^2$, and ∂_{t_*} is a Killing vector field, i.e. translation in t_* preserves the metric. These metrics are axisymmetric around the axis of rotation.

In general, for a manifold M with Σ_0 a codimension 1 hypersurface, the initial data are a Riemannian metric h and a symmetric 2-cotensor k which satisfy the constraint equations (needed for solvability), and one calls a Lorentzian metric g on M a solution of Einstein's equation with initial data (Σ_0, h, k) if the pull-back of g to Σ_0 is $-h$, and k is the second fundamental form of Σ_0 in M.

Our main result is the global non-linear asymptotic stability of the Kerr-de Sitter family for the initial value problem for small angular momentum a on the space

$$\Omega = [0,\infty)_{t_*} \times [r_- - \delta, r_+ + \delta]_r \times \mathbb{S}^2.$$

Theorem (Hintz-V '16; informal version)

Suppose (h, k) are smooth initial data on Σ_0, satisfying the constraint equations, which are close to the data (h_{b_0}, k_{b_0}) of a Schwarzschild–de Sitter spacetime, $b_0 = (m_0, 0)$, in a high regularity norm. Then there exist a solution g of Einstein's equation in Ω attaining these initial data at Σ_0, and black hole parameters $b = (m, a)$ which are close to b_0, so that

$$g - g_b = \mathcal{O}(e^{-\alpha t_*})$$

for a constant $\alpha > 0$ independent of the initial data; that is, g decays exponentially fast to the Kerr–de Sitter metric g_b. Moreover, g and b are quantitatively controlled by (h, k).

What the theorem states is that the metric 'settles down to' a Kerr-de Sitter metric at an exponential rate. Note that even if we perturb a Schwarzschild-dS metric, we get a KdS limit!

This 'settling down' means that gravitational waves are being emitted; far away observers can see this 'tail'. LIGO exactly aimed (successfully!) at capturing these waves, using numerical computations as a template to see what one would expect from the merger of binary black holes.

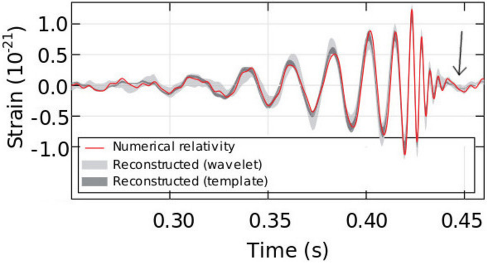

Figure: LIGO/Virgo collaboration 2016

For $\Lambda = 0$, at this point the strongest nonlinear result is that of Dafermos, Holzegel, Rodnianski and Taylor (2021) which is a finite codimension Schwarzschild stability result; this followed the earlier restricted (symmetry) stability result for Schwarzschild of Klainerman and Szeftel (2017).

Linearized $\Lambda = 0$ black hole results: Schwarzschild, plus Teukolsky in the slowly rotating case: Dafermos, Holzegel and Rodnianski (2016, 2017), as well as the stability result of Andersson, Bäckdahl, Blue and Ma (2019), also in the slowly rotating case, with also a more restricted general result, under a strong asymptotic assumption, and the slowly rotating stability result of Hintz-Häfner-V. (2019) which allows more slow decay on data.

Other works by Wald, Kay, Finster, Kamran, Smoller, Yau, Tataru, Tohaneanu, Marzuola, Metcalfe, Sterbenz, Donninger, Schlag, Soffer, Sá Barreto, Wunsch, Zworski, Wang, Bony, Dyatlov, Luk, Ionescu, Shlapentokh-Rothman, Giorgi, Teixera da Costa, Casals...

The analytic framework we use

- non-elliptic linear global analysis with coefficients of finite Sobolev regularity,
- with a simple global Nash-Moser iteration to deal with the loss of derivatives corresponding to both non-ellipticity and trapping

gives global solvability for quasilinear wave equations like the gauged Einstein's equation provided

- certain dynamical assumptions are satisfied (only trapping is normally hyperbolic trapping, with an appropriate subprincipal symbol condition) and
- there are no exponentially growing modes (with a precise condition on non-decaying ones), i.e. non-trivial solutions of the linearized equation at g_{b_0} of the form $e^{-i\sigma t_*}$ times a function of the spatial variables r, ω only, with $\operatorname{Im}\sigma > 0$.

Let us start by discussing the analytic aspects.

The main point that forces one to face major issues from the start is that we solve *all* the linear and non-linear problems *globally* on the underlying ‘physical space’ Ω.

The non-linear aspects can be reduced to a precise understanding of underlying linear problems, via linearization and an iteration such as Newton or Nash-Moser (requiring the finite coefficient regularity linear theory), so I will not talk about these.

The finite coefficient regularity version of the usual theory mostly simply requires slightly more care, so again I will not discuss this. A sharp version was worked out in Hintz’ thesis (cf. Beals and Reed in the 1980s), but if one uses Nash-Moser, one can be much more forgiving.

So let us discuss the smooth coefficient linear analysis briefly.

For this, one needs to specify some function spaces (usually with considerable freedom) X, Y, and consider the continuous map (the linearization)

$$P : X \to Y.$$

In spite of the considerable freedom, it is *crucial* to be able to fix these spaces. Note also that while many choices may be equivalent, other choices may result in very different operators (cf. boundary conditions)!

For us, X, Y are (almost) decaying exponentially weighted, in t_*, Sobolev spaces, $H^{s,\ell} = e^{-\ell t_*} H^s$ (with some extra information on the initial/final 'Cauchy surfaces'.

Solvability is a surjectivity statement...

Solvability is a surjectivity statement. The almost-surjective version is the semi-Fredholm estimate

$$\|v\|_{Y^*} \leq C(\|P^* v\|_{X^*} + \|v\|_Z),$$

where the inclusion map $X^* \to Z$ is compact. For us this has the form

$$\|v\|_{H^{-s+m-1,-\ell}} \leq C(\|P^* v\|_{-s,-\ell} + \|v\|_{-N,-\ell'});$$

then $Y = H^{s-m+1,\ell}$, $X = \{u \in H^{s,\ell} : \ Pu \in Y\}$.

It is proved in a 2-step process. First one proves

$$\|v\|_{H^{-s+m-1,-\ell}} \leq C(\|P^* v\|_{-s,-\ell} + \|v\|_{-N,-\ell});$$

here the error term on the right hand side hand is non-compact because it has not become weaker in decay. This uses microlocal analysis. Here all the ingredients are in principle unchanged for the full subextremal range, provided one checks the dynamical hypotheses... this is being written up with Lindblad Petersen for the full subextremal range (the basic writeup is done).

Then in Step 2 one proves an estimate for a model operator at infinity (this will be Kerr-de Sitter as our solution is decaying to it)

$$\|v\|_{H^{-s'+m-1,-\ell}} \leq C\|P_0^* v\|_{-s',-\ell};$$

Applying this to v (with $-s'+m-1 \geq -N$) or its localized to large t_* version, and using that $P-P_0$ has decaying coefficients, thus maps into a more decaying space, one gets the semi-Fredholm estimate

$$\|v\|_{H^{-s+m-1,-\ell}} \leq C(\|P^* v\|_{-s,-\ell} + \|v\|_{-N,-\ell'}).$$

In order to have the P_0 estimate, one conjugates it by the Fourier transform to obtain a family $\hat{P}_0(\sigma)$ where σ is the (complex!) Fourier dual of $-t_*$; this is where the stationarity is used. One then automatically has a(n analytic) Fredholm theory for $\hat{P}_0(\sigma)$, corresponding to the Step 1 estimate...

In order to have the P_0 estimate, one conjugates it by the Fourier transform to obtain a family $\hat{P}_0(\sigma)$ where σ is the (complex!) Fourier dual of $-t_*$; this is where the stationarity is used. One then automatically has a(n analytic) Fredholm theory for $\hat{P}_0(\sigma)$, corresponding to the Step 1 estimate. (Actually, there is a small wrinkle for large $|a|$ in that one would like a non-trapping Fredholm theory for simplicity: this requires a better choice of ϕ_*, i.e. ∂_{t_*}; this is possible due to work with Lindblad Petersen.)

Thus, the question is invertibility, i.e. whether $\hat{P}_0(\sigma)$ has a non-trivial nullspace (index 0 follows from large σ considerations); this is *how* the resonances play a role. The net result is that as long as $-\ell$ is not the imaginary part of a resonance σ, one has the desired estimate, and so the Fredholm theory for P for all but a discrete set of weights (at least as long as the Step 1 theory allows this: trapping!).

In combination, when one is a weaker (more growing) space than given by all the resonances, one gets surjectivity/solvability. As one makes the decay stronger, crossing the resonances, one loses surjectivity, but it is simple to explain what happens: the solution in general will have a quasinormal mode component, corresponding to the resonances crossed.

Thus, one can solve these wave equations in exponentially decaying spaces, modulo a finite quasinormal mode expansion. Of course, for the non-linear problem the latter is an issue; one would want solvability in decaying spaces...

This completes the analytic discussion, modulo the mode analysis. (Cf. recent work of Casals and Texeira da Costa!)

Unfortunately, in the harmonic/wave/DeTurck gauge, while the dynamical assumptions are satisfied, there *are* growing modes, although only a finite dimensional space of these. The key to proving the theorem (given the analytic background) is to overcome this issue.

The Kerr-de Sitter family automatically gives rise to non-decaying modes with $\sigma = 0$, but as these correspond to non-linear solutions, one may expect these not to be a problem with some work.

One might then expect that the other non-decaying (including growing!) modes come from the diffeomorphism invariance, i.e. gauge issues, but this is not true at this stage: there are growing modes! (Explicit for de Sitter.)

However, we can arrange for a partial success: we can modify Φ by changing δ_g^* by a 0th order term:

$$\widetilde{\delta}^* \omega = \delta^*_{g_0} \omega + \gamma_1 \, dt_* \otimes_s \omega - \gamma_2 g_0 \, \mathrm{tr}_{g_0}(dt_* \otimes_s \omega),$$

$$\Phi(g, \mathfrak{g}_0) = \widetilde{\delta}^* \Upsilon(g).$$

For suitable choices of $\gamma_1, \gamma_2 \gg 0$, this preserves the dynamical requirements, and while the gauged Einstein's equation does still have growing modes, it has a new feature:

$$\widetilde{\Box}^{\mathrm{CP}}_g = 2\delta_g G_g \widetilde{\delta}^*, \;\; g = g_{b_0}$$

has no non-decaying modes! (There was no reason to expect that the DeTurck gauge is well-behaved in any way except in a high differential order sense, relevant for the local theory!)

Such a change to the gauge term, called 'constraint damping' or 'stable constraint propagation' (SCP), has been successfully used in the numerical relativity literature by Pretorius and others, following the work of Gundlach et al, to damp numerical errors in $\Upsilon(g) = 0$; here we show rigorously why such choices work well.

SCP is useful because it means that, for $g = g_{b_0}$, any non-decaying mode h of the linearized gauge fixed Einstein equation is a solution of $D_g(\mathrm{Ric}(g) + \Lambda g)h = 0$.

Indeed this follows by applying $\delta_g G_g$ to the gauge fixed Einstein's equation, using Bianchi's second identity, giving that $\widetilde{\Box}_g^{\mathrm{CP}}(D_g \Upsilon)h$ and thus $(D_g \Upsilon)h$ vanish. Thus, properties of a gauge dependent equation are reduced to those of one independent of the gauge!

Growing modes are disastrous for non-linear equations, such as Einstein's, so we also need a statement that the above modes are actually pure gauge modes, i.e. given by linearized diffeomorphisms, so of the form $\delta_g^*\omega$ for a one-form ω, corresponding to infinitesimal diffeomorphisms. We call this, together with a precise treatment of the zero modes, UEMS, ungauged Einstein mode stability.

UEMS is actually well-established in the physics literature in a form that is close to what one needs for a precise theorem; this goes back to Regge-Wheeler, Zerilli and others; the invariant form we use is due to Ishibashi, Kodama and Seto.

Now, without the KdS-family zero modes (we call such a setting UEMS*, which holds for dS), we could easily have a framework to show global stability: namely consider

$$\Phi(g, \mathfrak{g}_0; \theta) = \widetilde{\delta}^*(\Upsilon(g) - \theta),$$

where θ is an unknown, lying in a finite dimensional space Θ of gauge terms of the form $D_{g_{b_0}}\Upsilon(\delta^*_{g_{b_0}}(\chi\omega))$, where $\chi \equiv 1$ for $t_* \gg 1$, $\chi \equiv 0$ near $t_* = 0$, and such that $\delta^*_{g_{b_0}}\omega$ is a non-decaying resonance of the gauged Einstein operator.

As $D_{g_{b_0}}\Upsilon(\delta^*_{g_{b_0}}(\omega)) = 0$ by SCP, $D_{g_{b_0}}\Upsilon(\delta^*_{g_{b_0}}(\chi\omega))$ is compactly supported, away from Σ_0, i.e. elements of Θ are also such.

Then we could solve

$$\mathrm{Ric}(g) + \Lambda g - \Phi(g, \mathfrak{g}_0; \theta) = 0$$

for g and θ, with $g - g_{b_0}$ in a decaying function space. Crucially θ is also treated as an unknown.

This can be seen by solving the linearized equation without θ in a space which is decaying apart from finitely many non-decaying resonant modes, and then subtracting away cut off versions of these resonant terms and checking the equation they satisfy.

However, it is not hard to actually deal with the full KdS family by modifying our equation by adding another term to it which corresponds to the family and somewhat enlarging the space Θ.

The result is that for an appropriate finite dimensional space $\overline{\Theta}$ the nonlinear equation

$$(\mathrm{Ric}(g) + \Lambda g) - \widetilde{\delta}^*(\Upsilon(g) - \Upsilon(g_{b_0,b}) - \theta) = 0$$

with prescribed initial condition is solvable for g, θ, b with $\theta \in \overline{\Theta}$, b near b_0, and $g - g_b$ exponentially decaying; here $g_{b_0,b} = (1-\chi)g_{b_0} + \chi g_b$ is the asymptotic Kerr-de Sitter metric with parameter b. Thus, *both b and θ are found along with g in the nonlinear iteration!* This proves the nonlinear stability of the KdS family with small a.

Thank you!