FLAT FLRW and KASNER BIG BANG SINGULARITIES

analysed on the level of scalar waves

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Two take home messages

- The homogeneous wave equation

$$\Box_g \psi = 0$$

can serve as a "poor" linear proxy for the full Einstein field equations

- Certain open sets of solutions ψ blow up as we approach the Big Bang singularity

Behavior of ψ near the Big Bang

Why

What

How

Sneak Preview:

- Stability of the **Big Bang** singularity
- Investigation of $\Box_g \psi = 0$ as a "poor man's" linearisation to the Einstein field equations,
- using renormalized energy estimates, commutation with spatial derivatives and control of error and bulk terms.

Friedmann-Lemaître-Robertson-Walker spacetimes

Spacetimes $(\mathbb{R}_+ \times \mathbb{T}^3, g)$ with metrics:

$$g_{\mathrm{FLRW}} = \quad -dt^2 + t^{\frac{4}{3\gamma}}(dx_1^2 + dx_2^2 + dx_3^2), \qquad \frac{2}{3} < \gamma < 2,$$

is a solution to the Einstein-Euler system for ideal fluids with

$p = (\gamma - 1)\rho,$ with p pressure and ρ energy density.

$$\begin{cases} \gamma = \frac{2}{3}, & \text{coasting universe without spacelike singularity} \\ \frac{2}{3} < \gamma \leq \frac{4}{3}, & \text{for the softer phase} \\ \frac{4}{3} \leq \gamma < 2, & \text{for the stiffer phase} \\ \gamma = 2, & \text{for stiff fluids} \Rightarrow p = \rho, \quad \text{incompressibility:} \quad c_s = c = 1 \end{cases}$$

Kasner spacetimes

Spacetimes $(\mathbb{R}_+ \times \mathbb{T}^3, g)$ with metrics:

$$g_{\text{Kasner}} = -dt^2 + \sum_{j=1}^{3} t^{2p_j} dx_j^2,$$

$$\sum_{j=1}^{3} p_j = 1, \quad \sum_{j=1}^{3} p_j^2 = 1, \quad p_j < 1,$$

is a solution to the Einstein vacuum equations.

For both spacetimes we have a Big Bang singularity at $t = 0$, where curvature blows up $|\text{Riem}| \sim t^{-2}$, as $t \to 0$.

Motivation

- Derive condition under which the Big Bang singularity is stable.

- Show that the singularity is not just an artefact of cosmology (isotropy, homogeneity).

- Establish the boundaries of validity of general relativity.

Previous investigations:

Stability of the Big Bang (and the Big Crunch)

- Rodninanski, I. & Speck, J. (2014) :
  Perturbations of FLRW data for the Einstein-scalar field, with spatial topology $\mathbb{T}^3$, linearized around generalized Kasner solutions,
  $\rightarrow$ *linear* stability result for Big Bang.
  Einstein-stiff-fluid systems, $\gamma = 2$, with spatial topology $\mathbb{T}^3$
  $\rightarrow$ *non-linear* stability result for Big Bang,
  $\rightarrow$ *asymptotically velocity term dominated behavior* close to singularity.

- Speck, J. (2017) :
  Perturbations of FLRW data for the Einstein-scalar field system with spatial topology $\mathbb{S}^3$.
  $\rightarrow$ *non-linear* stability result for Big Bang and Big Crunch,
  $\rightarrow$ *asymptotically velocity term dominated behavior* close to singularity.

$\Rightarrow$ Monotonic blow-up behavior might not hold for typical matter models

Previous investigations:

Linear stability of the Big Bang:

- Allen, P. T. & Rendall, A. D. (2010):
  Scalar perturbations for fixed Einstein-Euler background, in $\mathbb{T}^3$ topology
  → near the singularity and at late times

- Petersen, O. (2016):
  Kasner modes with $\mathbb{R}^3$ topology.
  → modes in *non-flat* Kasner spacetimes *grow logarithmically* for small times,
  → modes in *flat* Kasner spacetimes stay *bounded* for small times,
  → modes in *general* Kasner spacetimes *oscillate* with a polynomially decreasing amplitude for large times.

- Ringström, H. (2017):
  Linear systems of wave equations on cosmological backgrounds with convergent asymptotics
  → *asymptotically velocity term dominated behavior*

Goal:

Stability of Big Bang singularity?

analyze behaviour of smooth solutions to $\Box_g \psi = 0$ towards the singularity as an initial value problem

formulation

Characterize open sets of initial data at a given time $t_0 > 0$ for which such *blow up behaviour* occurs at $t = 0$.

method

Derive appropriate *energy estimates in physical space*, which may also prove useful for dynamical studies.

Main results

<u>Main Theorem</u> *[Asymptotic profile]*
Let ψ *be a smooth solution to the wave equation,* $\Box_g \psi = 0$, *for either of the metrics* $g_{\mathrm{FLRW}}, g_{\mathrm{Kasner}}$, *arising from initial data* $(\psi_0, \partial_t \psi_0)$ *on* Σ_{t_0}. *Then,* ψ *can be written in the following form:*

$$\psi_{\mathrm{FLRW}}(t,x) = A_{\mathrm{FLRW}}(x) t^{1-\frac{2}{\gamma}} + u_{\mathrm{FLRW}}(t,x), \tag{1}$$

$$\psi_{\mathrm{Kasner}}(t,x) = A_{\mathrm{Kasner}}(x) \log t + u_{\mathrm{Kasner}}(t,x), \tag{2}$$

where $A(x), u(t,x)$ *are smooth functions and* $u_{FLRW} t^{\frac{2}{\gamma}-1}, u_{Kasner} (\log t)^{-1}$ *tend to zero, as* $t \to 0$.

Main results

<u>Main Theorem</u> *[Blow-up]*

Let ψ be a smooth solution to the wave equation, $\Box_g \psi = 0$, for either of the metrics $g_{\mathrm{FLRW}}, g_{\mathrm{Kasner}}$, arising from initial data $(\psi_0, \partial_t\psi_0)$ on Σ_{t_0}, $t_0 > 0$. If $\partial_t\psi_0$ is non-zero in $L^2(\mathbb{T}^3)$, t_0 is sufficiently small such that

$$\frac{2t_0^{2-\frac{4}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i=1}^{3}\|\partial_t\partial_{x_i}\psi_0\|^2_{L^2(\mathbb{T}^3)} < \epsilon\|\partial_t\psi_0\|^2_{L^2(\mathbb{T}^3)}, \qquad \text{(FLRW)} \qquad (1)$$

$$\sum_{i=1}^{3}\frac{2t_0^{2-2p_i}}{(1-p_i)^2}\|\partial_t\partial_{x_i}\psi_0\|^2_{L^2(\mathbb{T}^3)} < \epsilon\|\partial_t\psi_0\|^2_{L^2(\mathbb{T}^3)}, \qquad \text{(Kasner)} \qquad (2)$$

and $\psi_0, \partial_t\psi_0$ satisfy the open conditions

$$(1-\epsilon)\|\partial_t\psi_0\|^2_{L^2(\mathbb{T}^3)} > t_0^{-\frac{4}{3\gamma}}\sum_{i=3}^{3}\|\partial_{x_i}\psi_0\|^2_{L^2(\mathbb{T}^3)} + \frac{2t_0^{2-\frac{8}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i,j=1}^{3}\|\partial_{x_j}\partial_{x_i}\psi_0\|_{L^2(\mathbb{T}^3)}, \qquad \text{(FLRW)} \qquad (3)$$

$$(1-\epsilon)\|\partial_t\psi_0\|^2_{L^2(\mathbb{T}^3)} > \sum_{i=3}^{3}t_0^{-2p_i}\|\partial_{x_i}\psi_0\|^2_{L^2(\mathbb{T}^3)} + \sum_{i,j=1}^{3}\frac{2t_0^{2-2p_i-2p_j}}{(1-p_i)^2}\|\partial_{x_j}\partial_{x_i}\psi_0\|_{L^2(\mathbb{T}^3)}, \qquad \text{(Kasner)} \qquad (4)$$

for some $0 < \epsilon < 1$, then $\|A(x)\|_{L^2(\mathbb{T}^3)} > 0$.

Preliminaries

Energy currents and vector field method

The wave equation

$$\Box_g \psi = 0$$

can be derived from the matter field Lagrangian:

$$\mathcal{L}(\psi, \mathrm{d}\psi, g^{-1}) = \int_{\mathcal{M}} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \mathrm{dVol}.$$

A symmetric stress energy-momentum tensor can be identified:

$$T_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi.$$

Energy conservation:

$$\nabla^\mu T_{\mu\nu} = (\Box_g \psi) \mathrm{d}\psi = 0.$$

Preliminaries

Define the current:

$$J^V_\mu(\tilde{\psi}) \doteq T_{\mu\nu}(\tilde{\psi}) V^\nu,$$

and the divergence:

$$\nabla^\mu J_\mu = \nabla^\mu (T_{\mu\nu} V^\nu) = K^V + \mathcal{E}^V,$$

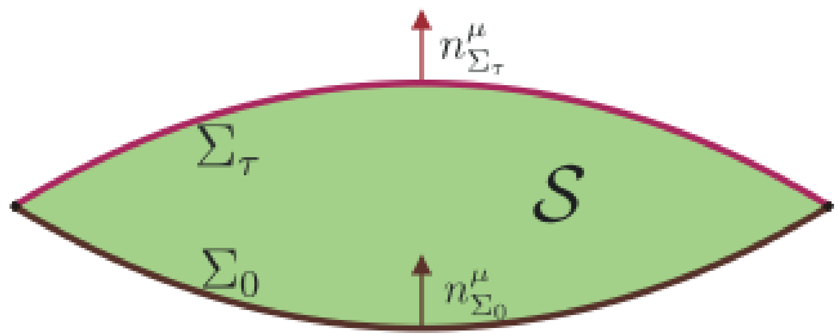
with the two scalar currents

$$\begin{aligned} K^V(\tilde{\psi}) &\doteq T(\tilde{\psi})(\nabla V) = \frac{1}{2}(\mathcal{L}_V g)^{\mu\nu} T_{\mu\nu}(\tilde{\psi}), \\ \mathcal{E}^V(\tilde{\psi}) &\doteq (\nabla^\mu T_{\mu\nu}) V^\nu = (\Box_g \tilde{\psi}) V(\tilde{\psi}). \end{aligned}$$

Preliminaries

The divergence theorem

To obtain Energy Theorem use versions of the divergence theorem. Consider a spacetime region $\mathcal{S}$ which is bounded by the homologous hypersurfaces Σ_τ and Σ_0 and obtain



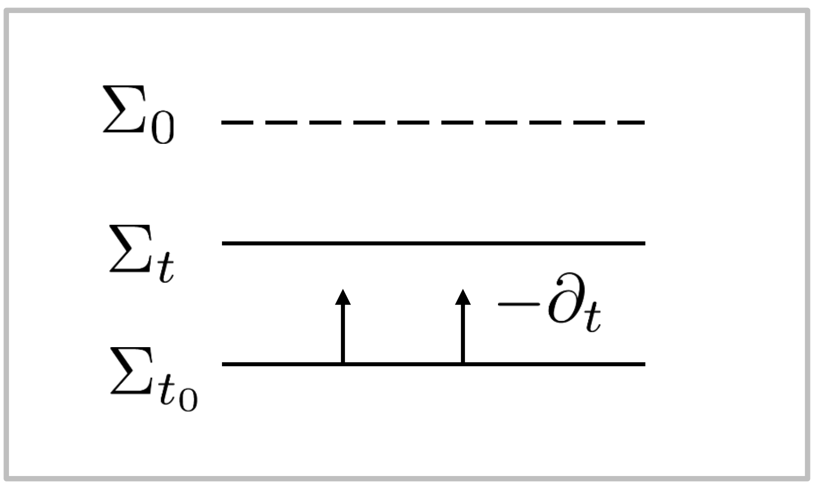
$$\int_{\Sigma_\tau} J^V_\mu(\tilde{\psi}) n^\mu_{\Sigma_\tau} \mathrm{dVol}_{\Sigma_\tau} + \int_{\mathcal{S}} \nabla^\mu J^V_\mu(\tilde{\psi}) \mathrm{dVol} = \int_{\Sigma_0} J^V_\mu(\tilde{\psi}) n^\mu_{\Sigma_0} \mathrm{dVol}\Sigma_0.$$

Sketch of the proof

Applying the divergence theorem to $\nabla^a J_a^X[\psi]$, over the spacetime domain $\{U_s\}_{s\in[t,t_0]}$, in the whole torus, $U_{t_0} = \Sigma_{t_0}$ we get

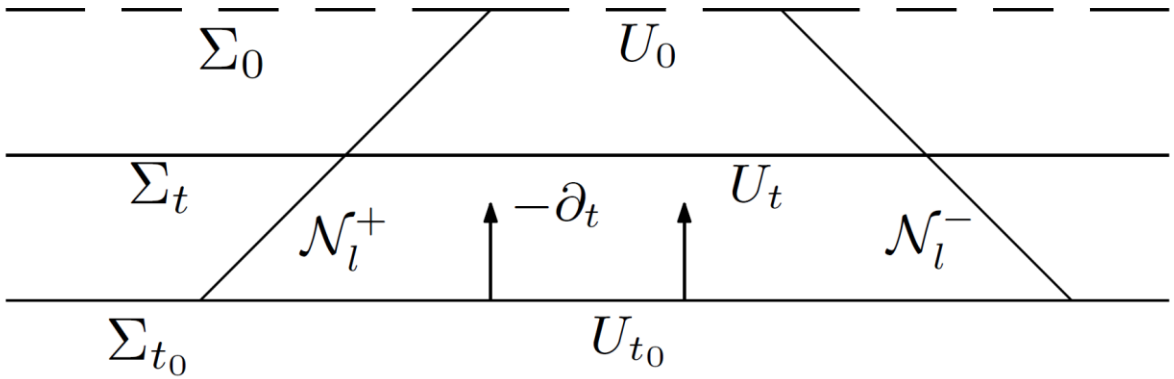
$$\int_{U_t} J_a^X[\psi] n^a_{U_t} \mathrm{vol}_{U_t} = \int_{U_{t_0}} J_a^X[\psi] n^a_{U_{t_0}} \mathrm{vol}_{U_{t_0}} \quad - \int_t^{t_0} \int_{U_s} \nabla^a J_a^X[\psi] \mathrm{vol}_{U_s} ds,$$

where $n_{U_t} = -\partial_t$, vol_{U_t} is the intrinsic volume form of U_t.



Sketch of the proof

For open initial conditions in a neighbourhood of Σ_{t_0}, U_{t_0} we obtain blow-up for the $L^2(U_0)$ norm



Applying the divergence theorem in the domain of dependence of an open neighborhood U_{t_0} of the initial hypersurface Σ_{t_0}, we get

$$\int_{U_t} J_a^X[\psi] n^a_{U_t} \mathrm{vol}_{U_t} + \sum_{l=1}^3 \int_{\cup \mathcal{N}_l^\pm} J_a^X[\psi] n^a_{\mathcal{N}_l^\pm} \mathrm{vol}_{\mathcal{N}_l^\pm} = \int_{U_{t_0}} J_a^X[\psi] n^a_{U_{t_0}} \mathrm{vol}_{U_{t_0}} - \int_t^{t_0} \int_{U_s} \nabla^a J_a^X[\psi] \mathrm{vol}_{U_s} ds,$$

Sketch of the proof

- We will choose the vector field X to be a suitable rescaling of $n_{U_t} = -\partial_t$:

$$n^a_{U_t} J^X_a[\psi] = J^X_0[\psi] = \frac{1}{2}[(\partial_t\psi)^2 + |\overline{\nabla}\psi|^2] \quad , \quad J^X_a[\psi] n^a_{\mathcal{N}^{\pm}_l} \geq 0$$

  where $\overline{\nabla}$ is the covariant derivative intrinsic to the level sets of t.

- Controlling the bulk in terms of $J^X_0[\psi]$, gives an energy estimate for ψ.

- Commuting the wave equation with spatial derivatives and applying the above energy argument, gives higher order energy estimates.

- Spatially homogeneous spacetimes: the spatial coordinate derivatives $\{\partial_{x_i}\}$ are Killing and hence $[\Box_g, \partial_{x_i}] = 0$, $i = 1, 2, 3$.

- This means that the energy estimates for ψ are also valid for $\partial^\alpha_x \psi = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ (multi-index notation).

- In this notation, the $H^k(\Sigma_t)$ norm of a smooth function $f : (0, +\infty) \times \mathbb{T}^3$ equals

$$\|f\|^2_{H^k(\Sigma_t)} = \sum_{|\alpha| \leq k} \int_{\Sigma_t} (\partial^\alpha_x f)^2 \mathrm{vol}_{Euc},$$

  where $\mathrm{vol}_{Euc} = dx_1 dx_2 dx_3$.

Flat FLRW

- Let ψ be a smooth solution to the scalar wave equation

$$\Box_{g_{\mathrm{FLRW}}}\psi = 0.$$

- Consider the orthonormal frame adapted to the constant t hypersurfaces Σ_t with the past normal vector field e_0 pointing towards the singularity

$$e_0 = -\partial_t, \qquad e_i = t^{-\frac{2}{3\gamma}}\partial_{x_i}.$$

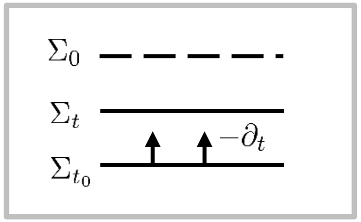
- Second fundamental form K_{ij} of Σ_t

$$K_{ii} := g(\nabla_{e_i} e_0, e_i) = -\frac{2}{3\gamma}\frac{1}{t}, \qquad i = 1, 2, 3.$$

- Intrinsic volume form on Σ_t

$$\mathrm{vol}_{\Sigma_t} = t^{\frac{2}{\gamma}}\mathrm{vol}_{Euc}.$$

Flat FLRW



Proposition (upper energy and pointwise bound):

The following energy inequality holds:

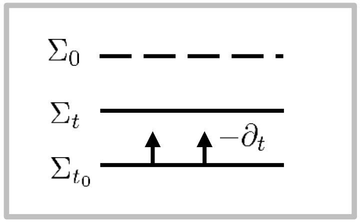
$$t^{\frac{2}{\gamma}} \int_{\Sigma_t} J_0^{e_0}[\partial_x^\alpha \psi] \mathrm{vol}_{\Sigma_t} \leq t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \psi] \mathrm{vol}_{\Sigma_{t_0}},$$

for all $t \in (0, t_0]$ and any multi-index α. Moreover, ψ satisfies the pointwise bound

$$|\psi(t,x)| \leq C \Big(\sum_{|\alpha| \leq 2} t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \psi] \mathrm{vol}_{\Sigma_{t_0}} \Big)^{\frac{1}{2}} \frac{t^{1-\frac{2}{\gamma}} - t_0^{1-\frac{2}{\gamma}}}{\frac{2}{\gamma} - 1} + |\psi(t_0, x)|,$$

where $C > 0$ is a constant independent of t_0, γ.

Flat FLRW



Proof (upper energy bound):

- Take $X = t^{\frac{2}{\gamma}} e_0$, and compute the divergence of $J_a^{t^{\frac{2}{\gamma}} e_0}[\psi]$:

$$\begin{aligned} \nabla^a J_a^{t^{\frac{2}{\gamma}} e_0}[\psi] &= \nabla^a (t^{\frac{2}{\gamma}} e_0)^b T_{ab}[\psi] \\ &= t^{\frac{2}{\gamma}} K^{ab} T_{ab}[\psi] - e_0 t^{\frac{2}{\gamma}} T_{00}[\psi] \\ &= \frac{4}{3\gamma}\frac{1}{t} t^{\frac{2}{\gamma}} |\overline{\nabla}\psi|^2. \end{aligned}$$

- From divergence theorem (holding for $\partial_x^\alpha \psi$ as well)

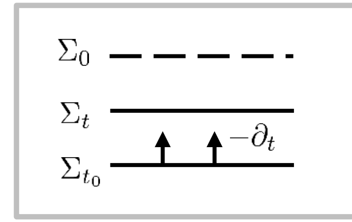
$$\begin{aligned} t^{\frac{2}{\gamma}} \int_{\Sigma_t} J_0^{e_0}[\psi] \mathrm{vol}_{\Sigma_t} &= t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\psi] \mathrm{vol}_{\Sigma_{t_0}} - \int_t^{t_0} \int_{\Sigma_s} \frac{4}{3\gamma} s^{\frac{2}{\gamma}-1} |\overline{\nabla}\psi|^2 \mathrm{vol}_{\Sigma_s} ds \\ &\leq t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\psi] \mathrm{vol}_{\Sigma_{t_0}}. \end{aligned}$$

- Bounds for $\partial_t \psi$:

$$t^{\frac{4}{\gamma}} \|\partial_t \partial_x^\alpha \psi\|_{L^2}^2 \leq 2 t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \psi] \mathrm{vol}_{\Sigma_{t_0}},$$

for all $t \in (0, t_0]$ and α.

Flat FLRW



Proof (upper pointwise bound):

- Integrating $\partial_t \psi$ in $[t, t_0]$ and Sobolev embedding $H^2(\mathbb{T}^3) \hookrightarrow L^\infty(\mathbb{T}^3)$:

$$\begin{aligned}
|\psi(t,x)| &= \Big| \int_{t_0}^{t} \partial_s \psi(s,x) ds + \psi(t_0,x) \Big| \\
&\leq C \int_{t}^{t_0} \|\partial_s \psi\|_{H^2} ds + |\psi(t_0,x)| \\
&\leq \frac{C}{\frac{2}{\gamma}-1} (t^{1-\frac{2}{\gamma}} - t_0^{1-\frac{2}{\gamma}}) (\sum_{|\alpha|\leq 2} t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha \psi] \mathrm{vol}_{\Sigma_{t_0}})^{\frac{1}{2}} + |\psi(t_0,x)|,
\end{aligned}$$

for $(\gamma < 2)$. $\square$

Remark :

- From the above we saw that $t^{1-\frac{2}{\gamma}}$ is the leading order of ψ at $t = 0$.
- Therefore, we derive energy bounds for the renormalised variable $\frac{\psi}{t^{1-\frac{2}{\gamma}}}$ with the wave equation:

$$\Box_{g_{\mathrm{FLRW}}} \left(\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right) = -\frac{2}{t} \left(1 - \frac{2}{\gamma} \right) e_0 \left(\frac{\psi}{t^{1-\frac{2}{\gamma}}} \right).$$

Flat FLRW

Proposition (renormalized energy estimates and asymptotic profile):

Let ψ be a smooth solution to the wave equation in FLRW backgrounds with $\frac{2}{3} < \gamma < 2$. Then, the following bounds hold uniformly in $t \in (0, t_0]$:

$$t^{4-\frac{6}{\gamma}} \int_{\Sigma_t} J_0^{e_0}\left[\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right] \mathrm{vol}_{\Sigma_t} \leq t_0^{4-\frac{6}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}\left[\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right] \mathrm{vol}_{\Sigma_{t_0}}, \qquad \frac{4}{3} \leq \gamma < 2,$$

$$t^{-\frac{2}{3\gamma}} \int_{\Sigma_t} J_0^{e_0}\left[\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right] \mathrm{vol}_{\Sigma_t} \leq t_0^{-\frac{2}{3\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}\left[\partial_x^\alpha \frac{\psi}{t^{1-\frac{2}{\gamma}}}\right] \mathrm{vol}_{\Sigma_{t_0}}, \qquad \frac{2}{3} < \gamma \leq \frac{4}{3},$$

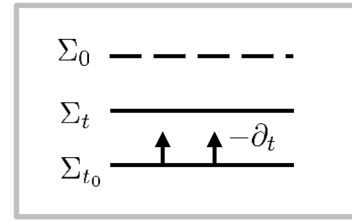
for all $t \in (0, t_0]$ and any multi-index α. Moreover, the limit

$$A(x) := \lim_{t \to 0} \frac{\psi}{t^{1-\frac{2}{\gamma}}}$$

exists, it is a smooth function and the difference $u(t,x) := \psi - A(x) t^{1-\frac{2}{\gamma}}$ satisfies

$$\lim_{t \to 0} t^{\frac{2}{\gamma}} \int_{\Sigma_t} J_0^{e_0}[\partial_x^\alpha u] \mathrm{vol}_{\Sigma_s} = 0.$$

Flat FLRW



Proof (renormalized energy estimates):

- Let $\eta > 0$. We compute

$$\nabla^a (J_a^{t^\eta e_0}[\frac{\psi}{t^{1-\frac{2}{\gamma}}}]) = t^{\eta-1}\Big[(\frac{1}{3\gamma}+\frac{\eta}{2})|\overline{\nabla}\frac{\psi}{t^{1-\frac{2}{\gamma}}}|^2 + (\frac{\eta}{2}+\frac{3}{\gamma}-2)[e_0(\frac{\psi}{t^{1-\frac{2}{\gamma}}})]^2\Big]$$

- This leads to different choices of η depending on the value of γ, given by

$$\begin{aligned} \eta &= 4-\frac{6}{\gamma}, \quad \text{for} \quad \frac{4}{3}\leq\gamma<2, \qquad \text{stiffer region},\\ \eta &= -\frac{2}{3\gamma}, \quad \text{for} \quad \frac{2}{3}<\gamma\leq\frac{4}{3}, \qquad \text{softer region}. \end{aligned}$$

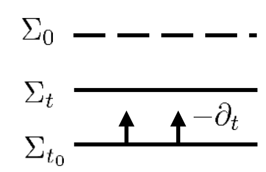
  The case $\gamma = \frac{4}{3}$ corresponds to radiation.

- For the two cases, the divergence reads

$$\begin{aligned} \nabla^a (J_a^{t^{4-\frac{6}{\gamma}} e_0}[\frac{\psi}{t^{1-\frac{2}{\gamma}}}]) &= (2-\frac{8}{3\gamma})t^{3-\frac{6}{\gamma}}|\overline{\nabla}\frac{\psi}{t^{1-\frac{2}{\gamma}}}|^2,\\ \nabla^a (J_a^{t^{-\frac{2}{3\gamma}} e_0}[\frac{\psi}{t^{1-\frac{2}{\gamma}}}]) &= (\frac{8}{3\gamma}-2)t^{-1-\frac{2}{3\gamma}}[e_0(\frac{\psi}{t^{1-\frac{2}{\gamma}}})]^2. \end{aligned}$$

- The divergence theorem for $J^X[\frac{\psi}{t^{1-\frac{2}{\gamma}}}]$, with $X = t^{4-\frac{6}{\gamma}}e_0$, $\frac{4}{3}\leq\gamma<2$, and $X = t^{-\frac{2}{3\gamma}}e_0$, $\frac{2}{3}<\gamma\leq\frac{4}{3}$, gives the first part of the proposition.

Flat FLRW



Proof (pointwise estimates and asymptotic profile):

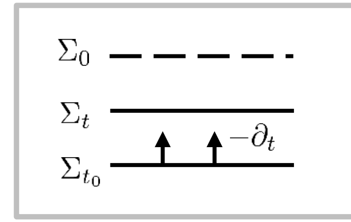
- In particular, taking into account the volume form $\mathrm{vol}_{\Sigma_t} = t^{\frac{2}{\gamma}}\mathrm{vol}_{Euc}$, we have the bounds:

$$|\partial_t \frac{\psi}{t^{1-\frac{2}{\gamma}}}| \leq C\|\partial_t \frac{\psi}{t^{1-\frac{2}{\gamma}}}\|_{H^2} \quad \leq \frac{C}{t^{2-\frac{2}{\gamma}}}\Big(\sum_{|\alpha|\leq 2} t_0^{4-\frac{6}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\frac{\psi}{t_0^{1-\frac{2}{\gamma}}}]\mathrm{vol}_{\Sigma_{t_0}}\Big)^{\frac{1}{2}}, \quad \frac{4}{3} \leq \gamma < 2$$

$$|\partial_t \frac{\psi}{t^{1-\frac{2}{\gamma}}}| \leq C\|\partial_t \frac{\psi}{t^{1-\frac{2}{\gamma}}}\|_{H^2} \quad \leq \frac{C}{t^{\frac{2}{3\gamma}}}\Big(\sum_{|\alpha|\leq 2} t_0^{-\frac{2}{3\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_x^\alpha\frac{\psi}{t^{1-\frac{2}{\gamma}}}]\mathrm{vol}_{\Sigma_{t_0}}\Big)^{\frac{1}{2}}, \quad \frac{2}{3} < \gamma \leq \frac{4}{3}$$

- $\partial_t\psi(t,x) \in L^1([0,t_0])$, uniformly in x, for all $\frac{2}{3} < \gamma < 2$.
- Thus, $\frac{\psi}{t^{1-\frac{2}{\gamma}}}$ has a limit function $A(x)$, as $t \to 0$.
- The smoothness of $A(x)$ follows by repeating the preceding argument for $\partial_x^\alpha\frac{\psi}{t^{1-\frac{2}{\gamma}}}$.

Flat FLRW



Proof (asymptotic profile $\lim_{t\to 0} t^{\frac{2}{\gamma}} \int_{\Sigma_t} J_0^{e_0}[\partial_x^\alpha u]\mathrm{vol}_{\Sigma_s} = 0$):

- Energy flux of $u(t,x) = \psi - A(x)t^{1-\frac{2}{\gamma}}$,

$$
\begin{aligned}
& t^{\frac{2}{\gamma}} \int_{\Sigma_t} |e_0(\psi - A(x)t^{1-\frac{2}{\gamma}})|^2 + |\overline{\nabla}(\psi - A(x)t^{1-\frac{2}{\gamma}})|^2 \mathrm{vol}_{\Sigma_t} \\
&= t^{\frac{2}{\gamma}} \int_{\Sigma_t} |t^{1-\frac{2}{\gamma}} e_0 \frac{\psi}{t^{1-\frac{2}{\gamma}}} + (1-\frac{2}{\gamma})t^{-\frac{2}{\gamma}}(\frac{\psi}{t^{1-\frac{2}{\gamma}}} - A(x))|^2 + t^{2-\frac{4}{\gamma}}|\overline{\nabla}(\frac{\psi}{t^{1-\frac{2}{\gamma}}} - A(x))|^2 \mathrm{vol}_{\Sigma_t} \\
\text{Cauchy Schwarz} \quad &\leq 2t^{\frac{2}{\gamma}} \int_{\Sigma_t} t^{2-\frac{4}{\gamma}} |e_0 \frac{\psi}{t^{1-\frac{2}{\gamma}}}|^2 + (1-\frac{2}{\gamma})^2 t^{-\frac{4}{\gamma}} |\frac{\psi}{t^{1-\frac{2}{\gamma}}} - A(x)|^2 + t^{2-\frac{4}{\gamma}} |\overline{\nabla} \frac{\psi}{t^{1-\frac{2}{\gamma}}}|^2 + t^{2-\frac{4}{\gamma}} |\overline{\nabla} A(x)|^2 \mathrm{vol}_{\Sigma_t} \\
\text{Volume elem.} \quad &\leq 4t^2 \int_{\Sigma_t} J_0^{e_0}[\frac{\psi}{t^{1-\frac{2}{\gamma}}}]\mathrm{vol}_{Euc} + 2(1-\frac{2}{\gamma})^2 \int_{\Sigma_t} |\frac{\psi}{t^{1-\frac{2}{\gamma}}} - A(x)|^2 \mathrm{vol}_{Euc} + 2t^2 \int_{\Sigma_t} |\overline{\nabla} A(x)|^2 \mathrm{vol}_{Euc} \\
\text{Previous Prop.} \quad &\leq o(1) + 2(1-\frac{2}{\gamma})^2 \int_{\Sigma_t} |\frac{\psi}{t^{1-\frac{2}{\gamma}}} - A(x)|^2 \mathrm{vol}_{Euc} + 2t^{2-\frac{4}{3\gamma}} \sum_{i=1}^{3} \int_{\Sigma_t} |\partial_{x_i} A(x)|^2 \mathrm{vol}_{Euc},
\end{aligned}
$$

- The third term in the preceding RHS clearly tends to zero, as $t \to 0$, and by the definition of $A(x)$, so does the second term.

- Since the above argument also applies to $\partial_x^\alpha[\psi - A(x)t^{1-\frac{2}{\gamma}}]$, the proposition validates the asymptotic profile of ψ, as stated earlier for flat FLRW.

□

Main results

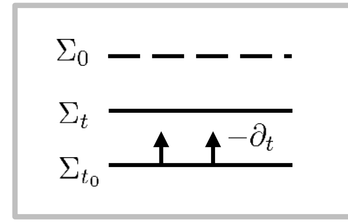
<u>Main Theorem</u> *[Asymptotic profile]*
Let ψ *be a smooth solution to the wave equation,* $\Box_g \psi = 0$*, for either of the metrics* $g_{\mathrm{FLRW}}, g_{\mathrm{Kasner}}$*, arising from initial data* $(\psi_0, \partial_t \psi_0)$ *on* Σ_{t_0}*. Then,* ψ *can be written in the following form:*

$$\psi_{\mathrm{FLRW}}(t,x) = A_{\mathrm{FLRW}}(x) t^{1-\frac{2}{\gamma}} + u_{\mathrm{FLRW}}(t,x), \tag{1}$$

$$\psi_{\mathrm{Kasner}}(t,x) = A_{\mathrm{Kasner}}(x) \log t + u_{\mathrm{Kasner}}(t,x), \tag{2}$$

where $A(x), u(t,x)$ *are smooth functions and* $u_{FLRW} t^{\frac{2}{\gamma}-1}, u_{Kasner} (\log t)^{-1}$ *tend to zero, as* $t \to 0$*.*

Flat FLRW



Remarks (improved control and asymptotic profile):

- The renormalised estimate yields an improved control over $\overline{\nabla}\psi$ for all $t \in (0, t_0]$

$$t^{2-\frac{4}{\gamma}} t^{\frac{2}{\gamma}} \int_{\Sigma_t} |\overline{\nabla}\psi|^2 \mathrm{vol}_{\Sigma_s} \leq t_0^{4-\frac{6}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\frac{\psi}{t^{1-\frac{2}{\gamma}}}] \mathrm{vol}_{\Sigma_s}, \qquad \frac{4}{3} \leq \gamma < 2,$$

$$t^{\frac{4}{3\gamma}-2} t^{\frac{2}{\gamma}} \int_{\Sigma_t} |\overline{\nabla}\psi|^2 \mathrm{vol}_{\Sigma_s} \leq t_0^{-\frac{2}{3\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\frac{\psi}{t^{1-\frac{2}{\gamma}}}] \mathrm{vol}_{\Sigma_s}, \qquad \frac{2}{3} < \gamma \leq \frac{4}{3},$$

- For $\gamma > \frac{2}{3}$, the main contribution of the energy flux generated by $J^{e_0}[\psi]$ comes from the $e_0\psi$ term.

$$t^{\frac{2}{\gamma}} \int_{\Sigma_t} J_0^{e_0}[\psi] \mathrm{vol}_{\Sigma_t} = \tfrac{1}{2} \int_{\Sigma_t} t^{\frac{4}{\gamma}} (\partial_t \psi)^2 \mathrm{vol}_{Euc} + O(t^\eta)$$

where $\eta = \frac{4}{\gamma} - 2 > 0$, for $\gamma \in [\frac{4}{3}, 2)$ and $\eta = 2 - \frac{4}{3\gamma} > 0$, for $\gamma \in (\frac{2}{3}, \frac{4}{3}]$.

Proof (blow-up):

- Hence, taking the limit $t \to 0$

$$\lim_{t\to 0} t^{\frac{2}{\gamma}} \int_{\Sigma_t} J_0^{e_0}[\psi] \mathrm{vol}_{\Sigma_s} = \frac{1}{2}(1 - \frac{2}{\gamma})^2 \int_{\Sigma_0} A^2(x) \mathrm{vol}_{Euc}$$

$$= t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\psi] \mathrm{vol}_{\Sigma_s} - \frac{4}{3\gamma} \int_0^{t_0} s^{\frac{2}{\gamma}-1} \int_{\Sigma_s} |\overline{\nabla}\psi|^2 \mathrm{vol}_{\Sigma_s} ds.$$

Flat FLRW

Lemma:
The following estimate for the L^2 norm of $\partial_{x_i}\psi$ holds:

$$\|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} \leq \|\partial_{x_i}\psi\|_{L^2(\Sigma_{t_0})} + \sqrt{2}\frac{(t_0^{1-\frac{2}{\gamma}} - t^{1-\frac{2}{\gamma}})}{1-\frac{2}{\gamma}} t_0^{\frac{2}{\gamma}} \left(\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_{x_i}\psi] \mathrm{vol}_{Euc} \right)^{\frac{1}{2}},$$

for all $t \in (0, t_0]$.

Proof:
Differentiating in e_0 we have:

$$\begin{aligned} \frac{1}{2} e_0 \|\partial_{x_i}\psi\|^2_{L^2(\Sigma_t)} &\leq \|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} \|e_0 \partial_{x_i}\psi\|_{L^2(\Sigma_t)} \\ &\leq \frac{1}{t^{\frac{2}{\gamma}}} \left(2 t_0^{\frac{2}{\gamma}} \int_{\Sigma_{t_0}} J_0^{e_0}[\partial_{x_i}\psi] \mathrm{vol}_{\Sigma_s} \right)^{\frac{1}{2}} \|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} \end{aligned}$$

or

$$e_0 \|\partial_{x_i}\psi\|_{L^2(\Sigma_t)} \leq \sqrt{2} \frac{t_0^{\frac{2}{\gamma}}}{s^{\frac{2}{\gamma}}} \left(\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_{x_i}\psi] \mathrm{vol}_{Euc} \right)^{\frac{1}{2}}.$$

Integrating the above on $[t, t_0]$ for $\gamma < 2$, proves the Lemma. $\square$

Flat FLRW

Proof (blow-up):

- Putting together improved estimate, asymptotic profile, the divergence theorem, and the L^2 norm of the spatial derivatives, we get

Divergence thm.

Use Lemma

$$
\begin{aligned}
\frac{1}{2}(1-\frac{2}{\gamma})^2\int_{\Sigma_0} A^2(x)\mathrm{vol}_{Euc} &= t_0^{\frac{2}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\psi]\mathrm{vol}_{\Sigma_s} - \frac{4}{3\gamma}\int_0^{t_0} s^{\frac{2}{\gamma}-1}\int_{\Sigma_s}|\overline{\nabla}\psi|^2\mathrm{vol}_{\Sigma_s}ds \\
&\geq \frac{1}{2}t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(\Sigma_{t_0})} + \frac{1}{2}t_0^{\frac{4}{\gamma}-\frac{4}{3\gamma}}\sum_{i=1}^{3}\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})} \\
&\quad -\frac{8}{3\gamma}\int_0^{t_0} s^{\frac{4}{\gamma}-1-\frac{4}{3\gamma}}ds\sum_{i=1}^{3}\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})} \\
&\quad -\frac{16}{3\gamma}\int_0^{t_0} s^{\frac{4}{\gamma}-1-\frac{4}{3\gamma}}\frac{(t_0^{1-\frac{2}{\gamma}}-s^{1-\frac{2}{\gamma}})^2}{(1-\frac{2}{\gamma})^2}ds\sum_{i=1}^{3}t_0^{\frac{2}{\gamma}}\int_{\Sigma_{t_0}} J_0^{e_0}[\partial_{x_i}\psi]\mathrm{vol}_{\Sigma_{t_0}} \\
&\geq \frac{1}{2}t_0^{\frac{4}{\gamma}}\|\partial_t\psi\|^2_{L^2(\Sigma_{t_0})} - \frac{1}{2}t_0^{\frac{8}{3\gamma}}\sum_{i=3}^{3}\|\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})} \\
&\quad -\frac{t_0^{2-\frac{4}{3\gamma}}}{1-(\frac{2}{3\gamma})^2}\sum_{i=1}^{3}\left[t_0^{\frac{4}{\gamma}}\|\partial_t\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})} + t_0^{\frac{8}{3\gamma}}\sum_{j=1}^{3}\|\partial_{x_j}\partial_{x_i}\psi\|^2_{L^2(\Sigma_{t_0})}\right]
\end{aligned}
$$

- If the assumptions of the blow-up Theorem for FLRW are satisfied, then $\|A(x)\|_{L^2(\mathbb{T}^3)} > 0$. $\square$

Main results

<u>Main Theorem</u> *[Blow-up]*
Let ψ *be a smooth solution to the wave equation,* $\Box_g \psi = 0$*, for either of the metrics* $g_{\mathrm{FLRW}}, g_{\mathrm{Kasner}}$*, arising from initial data* $(\psi_0, \partial_t \psi_0)$ *on* Σ_{t_0}*,* $t_0 > 0$*. If* $\partial_t \psi_0$ *is non-zero in* $L^2(\mathbb{T}^3)$*,* t_0 *is sufficiently small such that*

$$\frac{2t_0^{2-\frac{4}{3\gamma}}}{1-(\frac{2}{3\gamma})^2} \sum_{i=1}^{3} \|\partial_t \partial_{x_i} \psi_0\|^2_{L^2(\mathbb{T}^3)} < \epsilon \|\partial_t \psi_0\|^2_{L^2(\mathbb{T}^3)}, \qquad \text{(FLRW)} \qquad (1)$$

$$\sum_{i=1}^{3} \frac{2t_0^{2-2p_i}}{(1-p_i)^2} \|\partial_t \partial_{x_i} \psi_0\|^2_{L^2(\mathbb{T}^3)} < \epsilon \|\partial_t \psi_0\|^2_{L^2(\mathbb{T}^3)}, \qquad \text{(Kasner)} \qquad (2)$$

and $\psi_0, \partial_t \psi_0$ *satisfy the open conditions*

$$(1-\epsilon)\|\partial_t \psi_0\|^2_{L^2(\mathbb{T}^3)} > t_0^{-\frac{4}{3\gamma}} \sum_{i=3}^{3} \|\partial_{x_i} \psi_0\|^2_{L^2(\mathbb{T}^3)} + \frac{2t_0^{2-\frac{8}{3\gamma}}}{1-(\frac{2}{3\gamma})^2} \sum_{i,j=1}^{3} \|\partial_{x_j} \partial_{x_i} \psi_0\|_{L^2(\mathbb{T}^3)}, \qquad \text{(FLRW)} \qquad (3)$$

$$(1-\epsilon)\|\partial_t \psi_0\|^2_{L^2(\mathbb{T}^3)} > \sum_{i=3}^{3} t_0^{-2p_i} \|\partial_{x_i} \psi_0\|^2_{L^2(\mathbb{T}^3)} + \sum_{i,j=1}^{3} \frac{2t_0^{2-2p_i-2p_j}}{(1-p_i)^2} \|\partial_{x_j} \partial_{x_i} \psi_0\|_{L^2(\mathbb{T}^3)}, \qquad \text{(Kasner)} \qquad (4)$$

for some $0 < \epsilon < 1$*, then* $\|A(x)\|_{L^2(\mathbb{T}^3)} > 0$*.*

Related subsequent work

- <u>Bachelot, A. (2018):</u>
  Considers Klein-Gordon type equations on FLRW backgrounds. In practice, he considers warped product type geometries (in this sense, the results are more general than FLRW). Moreover, he considers Big Bang, Big Crunch, Big Rip, Big Brake and Sudden Singularities.

- <u>Ringström, H. (2018):</u>
  Considers different classes of Bianchi spacetimes and investigates singularities that are matter dominated as well as singularities that are vacuum dominated; and even when the asymptotics of the underlying Bianchi spacetime are oscillatory. Further, he analyzses Klein-Gordon type equations on flat Kasner backgrounds.

- <u>Girão, P., Natário, J. and Silva, J. (2018):</u>
  Show boundedness for a certain class of solutions approaching the FLRW Big Bang singularity on scalar wave equation level.

Conclusion

For certain open sets of smooth initial data, the resulting solutions ψ, to the wave equation for Friedmann-Lemaître-Robertson-Walker and Kasner spacetimes, blow up close to the Big Bang singularity.

773 campuses of US colleges as of August 24, 2021, have issued vaccination requirements for at least some students and/or employees.

⇩

Scientific evidence ?

⇩

Ronald N. Kostoff, Daniela Calina, Darja Kanduc, Michael B. Briggs, Panayiotis Vlachoyiannopoulos, Andrey A. Svistunov, Aristidis Tsatsakis
Why are we vaccinating children against COVID-19? Toxicology Reports, Volume 8, 2021, Pages 1665-1684

Ioannidis JPA. COVID-19
Vaccination in children and university students. *Eur J Clin Invest*. 2021;51:e13678.

Luckhaus, Stephan: Mathematical epidemiology : SIR models and COVID-19
MIS-Preprint: 60/2020

Luckhaus, Stephan: Corona, mathematical epidemiology, herd immunity, and data
MIS-Preprint: 105/2020 LINK: https://www.math.uni-leipzig.de/preprints/p2010.0010.pdf

All races, genders, social-, financial- and vaccination- statuses.