Positivity of mass for asymptotically hyperbolic initial data sets

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Asymptotically Euclidean initial data

Let $(\mathbb{R}^n, \delta)$ be the Euclidean space.

We say that an initial data set (M^n, g, k) is **asymptotically Euclidean** if there is a compact $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B_R}$ such that for $\varepsilon > 0$

- $e := \Phi_* g - \delta = O\left(|x|^{-\frac{n-2}{2}-\varepsilon}\right)$
- $\Phi_* k = O\left(|x|^{-\frac{n}{2}-\varepsilon}\right)$.

For AE manifolds with $\mathrm{Scal} \in L^1$ we can define **ADM mass**:

$$m_{ADM} = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \left(\operatorname{div}^\delta e - d(\operatorname{tr}^\delta e)\right)(\nu_r)\, d\mu^\delta.$$

Positive mass theorems in AE setting

Conjecture

Let (M^n, g, k) be an AE initial data set with $n \geq 3$ satisfying DEC $\mu \geq |J|_g$. Then $m_{ADM} \geq 0$ and $m_{ADM} = 0$ iff (M, g, k) is a slice of Minkowski.

Schoen-Yau 1979, 1981, 2017, Witten 1981, Eichmair 2013, Lohkamp 2006-2017, Eichmair-Huang-Lee-Schoen 2011, Huang-Lee 2017, Bray-Kazaras-Khuri-Stern 2019, Hirsch-Kazaras-Khuri 2020, Lesourd-Yau-Unger 2021 ...

We will refer to Riemannian positive mass theorem

Theorem (Schoen-Yau)

Suppose a complete AE manifold (M^n, g) with $3 \leq n \leq 7$ has $\mathrm{Scal}^g \geq 0$. Then $m_{ADM} \geq 0$ and $m_{ADM} = 0$ iff (M, g) is isometric to $(\mathbb{R}^n, \delta)$.

Origins of the Jang equation

Theorem (Jang 1978)

(M^n, g, k) is a slice of Minkowski spacetime if and only if there is $f : M \to \mathbb{R}$ such that

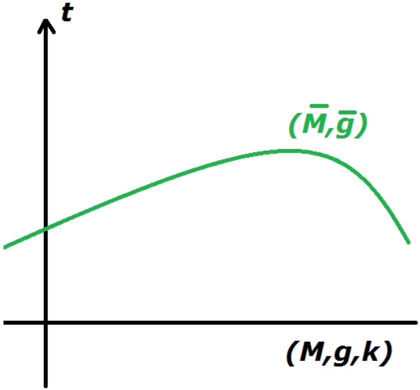
$$\begin{aligned} g_{ij} &= \delta_{ij} - \nabla_i f \nabla_j f \\ k_{ij} &= \frac{\operatorname{Hess}_{ij} f}{\sqrt{1 + |\nabla f|^2}} \end{aligned} \tag{1}$$

From (1) one obtains the **Jang equation**:

$$\left(g^{ij} - \frac{\nabla^i f \nabla^j f}{1 + |\nabla f|^2} \right) \left(\frac{\operatorname{Hess}_{ij} f}{\sqrt{1 + |\nabla f|^2}} - k_{ij} \right) = 0.$$

Important observation of Schoen and Yau (1981)

Given (M, g, k) consider a hypersurface $(\bar{M}, \bar{g})$ in $(M \times \mathbb{R}, g + dt^2)$. Set $k(\cdot, \partial_t) = 0$.



Important observation of Schoen and Yau

If $H_{\bar{M}} = \mathrm{tr}_{\bar{M}}\, k$ then

$$\mathrm{Scal}^{\bar{g}} = 2(\mu - J(w)) + |A - k|^2_{\bar{g}} + 2|q|^2_{\bar{g}} - 2\mathrm{div}^{\bar{g}} q,$$

for a 1-form q, so if $\mu \geq |J|_g$ holds then $\mathrm{Scal}^{\bar{g}}$ is 'almost nonnegative'.

Note that if $(\bar{M}, \bar{g}) = \mathrm{graph}\, f$ then $H_{\bar{M}} = \mathrm{tr}_{\bar{M}}\, k$ becomes

$$\left(g^{ij} - \frac{\nabla^i f \nabla^j f}{1 + |\nabla f|^2} \right) \left(\frac{\mathrm{Hess}_{ij} f}{\sqrt{1 + |\nabla f|^2}} - k_{ij} \right) = 0,$$

i.e. we recover the **Jang equation**.

The Jang equation reduction

Theorem (Schoen-Yau 1981, Eichmair 2013)

Let (M^n, g, k), $3 \leq n \leq 7$, be a complete asymptotically Euclidean initial data set satisfying $\mu \geq |J|_g$. Then $m_{ADM} \geq 0$ and $m_{ADM} = 0$ iff (M, g, k) is a slice of Minkowski spacetime.

Idea of the proof:

Find a solution $f \to 0$ as $|x| \to \infty$ of the Jang equation.

Then $\overline{g} = g + df \otimes df$ is asymptotically Euclidean with $\mathrm{Scal}^{\overline{g}} \geq 0$ (up to a conformal change). By Riemannian PMT we obtain

$$0 \leq m_{ADM}(\overline{g}) \leq m_{ADM}(g).$$

f may blow up in a compact set, but it is not a problem.

When $m_{ADM} = 0$, f gives the graphical embedding in Minkowski.

Other applications of the Jang equation

- Reduction arguments for other geometric inequalities in GR (Bray-Khuri, Khuri et al, . . .)
- Positivity of quasilocal mass (Liu-Yau, Wang-Yau, Alaee-Khuri-Yau, . . .)
- Detection of MOTS (Schoen-Yau, Yau, Andersson-Metzger, Eichmair, Moore, Bourni-Moore, . . .)
- Topology of the exterior region (Andersson-Dahl-Galloway-Pollack, . . .)

Asymptotically hyperbolic manifolds (chart-dependent)

Let $(\mathbb{H}^n, g_{hyp}) = (\mathbb{R} \times S^{n-1}, \frac{dr^2}{1+r^2} + r^2\sigma)$.

(M, g) is **asymptotically hyperbolic** if there is a compact $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \to \mathbb{H}^n \setminus \overline{B_R}$ such that $e := \Phi_* g - g_{hyp} = O(r^{-\frac{n}{2}-\varepsilon})$ for $\varepsilon > 0$.

Define $\vec{P} = (H_\Phi(\sqrt{1+r^2}), H_\Phi(x^1), \ldots, H_\Phi(x^n))$, where

$$H_\Phi(V) = \lim_{r\to\infty} \int_{S_r} \left(V(\operatorname{div} e - d \operatorname{tr} e) + (\operatorname{tr} e) dV - e(\nabla V, \cdot)\right)(\nu_r)\, d\mu.$$

Theorem (Chruściel-Herzlich 2003)

If $r\,\mathrm{Scal} \in L^1$ then Minkowskian length of $\vec{P}$ is a well-defined invariant. If (M^n, g) is complete and spin with $\mathrm{Scal} \geq -n(n-1)$ then $\vec{P}$ is timelike future directed unless $(M, g) \cong (\mathbb{H}^n, g_{hyp})$.

This generalizes an earlier definition and result of **X. Wang** 2001.

Further (non-spinor) results for AH manifolds

Positive mass theorem holds for AH manifolds in $3 \leq n \leq 7$ (Andersson-Cai-Galloway 2008, Chruściel-Galloway-Nguyen-Paetz 2018, Chruśchiel-Delay 2019).

Chruściel-Delay 2019: positive mass theorem for AH manifolds can be inferred from positivity of mass of asymptotically Euclidean initial data sets in all dimensions.

Huang-Jang-Martin 2019: rigidity can be inferred from positivity in all dimensions.

Asymptotically hyperbolic 'hyperboloidal' initial data

Prototype: hyperboloid $t = \sqrt{1 + r^2}$ in Minkowski, umbilic $k = g = g_{hyp}$.

For the mass vector to be well-defined, the following fall-off is required:

- $\Phi_* g - g_{hyp} = O\left(r^{-\frac{n}{2}-\varepsilon}\right)$
- $\Phi_*(k - g) = O\left(r^{-\frac{n}{2}-\varepsilon}\right)$
- $r\Phi_*(|\mu| + |J|_g) \in L^1$

Mass vector of AH initial data set

Mass vector of (M, g, k) is

$$\vec{P} = (H_\Phi(\sqrt{1+r^2}), H_\Phi(x^1), \ldots, H_\Phi(x^n))$$

where $H_\Phi(V)$ is defined as

$$\lim_{r\to\infty} \int_{S_r} \left(V(\operatorname{div} e - d \operatorname{tr} e) + (\operatorname{tr} e)dV - (e + 2\eta)(\nabla V, \cdot)\right)(\nu_r)\, dA,$$

where $e = \Phi_* g - g_{hyp}$, $\eta = \Phi_*(k - g)$.

This is Trautman-Bondi mass of Chruściel-Jezierski-Łęski 2004.

Positivity has been proven by spinor technique (Zhang 1999, Chruściel-Jezierski-Łęski 2004, ...), see also Chen-Wang-Yau 2016 ($n = 3$, extra assumptions on asymptotics).

Positive mass theorem for AH initial data

Theorem (S., CMP 2021)

Let (M^3, g, k) *be an asymptotically hyperbolic complete initial data set satisfying* $\mu \geq |J|_g$. *Then the mass vector of* (M, g, k) *is causal future directed, i.e.* $P_0 \geq \sqrt{\sum_{i=1}^{3} P_i^2}$.

If (M^3, g, k) *has 'simple' asymptotics and* $P_0 = 0$ *then* (M, g, k) *is a slice of Minkowski spacetime.*

The proof generalizes to dimensions $3 < n < 8$ (Lundberg, tbp, following Eichmair 2013).

Idea of the proof

The idea of the proof comes from Schoen and Yau 1982 *'Proof that the Bondi mass is positive'*:

Given an asymptotically hyperbolic initial data set (M, g, k) satisfying $\mu \geq |J|_g$ use the Jang equation to deform it to an **asymptotically Euclidean** manifold $(\bar{M}, \bar{g})$ such that Riemannian positive mass theorem applies.

Motivating example: hyperboloid in Minkowski

Consider $(M, g, k) = (\mathbb{H}^n, g_{hyp}, g_{hyp})$ where

$$(\mathbb{H}^n, g_{hyp}) = \left(\mathbb{R} \times S^{n-1}, \frac{dr^2}{1+r^2} + r^2\sigma\right).$$

Then $f = \sqrt{1+r^2}$ is a solution of the Jang equation.

The induced metric on $\mathrm{graph}\, f$ in $\mathbb{H}^n \times \mathbb{R}$ is

$$g_{hyp} + df \otimes df = dr^2 + r^2\sigma = \delta,$$

the Euclidean metric.

Proof of the theorem

- It suffices to show that the first component of the mass vector is nonnegative.
- By **density theorem** [Dahl and S., to appear in PAMQ] it suffices to assume that initial data has simple asymptotics and that $\mu > |J|_g$.

In particular,

$$g = \frac{dr^2}{1+r^2} + r^2(\sigma + \mathrm{m} r^{-3} + o(r^{-3})),$$

$$P_0 = \tfrac{1}{16\pi} \int_{S^2} \mathrm{tr}_\sigma(\mathrm{m} + 2\mathrm{k}) d\mu^\sigma,$$

where k comes from the expansion of the 'angular' components of k:

$$k_{\mu\nu} = \mathrm{k}_{\mu\nu} r^{-1} + o(r^{-1}).$$

Proof of main theorem

- For the above asymptotics one expects that the solution of the Jang equation expands at infinity as

$$f = \sqrt{1+r^2} + 2P_0 \ln r + \psi + o(1),$$

  where $\psi : S^2 \to \mathbb{R}$ is determined by the integrand in P_0.

- If f is as above then $g + df \otimes df$ has an asymptotically Euclidean end with $m_{ADM} = 2P_0$, Schoen and Yau's identity shows that we 'almost have' $\mathrm{Scal} \geq 0$.

- Consequently, the goal is to solve the Jang equation with the above asymptotics and make sure that we can apply the positive mass theorem for AE manifolds on its graph.

Solving the Jang equation: barriers

Barriers are super-/subsolutions $f_\pm$ on $\{r \geq R\}$ with $\partial_r f_\pm|_{\{r=R\}} = \mp\infty$.

In AH setting it is problematic to find them by inspection.

We rely on simple asymptotics and use a 'spherically symmetric substitution' to rewrite the Jang equation as a Riccati type ODE modulo correction terms.

Sub- and supersolutions to this ODE with certain initial values give rise to barriers.

Geometric solution

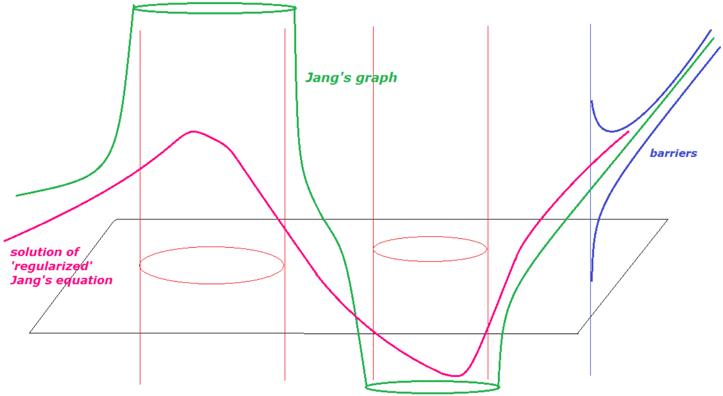
The **geometric solution** is constructed by taking the limit of graphs of solutions of regularized BVP

$$\begin{aligned} J(f_n) = \tau_n f_n \quad &\text{in} \quad \{r \leq R_n\} \\ f_n = \phi \quad &\text{on} \quad \{r = R_n\} \end{aligned}$$

where $\tau_n \to 0$, $R_n \to \infty$ as $n \to \infty$. Here $J(f)$ is the LHS of the Jang equation, $f_- < \phi < f_+$.

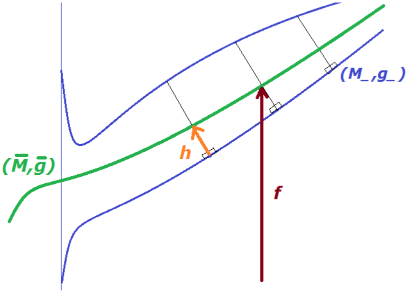
It is not an entire graph, but a union of graphical and cylindrical components. May have **asymptotically cylindrical ends**!

Geometric solution of the Jang equation



Asymptotically Euclidean structure of the end

We cannot directly apply rescaling technique to $J(f) = 0$ and need to rewrite the Jang equation $H_{\bar{M}} = \mathrm{tr}_{\bar{M}}\, k$ in terms the distance function to the lower barrier.



The graph of the lower barrier is asymptotically Euclidean so rescaling technique can be applied to the equation for h.

End of the proof of the main theorem

Further facts about the Jang graph $(\bar{M}, \bar{g})$:

- We have
$$\mathrm{Scal}^{\bar{g}} \geq 2|q|^2_{\bar{g}} - 2\mathrm{div}^{\bar{g}} q,$$
for a 1-form q.
- It has an asymptotically Euclidean end with $m_{ADM} = 2P_0$.
- Speaking about asymptotically cylindrical ends, bases of cylinders are topologically spheres.

These facts can be used to construct a complete asymptotically Euclidean manifold $(\tilde{M}, \tilde{g})$ with $\mathrm{Scal} \geq 0$ and $m_{ADM} \leq P_0$.

Then $P_0 \geq 0$ follows by positive mass theorem for asymptotically Euclidean manifolds.

Comments on rigidity

$P_0 = 0$ will imply that the solution of the Jang equation provides embedding into Minkowski.

The argument will not directly apply to general asymptotics, because we cannot solve the Jang equation due to the lack of barriers.

We do not know either how to show that (M, g, K) embeds in Minkowski if merely $P_0 = \sqrt{\sum_i P_i^2}$.

Thank you very much
for your attention!