The Memory Effect and Infrared Divergences

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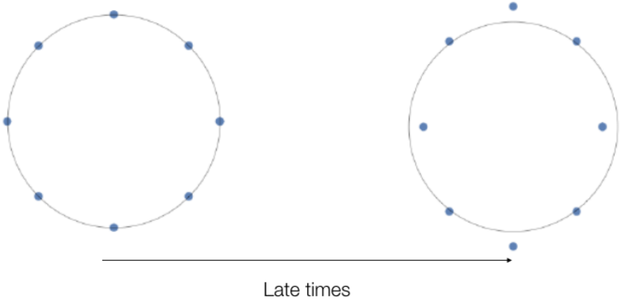
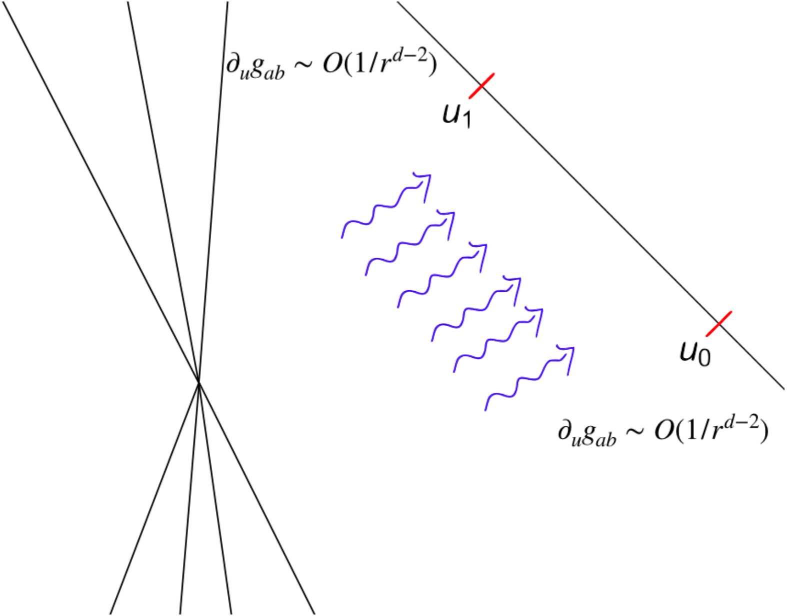
(hopefully, to appear soon)

The Gravitational Memory Effect

In an asymptotically flat spacetime, the memory effect is the permanent relative displacement at $O(1/r)$ of an array of test particles after the passage of a burst of gravitational radiation.



Set-up for Scattering



Memory Tensor

The change in the relative separation of test particles (ξ^a) is given in terms of their initial displacement $\xi^a_{(i)}$ by integrating the geodesic deviation equation,

$$\Delta\xi_a = \Delta_{ab}\xi^b_{(i)}$$

where

$$\Delta_{ab} = -\int_{-\infty}^{\infty} du' \int_{-\infty}^{u'} du'' E_{ab}$$

and

$$E_{ab} = C_{acbd}\left(\frac{\partial}{\partial u}\right)^c \left(\frac{\partial}{\partial u}\right)^d$$

At $O(1/r)$, Δ_{ab} is generically nonvanishing and is gauge invariant.

Memory in Bondi Coordinates; Electromagnetic Memory

In Bondi coordinates, let $h_{\mu\nu}$ denote the deviation of the metric from $\eta_{\mu\nu}$ at $O(1/r)$. Then

$$\Delta_{AB} = \frac{1}{2}\int_{-\infty}^{\infty} N_{AB} = \frac{1}{2}\left(h_{AB}|_{i^+} - h_{AB}|_{i^0}\right)$$

(N_{AB} is the Bondi news, where capital latin letters denote angular components.)

Similarly, in electromagnetism in flat spacetime, let A_μ be the vector potential at $O(1/r)$ and let E_A be the angular components of the electric field at $O(1/r)$. Define the electromagnetic memory as

$$\Delta_A = -\int_{-\infty}^{\infty} E_A = \left(A_A|_{i^+} - A_A|_{i^0}\right)$$

The presence of electromagnetic memory corresponds to a charged test particle receiving a net momentum kick.

Asymptotic Symmetries, Memory and Gauge Charges: EM Case

The asymptotic conditions on the vector potential are preserved under $A_\mu \to A_\mu + \nabla_\mu \chi$ where $\chi = \chi(x^A)$. Such "large gauge transformations" are actually symmetries—they are not degeneracies of the symplectic form. They have associated charges

$$Q_{\mathcal{C}}[\chi] = \frac{1}{4\pi} \int_{\mathcal{C}} \chi F^{(2)}_{ru} d\Omega$$

Memory corresponds to having A_A at asymptotically late and early retarded times differ by a large gauge transformation. Have

$$\frac{1}{4\pi} \int \Delta_A D^A \chi d\Omega = Q_{i^+}[\chi] - Q_{i^0}[\chi] + \int_{\mathscr{I}^+} J\chi$$

where J is the flux of charge through null infinity (which can be nonvanishing only if there are massless charged fields). Q_{i^+} is determined by the outgoing massive charged particles.

Asymptotic Symmetries, Memory and Gauge Charges: Gravitational Case

The asymptotic conditions on the metric are preserved under BMS transformations; in particular, under supertranslations $u \to u + \alpha(x^A)$. These are symmetries of the theory and have associated charges. In stationary eras, the supertranslation charges are

$$Q_{\mathcal{C}}[\alpha] = -\frac{1}{8\pi}\int_{\mathcal{C}} \alpha C^{(3)}_{ruru} d\Omega$$

Memory corresponds to having h_{AB} at asymptotically late and early retarded times differ by a supertranslation. Have

$$\frac{1}{8\pi}\int \Delta_{AB} D^A D^B \alpha d\Omega = Q_{i^+}[\alpha] - Q_{i^0}[\alpha] + \int_{\mathscr{I}^+} F\alpha$$

where F is the Bondi flux. Q_{i^+} is determined by the outgoing massive bodies.

Conservation of Charges at Spatial Infinity

For Maxwell fields satisfying suitable regularity conditions near spatial infinity, the large gauge charges at $\mathscr{I}^+$ and $\mathscr{I}^-$ match at i^0 under antipodal identification:

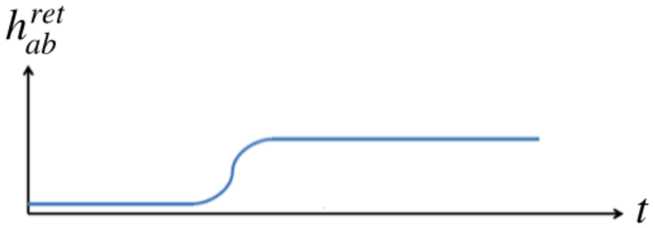
$$\lim_{\mathcal{C}^+ \to i^0} Q_{\mathcal{C}^+}[\chi] = \lim_{\mathcal{C}^- \to i^0} Q_{\mathcal{C}^-}[\tilde{\chi}]$$

where $\tilde{\chi}$ is the antipodal map on S^2 composed with χ. Similarly, in the gravitational case, if the metric satisfies (a slightly strengthened version of) Ashtekar-Hansen conditions near spatial infinity, the supertranslation charges match:

$$\lim_{\mathcal{C}^+ \to i^0} Q_{\mathcal{C}^+}[\alpha] = \lim_{\mathcal{C}^- \to i^0} Q_{\mathcal{C}^-}[\tilde{\alpha}]$$

Note: It would be good to prove this with a CK type of analysis. The CK initial data assumptions require the initial to be Schwarzschild ("$1 - 2m/r$") to order $1/r$, which implies vanishing supertranslation charges at spatial infinity. The initial data of interest here would be "$1 - 2m(\theta, \phi)/r$" to order $1/r$.

Memory and Fourier Transforms



$$\Delta_{ab} \neq 0 \implies \Delta h_{ab}^{(1)ret} \neq 0 \implies \hat{h}_{ab}^{ret}(\omega) \sim \frac{1}{\omega}$$

Memory and Infrared Divergences

The "out" quantum states are determined by the "radiation field"[1] h_{AB} at $O(1/r)$. The usual Fock space of such states is based upon a one-particle Hilbert space constructed by taking positive frequency data at $\mathscr{I}^+$ and using the Klein-Gordon (symplectic) norm

$$||h_{AB}||^2 = \int d\Omega \int_0^\infty d\omega \omega |\hat{h}_{AB}(\omega)|^2$$

But, as we have seen, for a state with memory, $\hat{h}_{AB}(\omega) \sim 1/\omega$ as $\omega \to 0$. States with memory have infinite norm in the usual Fock space. There is nothing physically or otherwise wrong with states with memory and, indeed, they unavoidably occur in scattering. But you will get divergences if you try to pretend that they are supposed to belong in the usual Fock space.

[1] The quantum operator $\mathbf{h}_{AB}$ on $\mathscr{I}^+$ actually is ill defined, but one can work with the news tensor $\mathbf{N}_{AB}$. I will ignore this issue here.

Infrared and Ultraviolet Divergences

Quantum fields are operators. In fact, it is healthier to think of them of them as elements of an abstract algebra; they would then be represented as operators in a Hilbert space representation of the algebra. States are positive (i.e., $\omega(A^*A) \geq 0$) linear functions on the algebra. The GNS construction shows that all states arise as vectors in some Hilbert space representation.

Quantum fields are distributional!

Ultraviolet divergences: Ill defined quantities resulting from taking products of the distributions at the same point (e.g., $\phi^2(x)$). A relatively sophisticated renormalization theory is needed to define these. This can be done in free (linear) quantum field theory, which enables one to do perturbation theory for nonlinear fields.

Infrared divergences: These arise from trying to put a state into a Hilbert space representation in which it doesn't belong.

Dealing with Infrared Divergences

As already noted, the usual “in” and “out” Fock spaces do not contain states of nonvanishing memory, but these arise in scattering. If you try to put an “out” state of nonvanishing memory in the standard “out” Fock space, you will get an infrared divergence in the expression for the state. People normally deal with this situation by putting in an infrared cutoff to obtain a state in the usual Fock space, calculating “inclusive cross-sections” or other quantities of interest where one sums over all “soft” photon states, and then removing the cutoff. This works for calculating cross-sections but clearly is quite unsatisfactory from a fundamental viewpoint—particularly if one views the S-matrix as fundamental—and one also cannot deal with issues such as entanglement of memory with other observables.

Memory Representations

States with memory are perfectly legitimate states. Can get Hilbert space representations of states with memory Δ_{AB} by starting with usual Fock representation and replacing $\mathbf{h}_{AB}$ by $\mathbf{h}_{AB} + H_{AB}\mathbf{I}$, where H_{AB} is a classical field with memory Δ_{AB}. (Note that H_{AB} can be chosen to be of arbitrarily low frequency.) The corresponding Fock space representations $\mathcal{F}_\Delta$ with different Δ are unitarily inequivalent.

What can we take for the "in" and "out" Hilbert spaces? The direct sum of the $\mathcal{F}_\Delta$ is way too large (non-separable) and won't work anyway (since a generic "in" state of definite memory should have nonvanishing amplitudes for all memories in the "out" state). It does not appear that any direct integral candidates work either.

Kulish-Faddeev Construction of the "in" (and "out") Hilbert Spaces in Massive QED

Key idea: The large gauge charges at spatial infinity are conserved. Therefore, can work in a sector of definite charges. However, charges are not invariant under rotations and Lorentz transformations, except for the sector of vanishing charges. Therefore, restrict to vanishing charges (including total electric charge—"put any extra charges behind the moon").

The incoming massive charged particle (improper) states $|p_1 \dots p_n\rangle_{\rm in}$ are eigenstates of the charges Q_{i^-}. Pair these states with the incoming photon states belonging to the representation with memory Δ_A such that

$$-\frac{1}{4\pi}\int \Delta_A D^A \chi d\Omega = Q_{i^-}[\chi]$$

Then take the direct integral over the p_i's and the direct sum over n to get the full "in" Hilbert space. This is referred to as *dressing the electrons*.

The Kulish-Faddeev Construction Fails in Massless QED

Again, restrict to the sector of vanishing charges at spatial infinity.
The incoming massless charged particle (improper) states $|k_1 \dots k_n\rangle_{\text{in}}$ are eigenstates of the integrated flux of charge through $\mathscr{I}^-$. So, again one could try to pair (or "dress") these states with incoming photon states belonging to the representation with memory Δ_A such that

$$\frac{1}{4\pi} \int \Delta_A D^A \chi d\Omega = \int_{\mathscr{I}^-} J\chi$$

However, for the states $|k_1 \dots k_n\rangle_{\text{in}}$, $\int J$ has angular delta-function singularities. This means that Δ_A must have correspondingly singular angular behavior. All states in such a memory representation will therefore have corresponding angular singularities in the electric field. All of the states in the Kulish-Faddeev Hilbert space are unphysical!

The Kulish-Faddeev Construction Fails Disastrously in Gravity

First, in gravity one cannot restrict to the sector of vanishing charges (including 4-momentum) at spatial infinity—unless one wants to consider only the Minkowski vacuum state. (Indeed, except for the vacuum, there are no eigenstates of all supertranslation charges, since such states would automatically have a time independent expected Bondi news.)

In massive or massless QED, one can "dress" any given charged particle incoming state of definite momenta with any photon state in the Hilbert space of the corresponding memory representation, and the Kulish-Faddeev construction will naturally yield a Hilbert space on which the quantum fields are represented. In massless QED, this construction can be done, but all states in this Hilbert space are singular. But in quantum gravity, the situation is much worse, because one would need to "dress" the incoming gravitons with other gravitons, which in turn would need to be dressed. One will not get a Hilbert space of states in this manner.

Hilbert Spaces and Quantum Field Theory

I view Hilbert spaces of states in QFT as analogous to coordinate patches on a manifold (except there are no overlap regions between different patches). A state is defined by giving all of its field correlation functions, which does not require a Hilbert space. For any given state, one can find a (separable) Hilbert space in which it sits, just as for any point in a manifold, one can find a coordinate patch in which it sits. But there is no reason why one should expect that all states of physical interest should lie in one (separable) Hilbert space, just as one shouldn't expect all points in a manifold to lie in one coordinate patch.

How Should One Do Quantum Scattering Theory?

Start with whatever physically reasonable "in" state one wishes to consider, as specified by all of its correlation functions of the "in" fields. (These can have whatever memory and/or charges at spatial infinity that one wishes to consider.) It then should be possible to calculate the corresponding "out" correlation functions by the same sort of LSZ perturbative methods as used in usual S-matrix calculations. This will yield the scattering map from "in" states to "out" states. There will be no infra-red divergences arising from trying to put the "out" state in a Hilbert space in which it does not belong.

It would be interesting to develop such a scattering theory!